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GEODESIC REFLECTIONS IN SEMI-RIEMANNIAN GEOMETRY

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1. INTRODUCTION.

The existence of a semi-Riemannian metric on a differentiable manifold M gives rise to a decomposition of the tangent bundle into a Whitney sum, $TM = T^+M \oplus T^-M$. This decomposition is given by two mutually orthogonal distributions on the manifold, in such a way that the restriction of the metric tensor to the distribution associated to T^+M is positive definite, and the restriction to the complementary distribution associated to T^-M is negative definite.

The influence of this split on the geometry of the manifold was studied by Dajczer and Nomizu [DN], with the aim of characterizing indefinite space forms. Kulkarni [K] used in a fundamental way this decomposition in order to prove the main theorem in that paper.

In the case of Lorentzian manifolds [BeE], there exists a (local) timelike vector field ξ, induced by this decomposition, in such a way that we may assume that (M, g) is time oriented.

U being a normal neigborhood of the point $m \in M$, we can consider the restrictions of the timecones:

$$I^-(m) = \{p = \exp_m(rv) / \langle v, v\rangle < 0, \langle \xi, v\rangle < 0\},$$
$$I^+(m) = \{p = \exp_m(rv) / \langle v, v\rangle < 0, \langle \xi, v\rangle > 0\}.$$

A special transformation defined in the normal neighborhood of each point is the local geodesic symmetry centered at m:

$$S_m : p = \exp_m(rx) \mapsto S_m(p) = \exp_m(-rx);$$

its study gives rise to characterizations of important kinds of (semi)-Riemannian manifolds.

For Lorentzian manifolds we have $S_m(I^+(m)) \subset I^-(m)$. Bearing in mind that $I^{(\pm)}(m)$ are open sets in M, we can study the restriction of the local geodesic symmetry S_m to $I^+(m) \cup I^-(m)$ and expect its properties to influence the geometry of the manifold.

In the more general framework of a semi-Riemannian manifold, we are interested in the study of different kinds of transformations, such as (local) geodesic symmetries and reflections with respect to submanifolds, in order to prove:

Theorem A. *A semi-Riemannian manifold (M, g) is locally symmetric if and only if the local symmetries along timelike geodesics are isometries.*

When we consider the particular case of Lorentzian manifolds, the previous theorem asserts that the restriction of the local geodesic symmetries S_m to the timecone are isometries if and only if (M, g) is locally symmetric.

Having in mind the strong relation between symplectic and indefinite almost Hermitian manifolds, we study symplectic and holomorphic geodesic reflections with respect to points and holomorphic surfaces. A result analogous to Theorem A is obtained, as well as the following characterizations of indefinite real and complex space forms:

Theorem B. *A semi-Riemannian manifold (M, g) is an indefinite real space form if and only if the geodesic reflection with respect to any timelike geodesic is an isometry.*

Theorem C. *An indefinite Hermitian manifold (M, g, J) of signature $(2p, 2q)$, $q > 1$ is a space of constant holomorphic sectional curvature if and only if the geodesic reflection with respect to any holomorphic timelike surface is symplectic.*

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2. Locally symmetric semi-Riemannian manifolds.

A natural condition to impose on a (semi)-Riemannian manifold is that its curvature tensor be parallel; that is, to have vanishing covariant differential, $\nabla R = 0$. Such a manifold is said to be locally symmetric. In particular, manifolds of constant curvature turn out to be locally symmetric.

In this section we will prove some lemmas which will be needed in the next section, in order to obtain some characterizations of locally symmetric semi-Riemannian manifolds in terms of timelike (or spacelike) geodesics.

In the remainder of this section, we will use the letters X, Y, Z for spacelike vectors, U, V, W for timelike vectors, and we will represent null vectors by T, L. An arbitrary tangent vector will be denoted by A, B, C.

Lemma 2.1. *Let (M, g) be a semi-Riemannian manifold. Then the following two conditions are equivalent:*

$$\text{i)}\quad \nabla R = 0;$$

$$\text{ii)}\quad \nabla_X R_{XAXA} = 0$$

for all unit tangent vectors X, A of M, with X spacelike.

Proof. Clearly i) implies ii). We will show that ii) implies i) in two steps. First we will prove that $\nabla_U R_{UAUA} = 0$ for all tangent vectors $U, A \in \mathfrak{X}(M)$, with U timelike; secondly, using ii) and this last condition, we will prove that $\nabla_T R_{TATA} = 0$ for all $T, A \in \mathfrak{X}(M)$, where T is a null vector, which will finish the proof.

Let U be a unit timelike vector, and X an arbitrary unit spacelike vector. If $g(X, U) = 0$, then $\lambda X + \mu U$, $(\lambda^2 - \mu^2 = 1)$, is a spacelike vector, and so, applying ii) to $\lambda X + \mu U$ and an arbitrary vector A, we have

$$\nabla_{\lambda X + \mu U} R_{\lambda X + \mu U, A, \lambda X + \mu U, A} = 0$$

which implies that $\nabla_U R_{UAUA} = 0$.

On the other hand, if $g(X, U) \neq 0$, we consider the vector $Z = \lambda_0 X + \mu U$, where $\lambda_0 = \frac{1}{2g(X,U)}$. We have $g(Z, Z) = \lambda_0^2 + \mu(1 - \mu)$, and so, if $\mu \in (0, 1)$, Z is a spacelike vector. Then, applying ii) to Z and A,

$$\nabla_{\lambda_0 X + \mu U} R_{\lambda_0 X + \mu U, A, \lambda_0 X + \mu U, A} = 0.$$

Linearizing previous expression, in the same way as before, we get

$$\nabla_U R_{UAUA} = 0,$$

for all vector A.

To finish the proof we only have to prove that $\nabla_T R_{TATA} = 0$, where T is a null vector. We consider $Z_\lambda = \frac{1}{\lambda} X + T$, for an arbitrary unit spacelike vector X, which is always a spacelike vector if $g(X, T) = 0$ or, if $g(X, T) \neq 0$, for $\lambda \neq -\frac{1}{2g(X,T)}$. Now, making use of the results proved before, and taking limits as $\lambda \mapsto \infty$, the result is obtained.

Thus, we have proved that $\nabla_A R_{ABAB} = 0, \forall A, B \in \mathfrak{X}(M)$. Then the result follows in the same way as in [VW], by using the Bianchi identities. □

Remark 2.2. By reversing the metric tensor, we transform spacelike into timelike vectors, and so condition ii) in previous lemma may be replaced by $\nabla_U R_{UAUA} = 0$ for all tangent vectors with U timelike.

In a similar way as before, we can prove the following:

Lemma 2.3. *Let (M, g) be a semi-Riemannian manifold. Then the following three conditions are equivalent:*

$$\text{i)} \quad \nabla R = 0;$$

$$\text{ii)} \quad \nabla_X R_{XYXY} = 0;$$

$$\text{iii)} \quad \nabla_U R_{UVUV} = 0$$

for all spacelike tangent vectors X, Y and timelike tangent vectors U, V.

An indefinite almost Hermitian manifold (M, g, J) is an almost complex manifold with an adapted semi-Riemannian metric, i.e.,

$$J^2 = -\mathrm{id} \qquad g(JX, JY) = g(X, Y).$$

Such a metric is necessarily of signature $(2p, 2q)$. The manifold is said to be indefinite Kähler if the almost complex structure is integrable and the Kähler form $\Omega(X, Y) = g(X, JY)$ is closed. These conditions are equivalent to $\nabla J = 0$. We refer to [BR] for some results related to the curvature of such manifolds.

Due to the special properties of the curvature tensor in the Kähler case, we can state

Lemma 2.4. *Let (M, g, J) be an indefinite Kähler manifold. Then (M, g, J) is locally symmetric if and only if $\nabla_X R_{XJXXJX} = 0$ for all spacelike tangent vectors.*

Proof. Let X, Y be arbitrary spacelike unit vectors. If $g(X, Y) \neq 0$ we consider the spacelike vector $Z = \lambda_0 X + \mu Y$ ($\mu > 0$, $\lambda_0 = \frac{1}{2g(X,Y)}$), and so

$$\nabla_Z R_{ZJZZJZ} = 0 = \nabla_{\lambda_0 X + \mu Y} R_{\lambda_0 X + \mu Y, \lambda_0 JX + \mu JY, \lambda_0 X + \mu Y, \lambda_0 JX + \mu JY}$$

Now, the coefficient of μ^4 in the expression above must vanish:

$$(1) \qquad 0 = \nabla_X R_{YJYYJY} + 2\nabla_Y R_{XJYYJY} + 2\nabla_Y R_{YJYYJX}.$$

If $g(X, Y) = 0$, $Z = \lambda X + \mu Y$ is a spacelike vector, and (1) is obtained in an analogous way as before. Using the second Bianchi identity, we get

$$0 = \nabla_X R_{YJYYJY} + \nabla_Y R_{JYXYJY} + \nabla_{JY} R_{XYYJY}$$

and substituing Y by JY in (1), we have:

$$(2) \qquad 0 = \nabla_X R_{JYYJYY} + 2\nabla_{JY} R_{XYJYY} - 2\nabla_{JY} R_{JYJXJYY}.$$

Adding (1) and (2) yields

$$\nabla_X R_{YJYYJY} = 0, \quad \text{for all spacelike vectors} \quad X, Y \in \mathfrak{X}(M).$$

Linearizing the expression above and applying the Kähler identity for the curvature tensor, after some computations (by using similar methods as those developed in [SV]), we get $\nabla_X R_{XYXY} = 0$ for all spacelike vectors X, Y tangent to M. So, the result follows as a consequence of Lemma 2.3. □

Remark 2.5. As in Remark 2.2, Lemma 2.4 holds also for timelike vectors, using $-g$ instead of g.

3. Geodesic symmetries and locally symmetric spaces.

Let (M, g) be a semi-Riemannian manifold of signature (p, q), and m an arbitrary point in M. Consider an orthonormal basis $\{e_1, \ldots, e_n\}$ of the tangent space $T_m M$. We denote by $(x_1, \ldots, x_n)$ the system of normal coordinates centered at m with $e_i = \frac{\partial}{\partial x_i}(m)$.

If ξ is a tangent vector at m and γ the geodesic through m with $\gamma(0) = m$ and $\xi = \gamma'(0)$, the geodesic symmetry with respect to m is defined by

$$S_m : p = \exp_m(r\xi) \mapsto S_m(p) = \exp_m(-r\xi).$$

By using the lemmas from the previous section, we have

Theorem A. *A semi-Riemannian manifold (M, g) is locally symmetric if and only if the local symmetries along timelike geodesics are isometries.*

Proof. We need only to prove the sufficiency. Let $\gamma(r) = \exp_m(r\xi)$ be a timelike geodesic. If the local geodesic symmetry is an isometry along γ, then we have

$$g_{ij}(\exp_m(r\xi)) = g_{ij}(\exp_m(-r\xi)), \quad 1 \leqslant i, j \leqslant n.$$

Now, in a normal coordinate neighborhood, the components of the metric tensor can be expressed by the following power series expansion [P, pg. 36]

$$g_{ij}(\exp_m(r\xi)) = g_{ij}(m) - \frac{r^2}{3} R_{\xi i \xi j}(m) - \frac{r^3}{6} \nabla_\xi R_{\xi i \xi j}(m) + O(r^4)$$

If we express the condition above using this power series expansion, we get

$$\nabla_\xi R_{\xi i \xi j} = 0, \quad i, j = 1, \ldots, n, \quad g(\xi, \xi) = -1$$

and so, the result follows from Lemma 2.1 and Remark 2.2. □

A geodesic reflection in an indefinite almost Hermitian manifold is said to be holomorphic (resp. symplectic) if it satisfies

$$(S_m)_* J = J(S_m)_* \quad \text{(respect. } S_m^* \Omega = \Omega).$$

If (M, g, J) is a locally symmetric Hermitian manifold, geodesic symmetries must be holomorphic isometries, and so, symplectic. The main purpose of the remainder of this section is to prove the converse.

Theorem 3.1. *Let (M, g, J) be an indefinite almost Hermitian manifold. Then M is locally symmetric Hermitian if and only if one of the following conditions is satisfied:*

i) *the local symmetries along timelike geodesics are holomorphic,*

ii) *the local symmetries along timelike geodesics are symplectic.*

Proof. If g is of signature $(2p, 2q)$, we consider at the point m the following basis for the tangent space:

$$\{e_1, \ldots, e_p, Je_1, \ldots, Je_p, e_{p+1}, \ldots, e_n, Je_{p+1}, \ldots, Je_n\}$$

where $g(e_i, e_j) = \varepsilon_i \delta_{ij}$, with $\varepsilon_i = 1$, for $i = 1, \ldots, p$ and $\varepsilon_i = -1$, for $i = p + 1, \ldots, n$; $(x_1, \ldots, x_{2n})$ denotes the system of local normal coordinates corresponding to such a basis.

By using similar methods as those developed in [G1] or [V], one can compute power series expansions for a general tensor field of type $(0, s)$ with respect to a system of normal coordinates in the semi-Riemannian case. If $\gamma(r) = \exp_m(r\xi)$ is a non-null geodesic through m, we have the following expansion for the components of the almost complex structure J:

$$
\begin{aligned}
J_a^b(r) &= (\varepsilon_c \Omega_{ac} g^{cb})(m) + r\varepsilon_c \left(g^{cb} \nabla_\xi \Omega_{ac}\right)(m) \\
&+ r^2 \varepsilon_c \left\{ \frac{1}{2} g^{cb} \nabla^2_{\xi\xi} \Omega_{ac} - \frac{1}{6} g^{cb} (R_{\xi a \xi Jc} - R_{\xi c \xi Ja}) + \frac{1}{3} R_{\xi c \xi b} \Omega_{ac} \right\}(m) \\
&+ r^3 \varepsilon_c \left\{ \frac{1}{6} g^{cb} \nabla^3_{\xi\xi\xi} \Omega_{ac} - \frac{1}{6} g^{cb} \sum_{t=1}^{n} (\varepsilon_t (R_{\xi a \xi t} \nabla_\xi \Omega_{tc} + R_{\xi t \xi c} \nabla_\xi \Omega_{at}) \right. \\
&\qquad - \frac{1}{12} g^{cb} (\nabla_\xi R_{\xi a \xi Jc} - \nabla_\xi R_{\xi c \xi Ja}) + \frac{1}{6} \nabla_\xi R_{\xi c \xi b} \Omega_{ac} \\
&\qquad \left. + \frac{1}{3} R_{\xi c \xi b} \nabla_\xi \Omega_{ac} \right\}(m) + O(r^4).
\end{aligned}
$$

Now, if the geodesic symmetry is holomorphic along any timelike geodesic, $\exp_m(rU)$, $(g(U,U) = -1)$, then

$$(S_m)_* J = J(S_m)_*.$$

Therefore, using the power series expansion of $J_a^b(\exp_m(rU))$, we obtain $\nabla_U \Omega_{ab} = 0$. Then $\nabla_U J = 0$ for all timelike vector U. Proceeding now as in the proof of Lemma 2.1, we obtain that $\nabla_X J = 0$, $\nabla_T J = 0$ for each spacelike vector X and null vector T. Hence (M, g, J) is an indefinite Kähler manifold.

If we consider the terms of degree three in the previous power series expansion, we get

$$\frac{1}{12} \varepsilon_b (\nabla_U R_{UaUJb} - \nabla_U R_{UbUJa}) + \frac{1}{6} \nabla_U R_{UJaUb} = 0.$$

Taking $a = U, b = JU$, the above expression gives $\nabla_U R_{UJUUJU} = 0$, and so, from Remark 2.5, (M, g, J) is a semi-Riemannian locally symmetric Hermitian manifold.

By the definition of a locally symmetric Hermitian manifold, geodesic symmetries are holomorphic maps, and so the converse in i) is trivial. Analogously, if (M, g, J) is locally symmetric Hermitian, then the geodesic symmetries are holomorphic isometries, and so, from the definition of the Kähler form, we obtain that they are symplectic.

Converseley, in order to prove that condition ii) is equivalent to i), it is enough to consider the power series expansion of the Kähler form, and to proceed in a similar way as before. (We delete the details.) □

The conditions on timelike geodesics in the previous theorems may be replaced by the analogous ones on spacelike geodesics, by considering analogous statements as in Section 2 for spacelike vectors, or by reversing the metric tensor and applying previous theorems for $(M, -g)$.

4. Geodesic reflections with respect to submanifolds and spaces of constant curvature.

Let (M, g) be a semi-Riemannian manifold of signature (p, q), and N a topologically embedded non-degenerate m-dimensional submanifold. We denote by U a neigborhood in M of a point $m \in N$ and let $\{E_1, \ldots, E_n\}$ be a local orthonormal frame field of (M, g) defined in U. We specialize the moving frame such that $\{E_1, \ldots, E_m\}$ are tangent vector fields and $\{E_{m+1}, \ldots, E_n\}$ normal vector fields of N. Further, let $(y_1, \ldots, y_m)$ be a system of coordinates in a neighborhood of m in N such that $\frac{\partial}{\partial y_i}(m) = E_i(m)$, $i = 1, \ldots, m$. Then the Fermi coordinates $(x_1, \ldots, x_n)$ with respect m, $(y_1, \ldots, y_m)$ and $\{E_{m+1}, \ldots, E_n\}$ are defined in a open neighborhood of m in M by

$$x_i\left(\exp_\nu\left(\sum_{m+1}^{n} t_\beta E_\beta\right)\right) = y_i, \quad i = 1, \ldots, n,$$

$$x_a\left(\exp_\nu\left(\sum_{m+1}^{n} t_\beta E_\beta\right)\right) = t_a, \quad a = n+1, \ldots, m,$$

where $\nu = T^\perp N$ is the normal bundle of N.

We will call a (local) reflection with respect to N the local diffeomorphism φ whose expression in local Fermi coordinates is given by

$$\varphi(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n) = (x_1, \ldots, x_m, -x_{m+1}, \ldots, -x_n).$$

We will restrict ourselves to the case of curves and holomorphic surfaces in real and Hermitian manifolds, with the aim of giving some new characterizations for real and complex indefinite space forms.

In order to characterize indefinite real space forms, we will use the following result of Dajczer and Nomizu [DN].

Theorem 4.1. *A semi-Riemannian manifold (M, g) of $\dim M \geqslant 3$ is a space of constant curvature if and only if $R(X, Y, Z, X) = 0$ for all $\{X, Y\}$ orthonormal of signature $(-, +)$ and $g(X, Z) = 0 = g(Y, Z)$.*

The signature of the plane $\{X, Y\}$ may be replaced by $(+, +)$ or $(-, -)$, and the vector Z may be restricted to range over timelike vectors or spacelike vectors if the metric g is of signature (p, q), with $p, q > 1$ [DN].

We state our Theorem B.

Theorem B. *A semi-Riemannian manifold (M, g) of dimension greater or equal than two is a space of constant curvature if and only if the geodesic reflection with respect to any timelike geodesic is an isometry.*

(Timelike geodesics may be replaced by spacelike geodesics just by reversing the metric tensor.)

Proof. First of all, we will prove that if the geodesic reflection with respect to a unit speed curve σ is an isometry, then it must be a geodesic. Let $m = \sigma(t)$ be an arbitrary point and $\gamma(r) = \exp_m(r\xi)$ an arbitrary geodesic through m meeting σ orthogonally, where $\exp_m$ is the exponential map and ξ an arbitrary unit normal vector at m.

If $(x_1, x_2, \ldots, x_n)$ is a system of Fermi coordinates with respect to σ, the geodesic reflection is an isometry if and only if the components of the metric tensor satisfy

$$(3) \qquad g_{11}(x) = g_{11}(\varphi(x)) \quad g_{ab}(x) = g_{ab}(\varphi(x)) \quad g_{1a}(x) = -g_{1a}(\varphi(x)).$$

Now, if we consider the power series expansions of the components of the metric tensor along γ, in an analogous way as in [GV], we obtain

$$g_{11}(\exp_m(r\xi)) = \varepsilon_1 - 2r\kappa_\xi(m) - r^2(R_{\xi 1\xi 1} - \varepsilon_1\kappa_\xi^2)(m) + \frac{r^3}{3}(\nabla_\xi R_{\xi 1\xi 1} - 4\varepsilon_1\kappa_\xi R_{\xi 1\xi 1})(m) + o(r^4)$$

where $\kappa_\xi = g(\xi, \ddot{\sigma})$ is the geodesic curvature of σ.

Applying the isometry condition (3) to the power series expansion of g_{11}, yields $\kappa_\xi = 0$ and therefore σ is a geodesic.

Using the fact that σ is a geodesic, the power series expansion of the components g_{1a} of the metric tensor have the expression

$$g_{1a}(\exp_m(r\xi)) = \frac{2r^2}{3}R_{\xi 1\xi a}(m) - \frac{r^3}{4}(3\nabla_\xi R_{\xi 1\xi a})(m) + o(r^4).$$

Therefore, using again the isometry condition (3), we get $R_{\xi 1\xi a} = 0$, and, by applying Theorem 4.1, it follows that (M, g) is a space of constant curvature for $\dim M \geqslant 3$.

If M is a 2-dimensional manifold, then it is a Lorentz manifold, and therefore, the scalar curvature is given by $\tau = 2R_{\xi 1\xi 1}$, where $\{\xi, X_1\}$ is an orthonormal basis of signature $(-, +)$. Now, if we consider again the power series expansion of the component g_{11} of the metric tensor along the geodesic γ and the hypothesis, we get $\nabla_\xi R_{\xi 1\xi 1} = 0 = \nabla_\xi \tau$, and so the scalar curvature is constant along timelike geodesics. Proceeding as in the lemmas in Section 2 we obtain that the scalar curvature is constant, and so also the sectional curvature.

In order to prove the converse, if (M, g) is a space of constant curvature, we can solve explicitly the Jacobi equation $Y'' + RY = 0$, and in an analogous way to that developed in [V] it is possible to find global expressions for the components of the metric tensor. So, if (M, g) has constant sectional curvature $c > 0$ and σ is an arbitrary unit speed geodesic along a spacelike normal geodesic $\gamma(r) = \exp_m(rx))$, we have the particular expressions for the metric tensor

$$\begin{aligned} g_{11}(\exp_m(rx)) &= \varepsilon_1 \left(\frac{\cos \sqrt{c} r}{\sqrt{c}} \right)^2, \\ g_{1a}(\exp_m(rx)) &= 0, \\ g_{ab}(\exp_m(rx)) &= \varepsilon_a \left(\frac{\sin \sqrt{c} r}{r \sqrt{c}} \right)^2 \delta_{ab}. \end{aligned}$$

If γ is a timelike geodesic, trigonometric functions in the expressions above must be replaced by hyperbolic functions. If the curvature c is negative, analogous formulas can be obtained by reversing the metric, having in mind that spacelike geodesics are transformed into timelike ones.

As a consequence, if (M, g) is a space of constant curvature, then geodesic reflections with respect to non-null geodesics are isometries along normal non-null geodesics. By a process analogous to that of Le mma 2.1, the result for null geodesics is obtained by taking limits. □

Remark 4.2. In the same way as the theorem before, by applying the result of [DN], one can prove that (M, g) is a space of constant curvature if and only if all geodesic reflections with respect to spacelike (resp. timelike) geodesics are isometries along all normal spacelike (resp. timelike) geodesics.

A strictly weaker condition to impose on a geodesic reflection is to be volume-preserving (up to sign). Then we obtain

Theorem 4.3. *A semi-Riemannian manifold (M, g) is locally symmetric if and only if the geodesic reflection with respect to any timelike (resp. spacelike) geodesic is volume-preserving (up to sign).*

Proof. Let $m = \sigma(t)$ be an arbitrary point, and $\gamma(r) = \exp_m(r\xi)$ an arbitrary non-null geodesic through m meeting σ orthogonally, where ξ is an arbitrary unit normal vector at m. In order to prove our result we need the power series expansion of the volume density function $\omega_{1,\dots,n} = \omega(X_1, \dots, X_n)$ along a non-null geodesic γ normal to σ.

If R is the curvature of the metric connection ∇, then the Ricci tensor is defined by

$$\varrho_{ij} = \varrho(X_i, X_j) = \sum_{k=1}^{n} \varepsilon_k R_{ikjk}$$

where $\{X_i, i = 1, \ldots, n\}$ is an orthonormal local basis of vector fields, and in an analogous way as in [GV], we obtain the expression:

$$\omega_{1,\ldots,n}(x = \exp_m(r\xi)) = 1 - r\varepsilon_1\kappa_\xi(m) - \frac{r^2}{6}(\varrho_{\xi\xi} + 2\varepsilon_1 R_{\xi 1\xi 1})(m) - \frac{r^3}{12}(\nabla_\xi \varrho_{\xi\xi} - 2\varepsilon_1\kappa_\xi\varrho_{\xi\xi} + \varepsilon_1\nabla_\xi R_{\xi 1\xi 1})(m) + O(r^4)$$

where $\varepsilon_1 = 1$ (resp. -1) if σ is a spacelike (resp. timelike) curve.

Now, if the geodesic reflection with respect to any timelike geodesic is volume-preserving, we get

$$\nabla_\xi \varrho_{\xi\xi} + \varepsilon_1 \nabla_\xi R_{\xi U\xi U} = 0 \tag{4}$$

for all unit timelike vectors U orthogonal to ξ.

If X, U are arbitrary orthonormal spacelike respectively timelike vectors orthogonal to ξ, we have that $V = \lambda X + \mu U$ is a timelike vector when $\lambda^2 < \mu^2$, which is orthogonal to ξ.

Applying (4) to the unit vector $\frac{\lambda X + \mu U}{\sqrt{\mu^2 - \lambda^2}}$, we have

$$(-\lambda^2 + \mu^2)\nabla_\xi\varrho_{\xi\xi} + \lambda^2\varepsilon_1\nabla_\xi R_{\xi X\xi X} + \mu^2\varepsilon_1\nabla_\xi R_{\xi U\xi U} + 2\lambda\mu\varepsilon_1\nabla_\xi R_{\xi X\xi U} = 0.$$

Writing the last expression for $-\lambda$, μ it follows that $\nabla_\xi R_{\xi X\xi U} = 0$, and consequently, we obtain that:

$$\nabla_\xi\varrho_{\xi\xi} - \varepsilon_1\nabla_\xi R_{\xi X\xi X} = 0 \tag{5}$$

for all unit spacelike vectors X orthogonal to ξ.

Let us complete $\{\xi\}$ to obtain an orthonormal basis of $T_m M$, and let $\{\xi, X_1, \ldots, X_{n-1}\}$ denote the parallel frame obtained along $\gamma(r)$ by parallel displacement. Then, for the derivative of the Ricci tensor, we have

$$\nabla_\xi\varrho_{\xi\xi} = \sum_{k=1}^{n-1}\varepsilon_k\nabla_\xi R_{\xi X_k\xi X_k}.$$

Now, using (4) and (5) together with the fact that σ is a timelike geodesic ($\varepsilon_1 = -1$), it follows $\nabla_\xi\varrho_{\xi\xi} = -(p+q-1)\nabla_\xi\varrho_{\xi\xi}$.

Hence, $\nabla_\xi \varrho_{\xi\xi}(m) = 0$ and so, the result follows from (4) by applying Remark 2.2.

Conversely, Let (M, g) be locally symmetric. If we use an analogous method as in [GV], having in mind that the coefficients of the power series expansion of the volume function are determined by the covariant derivatives of the curvature tensor, then one can see that the power series expansion of the volume function along a non-null geodesic is an even function and so, the geodesic reflection is volume-preserving (up to sign). Null geodesics are treated in the same way as in previous theorems. □

We note that, in an analogous way as in Theorem B, we must restrict to geodesics because if the geodesic reflection with respect to a non-degenerate curve (i.e., the tangent vector field is non-null) is volume-preserving (up to sign), then it must be a geodesic.

Aiming to obtain a characterization of indefinite complex space forms, we will consider symplectic geodesic reflections with respect to surfaces. First of all, we note that, if P is a non-degenerate submanifold in an indefinite almost Hermitian manifold (M, g, J) such that the geodesic reflection φ with respect to P is symplectic ($\varphi^*\Omega = \Omega$), then the fixed submanifold P must be holomorphic. (The proof is similar to that of Theorem 7 in [CV1], having in mind that the restriction of the metric tensor to P is non-degenerate.)

Therefore, we restrict to geodesic reflections with respect to holomorphic surfaces. In order to prove Theorem C, we will use the following result of [BR]:

Theorem 4.4. *Let (M, g, J) be an indefinite Kähler manifold. Then the sectional curvature of all non-degenerate holomorphic planes is constant at a point m if and only if*

$$R(X, JX, X, Y) = 0 \qquad (6)$$

where X, JX, Y are orthogonal tangent vectors at m.

We note that if we consider condition (6) for timelike (resp. spacelike) vectors only, then the result remains true if (M, g, J) is of signature $(2p, 2q)$, $q > 1$ (resp. $p > 1$).

Now, we are in a condition to state our theorem:

Theorem C. *An indefinite Hermitian manifold (M, g, J) of signature $(2p, 2q)$, $q > 1$ is a space of constant holomorphic sectional curvature if and only if the geodesic reflection with respect to any holomorphic timelike surface is symplectic.*

P r o o f. Let us consider an arbitrary spacelike geodesic normal to a holomorphic surface P through m, satisfying $\gamma(r) = \exp_m(r\xi)$, $\langle\xi, \xi\rangle = 1$. With respect to a system of Fermi coordinates adapted to P, we can express for the reflection along γ the condition of being symplectic:

$$\Omega_{ij}(p) = \Omega_{ij}(\varphi(p)) \quad \Omega_{ia}(p) = -\Omega_{ia}(\varphi(p)) \quad \Omega_{ab}(p) = \Omega_{ab}(\varphi(p)) \tag{7}$$

where $\frac{\partial}{\partial x_i}(m) = e_i$ are tangent vectors to P, and $\frac{\partial}{\partial x_a}(m) = e_a$ are normal vectors to P.

Considering the first condition in (7) in an analogous way as in [CV2], we obtain that $(\nabla_\xi J)\xi$ must be tangent to the submanifold. Now, M being of complex dimension greater than two and $q > 1$, there exists a timelike holomorphic surface normal to P and to $\{\xi, J\xi\}$. In the same way as before, one obtains that $(\nabla_\xi J)\xi$ must be normal to P, and so it vanishes.

In the same way as we did in Theorem 3.1, one can prove that $(\nabla_X J)X = 0$ for any tangent vector field to M, and so (M, g, J) is an indefinite nearly Kähler manifold. By using the fact that the almost complex structure J is integrable, it follows that M is an indefinite Kähler manifold.

In order to prove the constancy of the holomorphic sectional curvature, we consider the following power series expansion, obtained in an analogous way as in [CV2]:

$$\Omega_{ia}(\exp_m(r\xi)) = -r^2\left(\frac{1}{2}R_{\xi e_i \xi J e_a} - \frac{1}{6}R_{\xi e_a \xi J e_i}\right)(m) + O(r^3).$$

If we consider the second condition in (7) for the special choice of coordinates with $e_a = J\xi$, we obtain that $R(\xi, J\xi, \xi, Je_i) = 0$. Then, the constancy of the holomorphic sectional curvature follows directly from the theorem before.

In order to prove the converse, we solve the Jacobi equation, using the fact that the curvature tensor in a Kähler manifold of constant holomorphic sectional curvature is expressed by

$$\begin{aligned} R(X,Y,Z,W) = \frac{c}{2}\{&g(X,Z)g(Y,W) - g(X,W)g(Y,Z) + \Omega(X,Z)\Omega(Y,W) \\ &-\Omega(X,W)\Omega(Y,Z) + 2\Omega(X,Y)\Omega(Z,W)\}. \end{aligned}$$

If we proceed in an analogous way as in [CV1], we obtain the following expressions:

$$\begin{aligned}\Omega_{ij}(\exp_m(r\xi)) &= (\cos r\sqrt{c})^2\Omega_{ij}(m) + \Big(\frac{2}{\sqrt{c}}\cos r\sqrt{c}\sin r\frac{\sqrt{c}}{2}\Big)g(e_i, TJe_j)(m) \\ &\quad + \Big(\frac{2}{\sqrt{c}}\cos r\sqrt{c}\sin r\frac{\sqrt{c}}{2}\Big)g(Te_i, Je_j)(m) \\ &\quad + \Big(\frac{2}{\sqrt{c}}\sin r\frac{\sqrt{c}}{2}\Big)^2 g(Te_i - {}^t\perp e_i, TJe_j - {}^t\perp Je_j)(m), \\ \Omega_{ia}(\exp_m(r\xi)) &= -\Big(\frac{2}{\sqrt{c}}\sin r\frac{\sqrt{c}}{2}\Big)^2 g({}^t\perp e_i, Je_a)(m), \\ \Omega_{ab}(\exp_m(r\xi)) &= -\Big(\frac{2}{\sqrt{c}}\sin r\frac{\sqrt{c}}{2}\Big)^2 \Omega_{ab}(m)\end{aligned}$$

where T and $\perp$ are the shape operator and the normal connection of the surface P, defined in the same way as in [V].

If γ is a timelike geodesic, trigonometric fuctions must be replaced by hyperbolic ones. In the same way as for Theorem B, if the holomorphic sectional curvature is negative, by reversing the metric tensor, the formulas can be obtained from the above ones, having in mind that timelike geodesics are transformed into spacelike ones.

In any case, from the above formula it is clear that $(\varphi^*\Omega)(\exp_m(r\xi)) = \Omega(\exp_m(r\xi))$ and so the geodesic reflection is symplectic along any non-null geodesic. By using limits, analogously as in Theorem B, one obtain the result for null geodesics. □

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