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A TOPOLOGICAL CHARACTERIZATION OF STATIONARY SETS

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If X is a topological space and $x \in X$, then let $t(x, X)$ denote the 'tightness of X at x'. Recall that $t(x, X)$ is the smallest cardinal number τ such that if $A \subseteq X$ and $x \in \overline{A}$, then there exists $B \subseteq A$ such that $|B| \leqslant \tau$ and $x \in \overline{B}$. The 'tightness' of X is defined as

$$t(X) = \sup\{t(x, X) \colon x \in X\}.$$

Let κ denote an uncountable regular cardinal with the usual order topology. A subset S of κ is called 'stationary' if for every closed unbounded (cub) subset C of κ, $C \cap S \neq \varphi$. Stationary sets, with the subspace topology, are interesting topological spaces and their properties have been extensively studied in such articles as [F] and [ν DL]. The aim of this article is to prove the following topological characterization of stationary sets.

Theorem. *For a subset S of κ, the following are equivalent:*

(a) *S is stationary*

(b) *If X is a compact Hausdorff space and for each $x \in X$, $t(x, X) < \kappa$, then for any continuous function $f \colon S \to X$, there exists a unique point $x \in X$ such that $|f^{-1}(x)| = \kappa$.*

(c) *If X is a compact Hausdorff space which contains a subspace homeomorphic to S, then there exists $x \in X$ such that $t(x, X) \geqslant \kappa$.*

Proof. (a) $\to$ (b) Let S be stationary and let $f \colon S \to X$ be a continuous function, where X is a compact Hausdorff space and for each $x \in X$, $t(x, X) < \kappa$.

For each $\beta < \kappa$, let $S_\beta = \{\alpha \in S \colon \alpha \geqslant \beta\}$ and let $Y_\beta = \overline{f(S_\beta)}$. Then $\{Y_\beta \colon \beta < \kappa\}$ is a decreasing family of nonempty closed subsets of X. Therefore, $\bigcap_{\beta<\kappa} Y_\beta \neq \varphi$. Let $x \in \bigcap_{\beta<\kappa} Y_\beta$.

Let us show that $|f^{-1}(x)| = \kappa$. Assume the contrary and let $|f^{-1}(x)| < \kappa$. Then there exists $\gamma < \kappa$ such that $f^{-1}(x) \cap S_\gamma = \varphi$. Then for each $\alpha \in S_\gamma$, there exists a neighborhood U_α of $f(\alpha)$ in X such that $x \notin \overline{U_\alpha}$. By the continuity of f, there exists $\beta_\alpha < \alpha$ such that $f((\beta_\alpha, \alpha] \cap S) \subseteq U_\alpha$. Then $x \notin \overline{f((\beta_\alpha, \alpha] \cap S)}$. Since the map $\alpha \to \beta_\alpha$ is a regressive map of the stationary set S_γ into κ, by Fodor's theorem for stationary sets (otherwise known as the Pressing Down Lemma) (See [F] or [J, Theorem 22, p. 59]), there is a stationary subset A of S_γ and a $\beta < \kappa$ such that $\beta_\alpha = \beta$, for each $a \in A$. Thus $x \notin \overline{f((\beta, \alpha] \cap S)}$, for each $\alpha \in A$. Therefore, if $B \subseteq S_\beta$ and $|B| < \kappa$, then, since there exists $\alpha \in A$ such that $f(B) \subseteq f((\beta, \alpha] \cap S)$, $x \notin \overline{f(B)}$. This is a contradiction since $x \in Y_\beta = \overline{f(S_\beta)}$ and $t(x, X) < \kappa$. Hence $|f^{-1}(x)| = \kappa$.

Note that the point x is unique. Indeed, if $y \in X$ and $y \neq x$, then since $f^{-1}(x)$ and $f^{-1}(y)$ are disjoint closed subsets of S, they cannot be both unbounded. Hence $|f^{-1}(y)| < \kappa$.

(b) $\to$ (c) is obvious

(c) $\to$ (a)

Let S be a nonstationary subset of κ. Then $S \subseteq \kappa \setminus C$, for some cub subset C of κ. We will show that $\kappa \setminus C$, and hence S, can be embedded in a compact Hausdorff space X such that $t(x, X) < \kappa$, for each $x \in X$.

Let $\{P_\alpha : \alpha \in J\}$ be the collection of all maximal convex subsets of $\kappa \setminus C$. Note that $P_\alpha s$ are disjoint clopen subsets of $\kappa \setminus C$ and $\kappa \setminus C = \bigcup_{\alpha \in J} P_\alpha$. Hence $\kappa \setminus C = \oplus\{P_\alpha : \alpha \in J\}$, the disjoint sum of $P_\alpha s$. Since each P_α is a bounded subset of κ, it can be embedded in a compact Hausdorff space Y_a such that $t(y, Y_\alpha) < \kappa$, for each $y \in Y_\alpha$. For example, if $P_\alpha \subseteq [0, \beta]$, for some $\beta < \kappa$, then $[0, \beta]$ is one such space. Let Y be the disjoint sum of $Y_\alpha s$. Then Y is a locally compact Hausdorff space and $\kappa \setminus C$ is embedded in Y. Let $X = Y \cup \{x\}$ be the one-point compactification of Y. Note that $t(x, X) \leqslant \omega_0 < \kappa$ and for each $y \in Y$, $t(y, X) = t(y, Y) < \kappa$. Hence X is the desired compact Hausdorff space. □

From (c) $\to$ (a), it follows that if $\kappa = \nu^+$ is a successor cardinal, then every nonstationary subset of κ can be embedded in a compact Hausdorff space X such that $t(X) \leqslant \nu < \kappa$. Also, if $\kappa = \omega_1$, each nonstationary subset of κ can be embedded in a disjoint sum of first countable compact Hausdorff spaces. The one-point compactification of such a disjoint sum is sequential. We, therefore, obtain the following.

Theorem. *If κ is a successor cardinal (respectively, if $\kappa = \omega_1$), then a subset T of κ is nonstationary if and only if T can be embedded in a compact Hausdorff space X such that $t(X) < \kappa$ (respectively, X is sequential).*

The following example shows that the above theorem is not true for regular limit cardinals.

E x a m p l e . Let κ be an uncountable regular limit cardinal, and let C be the set of all limit cardinals less that κ. Then C is a cub subset of κ. Let $T = \kappa \setminus C$. Then T is nonstationary. Let us show that $t(T) = \kappa$.

Let ν be an arbitrary cardinal less that κ and let $A = \{\alpha \in T\colon \nu < \alpha < \nu^+\}$. Then $\nu^+ \in \overline{A}$ in T, but for any subset B of A such that $|B| \leqslant \nu$, $\nu^+ \notin \overline{B}$ in T. Therefore, $t(T) \geqslant \nu^+$. Since ν is arbitrary, it follows that $t(T) = \kappa$. Hence, T cannot be embedded in any space of tightness less than κ.

E x a m p l e . For every uncountable regular cardinal κ, there exists a first countable countably compact Hausdorff space of cardinality κ which cannot be embedded in a compact Hausdorff space of tightness less than κ. Let $S = \{\alpha < \kappa\colon cf(\alpha) = \omega_0\}$. Then S is such a space. Note that S is a stationary subset of κ.

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