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OSCILLATORY PROPERTIES OF FUNCTIONAL DIFFERENTIAL SYSTEMS OF NEUTRAL TYPE

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1. In this paper we are concerned with the oscillatory and nonoscillatory behavior of functional differential systems of the form

$$(S,\sigma)\qquad \begin{aligned} {[y_1(t)-a(t)y_1(h(t))]'} &= p_1(t)f_1(y_2(g_2(t))),\\ y_i'(t) &= p_i(t)f_i(y_{i+1}(g_{i+1}(t))), \quad i=2,\dots,n-1,\\ y_n'(t) &= \sigma p_n(t)f_n(y_1(g_1(t))), \end{aligned}$$

where $n \geqslant 2$, $\sigma = 1$ or $\sigma = -1$ and

(C_1) $a\colon [0,\infty) \to R$ is a continuous function satisfying

$$|a(t)| \leqslant \beta < 1,\ a(t)a(h(t)) \geqslant 0 \text{ on } [0,\infty),\ \text{where } \beta \text{ is a constant;}$$

(C_2) $p_i\colon [0,\infty) \to [0,\infty)$, $i = 1,2,\dots,n$ are continuous functions not identically zero on any subinterval $[T,\infty) \subset [0,\infty)$,

$$\int^{\infty} p_i(t)\,\mathrm{d}t = \infty, \quad i = 1,2,\dots,n-1;$$

(C_3) $h\colon [0,\infty) \to R$ is a continuous function, $h(t) \leqslant t$ on $[0,\infty)$, h is nondecreasing on $[0,\infty)$ and $\lim\limits_{t\to\infty} h(t) = \infty$;

(C_4) $g_i\colon [0,\infty) \to R$, $i = 1,2,\dots,n$ are continuous functions and $\lim\limits_{t\to\infty} g_1(t) = \infty$, $i = 1,2,\dots,n$;

(C_5) $f_i\colon R \to R$, $i = 1,2,\dots,n$ are continuous functions, $uf_i(u) > 0$ for $u \neq 0$, $i = 1,2,\dots,n$;

(C_6) g_i, $i = 1,2,\dots,n$ are increasing functions on $[0,\infty)$;

(C_7) f_i, $i = n-1,n$ are nondecreasing functions on R.

Remark 1. Let $g_i(t) = t$, $i = 2, \ldots, n$, $p_i(t) > 0$ on $[0, \infty)$, $i = 1, 2, \ldots, n-1$, $f_i(u) = u$, $u \in R$, $i = 1, 2, \ldots, n-1$. Then the system (S, σ) is equivalent to the n-th order differential equation of neutral type with quasiderivatives:

(E, σ)

$$\Big(\frac{1}{p_{n-1}(t)} \cdots \Big(\frac{1}{p_2(t)}\Big(\frac{1}{p_1(t)}(y(t) - a(t)y(h(t)))'\Big)'\Big)' \cdots\Big)' = \sigma p_n(t) f_n(y(g(t))).$$

Recently there has been a growing interest in the study of oscillatory solutions of neutral differential equations of n-th order, see, for example, the papers [1, 4–6, 10] and the references cited therein. As far as is known to the author, the oscillatory theory of systems of neutral differential equations is studied only in the papers [2, 3, 9].

The purpose of this paper is to establish some new criteria for the oscillation of the system (S, σ). These criteria extend and improve those introduced in [7]. Our results are new even when $a(t) \equiv 0$.

Let $t_0 \geqslant 0$. Denote

$$t_1 = \min\big\{ \inf_{t \geqslant t_0} h(t), \inf_{t \geqslant t_0} g_i(t),\ i = 1, 2, \ldots, n\big\}.$$

A function $y = (y_1, \ldots, y_n)$ is a solution of the system (S, σ) if there exists a $t_0 \geqslant 0$ such that y is continuous on $[t_1, \infty)$, $y_1(t) - a(t)y_1(h(t))$, $y_i(t)$, $i = 2, \ldots, n$ are continuously differentable on $[t_0, \infty)$ and y satisfies (S, σ) on $[t_0, \infty)$.

Denote by W the set of all solutions $y = (y_1, \ldots, y_n)$ of the system (S, σ) which exist on some ray $[T_y, \infty) \subset [0, \infty)$ and satisfy

$$\sup\Big\{\sum_{i=1}^{n} |y_i(t)| \colon t \geqslant T\Big\} > 0 \quad \text{for any } T \geqslant T_y.$$

A solution $y \in W$ is nonoscillatory if there exists a $T_y \geqslant 0$ such that its every component is different from zero for all $t \geqslant T_y$. Otherwise a solution $y \in W$ is said to be oscillatory.

2. Denote

$$\begin{aligned} &\gamma_i(t) = \sup\{s \geqslant 0 \colon g_i(s) \leqslant t\}, \quad t \geqslant 0, \quad i = 1, 2, \ldots, n; \\ &\gamma_h(t) = \sup\{s \geqslant 0 \colon h(s) \leqslant t\}, \quad t \geqslant 0; \\ &\gamma(t) = \max\{\gamma_h(t), \gamma_1(t), \ldots, \gamma_n(t)\}, \quad t \geqslant 0. \end{aligned} \tag{1}$$

For any $y_1(t)$ we define $z(t)$ by

$$z(t) = y_1(t) - a(t)y_1(h(t)), \quad t \geqslant \gamma_h(t_0) = t_1 > 0. \tag{2}$$

The inequality (2) implies that

$$y_1(t) = z(t) + a(t)y_1(h(t)) \quad t \geqslant t_1, \tag{3}$$

$$y_1(t) = z(t) + a(t)z(h(t)) + a(t)a(h(t))y_1(h((h(t)), \quad t \geqslant \gamma_h(t_1) = t_2. \tag{4}$$

Lemma 1. *Let* (C_1)–(C_5) *hold and let* $y \in W$ *be a solution of the system* (S, σ) *with* $y_1(t) \neq 0$ *on* $[t_0, \infty)$, $t_0 > 0$. *Then* y *is nonoscillatory and* $z(t)$, $y_2(t)$, …, $y_n(t)$ *are monotone on some ray* $[T, \infty)$, $T \geqslant t_0$.

Proof. Let $y \in W$ and let $y_1(t) \neq 0$ on $[t_0, \infty)$, $t_0 \geqslant 0$. Then in view of (C_3)–(C_5) the n-th equation of (S, σ) implies that either $y_n'(h(t)) \geqslant 0$ or $y_n'(h(t)) \leqslant 0$ for $t \geqslant \gamma(t_0) = T_1$, and $y_n'(t)$, $y_n(t)$ are not identically zero on any infinite subinterval of $[T_1, \infty)$. Thus y_n is a monotone function on $[T_1, \infty)$ and hence there exists a $T_2 \geqslant T_1$ such that $y_n(t) \neq 0$ on $[T_2, \infty)$. Analogously we can prove that $y_{n-1}(t), \ldots, y_2(t)$, $z(t)$ are nonoscillatory and monotone functions on an interval $[T, \infty)$, $T \geqslant T_2$. □

Lemma 2. *Suppose that* (C_1)–(C_5) *hold. Let* $y = (y_1, \ldots, y_n) \in W$ *be a nonoscillatory solution of* (S, σ) *and let* $\lim_{t\to\infty} z(t) = L_1$, $\lim y_k(t) = L_k$, $k = 2, \ldots, n$. *Then*

(5) *if* $k \geqslant 2$, $|L_k| > 0$ *implies* $\lim_{t\to\infty} y_i(t) = \delta\infty$, $i = 1, \ldots, k-1$, *where* $\delta = \operatorname{sign} L_k$;

(6) *if* $1 \leqslant k < n$, $|L_k| < \infty$ *implies* $\lim_{t\to\infty} y_i(t) = 0$, $i = k+1, \ldots, n$.

Proof. Lemma 1 implies that $z(t)$, $y_k(t)$, $k = 2, \ldots, n$ are monotone functions for large t and therefore there exist finite or infinite limits: $\lim_{t\to\infty} z(t) = L$, $\lim_{t\to\infty} y_k(t) = L_k$, $k = 2, \ldots, n$.

(i) Let $k \geqslant 2$, $L_k > 0$. Similarly we proceed if $L_k < 0$. Then there exists a $t_0 \geqslant 0$ such that $y_k(t) \geqslant L_k/2$ for $t \geqslant t_1$. From the $(k-1)$-st, …, the first equations of (S, σ), taking into account (C_2), (C_4), (C_5), we get that $y_{k-1}(t)$, …, $y_2(t)$, $z(t)$ are increasing functions and $\lim_{t\to\infty} y_i(t) = \infty$, $i = k-1, \ldots, 2$, $\lim_{t\to\infty} z(t) = \infty$.

By virtue of monotonicity of $z(t)$ (> 0), (4) and (C_1) we conclude that

$$y_1(t) \geqslant z(t) + a(t)z(h(t)) \geqslant z(t) - \beta z(h(t)) \geqslant (1-\beta)z(t).$$

If $\lim_{t\to\infty} z(t) = \infty$, then $\lim_{t\to\infty} y_1(t) = \infty$.

(ii) Let $1 \leqslant k < n$, $0 \leqslant L_k < \infty$. Suppose that $L_i > 0$ for some $i \in \{k+1, \ldots, n\}$. Then by (5) $\lim_{t\to\infty} y_i(t) = \infty$, $i = 1, \ldots, i-1$. This contradicts the fact that $L_k < \infty$. Therefore $L_i = 0$, $i = k+1, \ldots, n$. □

If $a(t) \equiv 0$ on $[0,\infty)$, then we denote the system (S,σ) by (S_0,σ). It is then a system of differential equations with deviating arguments. For the system (S_0,σ) the following lemma holds:

Lemma 3 [8, Lemma 1]. *Suppose that* (C_2), (C_4) *and* (C_5) *hold. Let* $y = (y_1,\dots,y_n)$ *be a nonoscillatory solution of* (S_0,σ) *on* $[0,\infty)$. *Then there exist an integer* $l \in \{1,\dots,n\}$, $\sigma(-1)^{n+l+1} = 1$ *or* $l = n$, *and a* $t_0 \geqslant 0$ *such that for* $t \geqslant t_0$

$$y_i(t)y_1(t) > 0, \quad i = 1,2,\dots,l,$$
$$(-1)^{l+i}y_i(t)y_1(t) > 0, \quad i = l,l+1,\dots,n.$$

We now generalize this lemma to the system (S,σ).

Lemma 4. *Suppose that* (C_1)–(C_5) *hold. Let* $y = (y_1,\dots,y_n)$ *be a nonoscillatory solution of* (S,σ) *on* $[0,\infty)$. *Then there exist an integer* $l \in \{1,2,\dots,n\}$, $\sigma(-1)^{n+l+1} = 1$ *or* $l = n$, *and a* $t_0 \geqslant 0$ *such that for* $t \geqslant t_0$ *either*

$$y_1(t)z(t) > 0, \tag{7}$$
$$y_1(t)y_i(t) > 0, \quad i = 1,2,\dots,l, \tag{8}$$
$$(-1)^{l+i}y_i(t)y_1(t) > 0, \quad i = l,l+1,\dots,n \tag{9}$$

or

$$y_1(t)z(t) < 0, \tag{10}$$
$$(-1)^i y_i(t)y_1(t) > 0, \quad i = 2,\dots,n, \quad \textit{where} \quad \sigma(-1)^n = -1. \tag{11}$$

Proof. Let $y = (y_1,\dots,y_n) \in W$ be a nonoscillatory solution of (S,σ). Without loss of generality we suppose that $y_1(g_1(t)) > 0$ for $t \geqslant T_0 \geqslant a$. Then Lemma 1 implies that $z(t)$ $(\neq 0)$ and $y_i(t)$, $i = 2,\dots,n$ are monotone on $[T_1,\infty)$, $T_1 \geqslant T_0$. Therefore either (7) or (10) hold on $[T_1,\infty)$.

I) Let (7) hold on $[T_1,\infty)$. In this case we can use Lemma 3 which implies that there exist $l \in \{1,2,\dots,n\}$, $\sigma(-1)^{n+l+1} = 1$ or $l = n$ and a $t_0 \geqslant T_1$ such that (8), (9) hold for $t \geqslant t_0$.

IIa) Let (10) hold and let $y_2(t) < 0$ on $[T_1,\infty)$. Then in view of (C_2), (C_4), (C_5), the first equation of (S,σ) implies that $z(t)$ is decreasing on $[T_2,\infty)$, $T_2 \geqslant \gamma(T_1)$. We now show that this case cannot occur. Indeed, taking into account that $y_1(t) > 0$, $z(t) < 0$ on $[T_2,\infty)$ and (C_1), we obtain from (3) that $y_1(h(t)) \geqslant y_1(t)$ on $[T_2,\infty)$.

Then with regard to the monotonicity of y_1, z, there exist $\lim_{t\to\infty} y_1(t) = c \geqslant 0$, $\lim_{t\to\infty} z(t) = L < 0$. Then (2) together with (C_1) implies

$$L = \lim_{t\to\infty} \left(y_1(t) - a(t)y_1(h(t))\right) \geqslant c(1-\beta) \geqslant 0.$$

This contradicts the inequality $L < 0$.

IIb) Let (10) hold and let $y_2(t) > 0$ on $[T_1, \infty)$. Then in view of (C_2), (C_4) and (C_5) the first equation of (S,σ) implies that $z(t)$ is increasing on $[T_2,\infty)$, $T_2 \geqslant \gamma(T_1)$. If $n \geqslant 3$ we now show that $y_3(t) < 0$ on $[T_3,\infty)$, $T_3 \geqslant T_2$. In the opposite case by virtue of (C_2), (C_4) and (C_5) the second equation of (S,σ) gives that there exist an $L_2 > 0$ and a $T_4 \geqslant T_3$ such that $y_2(t) \geqslant L_2$ on $[T_4,\infty)$. With regard to the system (S,σ) we conclude that $z(t) \geqslant z(T_4) + f_1(c)\int_{T_4}^{t} p_1(t)\,dt \to \infty$ for $t \to \infty$. This contradicts the negativeness of $z(t)$ on $[T_1,\infty)$. If $n > 3$ we similarly prove that $y_4(t) > 0$, $y_5(t) < 0$, $\ldots$, $(-1)^n y_n(t) > 0$ for $t \geqslant t_0 \geqslant T_4$, where $\sigma(-1)^n = -1$.

The proof of Lemma 4 is complete. □

Remark 2. The case $y_1(t)z(t) < 0$ on $[t_0,\infty) \subset [0,\infty)$ can occur only if $a(t) > 0$ on $[t_1,\infty)$ and $\sigma(-1)^n = -1$.

We denote by N_l^+ or N_2^- the set of all nonoscillatory solutions of (S,σ) which satisfy (7)–(9) or (10), (11), respectively. Denote by N the set of all nonoscillatory solutions of (S,σ). Then by Lemma 4 the following classification holds.

$$\begin{aligned}
(12)\qquad & N = N_n^+ \cup N_{n-1}^+ \cup \ldots \cup N_3^+ \cup N_1^+ \quad \text{for } \sigma = 1,\ n \text{ even},\\
& N = N_n^+ \cup N_{n-1}^+ \cup \ldots \cup N_4^+ \cup N_2^+ \cup N_2^- \quad \text{for } \sigma = 1,\ n \text{ odd},\\
& N = N_n^+ \cup N_{n-2}^+ \cup \ldots \cup N_2^+ \cup N_2^- \quad \text{for } \sigma = -1,\ n \text{ even},\\
& N = N_n^+ \cup N_{n-2}^+ \cup \ldots \cup N_3^+ \cup N_1^+ \quad \text{for } \sigma = -1,\ n \text{ odd}.
\end{aligned}$$

Lemma 5. I) *Let $y \in N_l^+$, $l \geqslant 2$. Then*

$$(13)\qquad |y_1(t)| \geqslant (1-\beta)|z(t)| \quad \textit{for large } t.$$

II) *Let $y \in N_1^+$.*

(i) *If $\lim_{t\to\infty} z(t) = L > 0$, then there exists an a_0: $0 < a_0 < 1$ such that*

$$(14)\qquad |y_1(t)| \geqslant a_0|z(t)| \quad \textit{for large } t;$$

(ii) *If $\lim_{t\to\infty} z(t) = 0$ then $\liminf_{t\to\infty} y_1(t) = 0$, $\lim_{t\to\infty} y_i(t) = 0$, $i = 2,\ldots,n$.*

Proof. Without loss of generality we suppose that $y_1(t) > 0$ on $[t_0, \infty)$, $t_0 \geqslant 0$.

I) The relation (13) is derived in the proof of Lemma 2.

II) (i) Let $y \in N_1^+$, $y_1(t) > 0$ on $[t_0, \infty)$ and let $\lim_{t\to\infty} z(t) = L > 0$. Then the first equation of (S, σ) together with (C_2), (C_5) implies that $z(t)$ (> 0) is a decreasing function on $[t_1, \infty)$, $t_1 \geqslant \gamma(t_0)$. We choose δ: $1 < \delta < 1/\beta$, where β is defined by (C_1). Then there exists a $t_2 \geqslant t_1$ such that $L \leqslant z(t) \leqslant z(h(t)) \leqslant \delta L$ for $t \geqslant t_2$. The last inequality implies

$$z(h(t)) \leqslant \delta L \leqslant \delta z(t) \quad \text{for} \quad t \geqslant t_o. \tag{16}$$

Taking into account (16), (C_1) we obtain from (4) that

$$y_1(t) \geqslant z(t) + a(t)z(h(t)) \geqslant z(t) - \beta z(h(t)) \geqslant (1 - \beta\delta)z(t) = a_0 z(t)$$

for $t \geqslant t_2$, where $a_0 = 1 - \beta\delta > 0$.

(ii) Let $\lim_{t\to\infty} z(t) = 0$ and $\liminf_{t\to\infty} y_1(t) = L_1 > 0$. Then (3) yields

$$0 < L_1 \leqslant \lim_{t\to\infty} z(t) + \beta \liminf_{t\to\infty} y_1(h(t)) \leqslant \beta L_1.$$

This contradicts the fact that $0 < \beta < 1$ and proves that $L_1 = 0$. Using Lemma 2 we obtain $\lim_{t\to\infty} y_1(t) = 0$, $i = 2, \dots, n$. □

Lemma 6. *Let $y \in N_2^-$. Then*

$$\lim_{t\to\infty} z(t) = 0, \quad \lim_{t\to\infty} y_1(t) = 0, \quad i = 1, 2, \dots, n. \tag{17}$$

Proof. Let $y \in N_2^-$. We may suppose that $y_1(t) > 0$, $z(t) < 0$ on $[t_0, \infty)$, $t_0 \geqslant 0$. In view of the first equation of (S, σ), (C_2), (C_5) we conclude that $z(t)$ is an increasing function on $[t_0, \infty)$. From (3), taking into account the inequality $z(t) < 0$ and (C_1) we have $y_1(t) \leqslant y_1(h(t))$, $t \geqslant t_0$. Then there exists $\lim_{t\to\infty} z(t) = L \leqslant 0$, $\lim_{t\to\infty} y_1(t) = c \geqslant 0$. Let $c > 0$. Then the inequality $y_1(t) \leqslant \beta y_1(h(t))$ implies $c \leqslant \beta c$. This contradicts the fact that $\beta < 1$. Thus we conclude that $c = 0$. From (2) we obtain $\lim_{t\to\infty} z(t) = 0$. Then using Lemma 2 we have $\lim_{t\to\infty} y_1(t) = 0$, $i = 2, \dots, n$. □

In the sequel we will use the following notation:

(18) $\quad G_1(t) = g_1(t), \quad G_i(t) = g_i(G_{i-1}(t)), \quad i = 2, \dots, n;$

$\quad g_i^{-1}(t)$ denotes the inverse function to $g_i(t)$, $\quad i = 1, \dots, n$.

(19) $\quad t_{k-1} = \max\{t_k, \gamma_k(t_k)\}, \quad s_k = \max\{s_{k-1}, g_k(s_{k-1})\}, \quad k = 2, \dots, n.$

We now put

$$f_i(x) \equiv x, \ i = 1, 2, \dots, n-2 \quad (\text{if } n \geqslant 3), \tag{20}$$

$$P_{i-1}(t) = p_{i-1}(t) f_{i-1}(|y_i(g_i(t))|), \quad i = 2, \dots, n; \tag{21}$$

$$\bar{y}_1(t) = z(t), \quad \bar{y}_i(t) = y_i(t), \quad i = 2, \dots, n.$$

The system (S, σ) in which the functions f_i, $i = 1, 2, \dots, n-2$ satisfy (20) will be denoted by $(\bar{S}, \sigma)$.

Lemma 7. *Let the assumptions* (C_1)–(C_7) *hold and let* $y = (y_1, \dots, y_n) \in W$ *be a nonoscillatory solution of* $(\bar{S}, \sigma)$ *on* $[t_0, \infty)$, $t_0 \geqslant 0$. *Then there exist a* $t_1 \geqslant t_0$ *and an integer* $l \in \{1, 2, \dots, n\}$, $\sigma(-1)^{n+l+1} = 1$ *or* $l = n$, *such that*

$$|\bar{y}_k(g_k(t))| \geqslant \int_{g_k(t)}^{s_k} p_k(x_k) \dots \int_{g_{n-2}(x_{n-3})}^{s_{n-2}} p_{n-2}(x_{n-2}) \times \int_{g_{n-1}(x_{n-2})}^{s_{n-1}} P_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} \mathrm{d}x_{n-2} \dots \mathrm{d}x_k, \tag{22_k}$$

for $t_1 \leqslant t \leqslant s_k$, $1 \leqslant k \leqslant n-1$,

$$|\bar{y}_i(g_i(t))| \geqslant \int_{t_i}^{g_i(t)} p_i(x_i) \dots \int_{t_{l-2}}^{g_{l-2}(x_{l-3})} p_{l-2}(x_{l-2}) \int_{t_{l-1}}^{g_{l-1}(x_{i-2})} p_{l-1}(x_{l-1}) \mathrm{d}x_{l-1} \mathrm{d}x_{l-2} \dots \mathrm{d}x_i, \tag{23_l}$$

for $t \geqslant t_i \geqslant \gamma(t_0)$, $i = 1, 2, \dots, l-1$, $l \leqslant n$.

Proof. The proof of this lemma is analogous to the proof of Lemma 3 in [8] and therefore we omit it. □

Remark 3. Putting (22_l) into (23_l), where $l \leqslant n-2$, we obtain

$$|\bar{y}_i(g_i(t))| \geqslant \int_{t_i}^{g_i(t)} p_i(x_i) \dots \int_{t_{l-1}}^{g_{l-1}(x_{l-2})} p_{l-1}(x_{l-1}) \int_{g_l(x_{l-1})}^{s_l} p_l(x_l) \dots \int_{g_{n-2}(x_{n-3})}^{s_{n-2}} p_{n-2}(x_{n-2}) \times \int_{g_{n-1}(x_{n-2})}^{s_{n-1}} P_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} \mathrm{d}x_{n-2} \dots \mathrm{d}x_l \mathrm{d}x_{l-1} \dots \mathrm{d}x_1, \tag{24_i}$$

$t \geqslant t_1 \geqslant t_0$, $i = 1, 2, \dots, l$, $l \leqslant n-1$.

Denote

$$(25_n)\qquad D^n_{n-1}(G_{n-1}(t),t_{n-1};p)=\int\limits_{t_{n-1}}^{G_{n-1}(t)} p_{n-1}(x_{n-1})$$

$$\times \int\limits_{g^{-1}_{n-1}(x_{n-1})}^{G_{n-2}(t)} p_{n-2}(x_{n-2})\ldots\int\limits_{g_2^{-1}(x_2)}^{G_1(t)} p_1(x_1)\mathrm{d}x_1\ldots\mathrm{d}x_{n-2}\mathrm{d}x_{n-1};$$

$$(25_1)\qquad D^1_{n-1}(G_{n-1}(t),t_{n-1};p)=\int\limits_{t_{n-1}}^{G_{n-1}(t)} p_{n-1}(x_{n-1})$$

$$\times \int\limits_{t_{n-2}}^{g^{-1}_{n-1}(x_{n-1})} p_{n-2}(x_{n-2})\ldots\int\limits_{t_1}^{g_2^{-1}(x_2)} p_1(x_1)\mathrm{d}x_1\ldots\mathrm{d}x_{n-2}\mathrm{d}x_{n-1},\quad n\geqslant 3;$$

$$(25_l)\qquad D^l_{n-1}(G_{n-1}(t),t_{n-1};p)=\int\limits_{t_{n-1}}^{G_{n-1}(t)} p_{n-1}(x_{n-1})$$

$$\times \int\limits_{t_{n-2}}^{g^{-1}_{n-1}(x_{n-1})} p_{n-2}(x_{n-2})\ldots\int\limits_{t_{l-1}}^{g_l^{-1}(x_l)} p_{l-1}(x_{l-1})\int\limits_{g^{-1}_{l-1}(x_{l-1})}^{G_{l-2}(t)} p_{l-2}(x_{l-2})$$

$$\ldots\int\limits_{g_2^{-1}(x_2)}^{G_l(t)} p_1(x_1)\mathrm{d}x_1\ldots\mathrm{d}x_{l-2}\mathrm{d}x_{l-1}\ldots\mathrm{d}x_{n-2}\mathrm{d}x_{n-1},$$

$2\leqslant l\leqslant n-1$, $t_k=g_k(t_{k-1})$, $k=l,\ldots,n-1$;

$$(25^1)\qquad D^1_1(G_1(t),t_1;p)=\int\limits_{t_1}^{G_1(t)} p_1(t)\mathrm{d}t.$$

We will say that the system (S,σ) has the property A_0 if every solution

$$y=(y_1,\ldots,y_n)\in W$$

is either oscillatory or

(P_1) $\quad z(t),\ y_i(t),\quad i=2,\ldots,n\quad$ tend monotonically to zero as $\quad t\to\infty$.

We will say that the system (S, σ) has the property B_0 if every solution

$$y = (y_1, \dots, y_n) \in W$$

is either oscillatory or (P_1) holds or

$$(P_2) \qquad \lim_{t \to \infty} y_i(t) = \delta\infty, \quad i = 1, 2, \dots, n,$$

where $\delta = \operatorname{sign} y_1(t)$.

Remark 4. (i) If the system (S, σ) has the property A_0 (the property B_0), where (P_1) holds iff $\sigma(-1)^n = 1$, then we say that the system (S, σ) has the property A (the property B).

(ii) In view of Lemma 5 and Lemma 6 the property (P_1) can be replaced by

$\liminf_{t \to \infty} y_1(t) = 0$ and $y_i(t)$ $(i = 2, \dots, n)$ tend monotonically to zero as $t \to \infty$.

Theorem 1. *Let the assumptions* (C_1)–(C_7) *hold and let there exist a continuous nondecreasing function* $g\colon [0, \infty) \to R$ *such that*

$$(26) \qquad g_n(t) \leqslant g(t), \quad g(G_{n-1}(t)) \leqslant t.$$

Let

$$(27) \qquad f_n(uv) \geqslant K f_n(u) f_n(v), \quad u > 0,\ v > 0 \quad (0 < K = const.),$$

$$(28) \qquad \int_0^{\alpha} \frac{dx}{f_n(f_{n-1}(x))} < \infty, \quad \int_0^{-\alpha} \frac{dx}{f_n(f_{n-1}(x))} < \infty$$

for every constant $\alpha > 0$.

If

$$(29) \qquad \lim_{u \to \infty} \int_T^u p_n(t) f_n(D_{n-1}^l(G_{n-1}(t), T; p))dt = \infty$$

for $l = 1, 2, \dots, n$, *where* $\sigma(-1)^{n+l+1} = 1$ *or* $l = n$, *then the system* $(\bar{S}, -1)$ *has the property* A_0 *and the system* $(\bar{S}, 1)$ *has the property* B_0.

P r o o f. Let $y=(y_1,\dots,y_n)\in W$ be a nonoscillatory solution of $(\bar{S},\sigma)$ on $[0,\infty)$. Then by Lemma 4 there exist $l\in\{1,\dots,n\}$, $\sigma(-1)^{n+l+1}=1$ or $l=n$ and a $t_0\geqslant 0$ such that the classification (12) holds. Without loss of generality we suppose that $y_1(t)>0$ for $t\geqslant t_0$.

Ia) Let $\sigma=-1$, $y\in N_n^+$ ($n+1$ is even). We prove that $N_n^+=\emptyset$. From (23_n) for $i=1$ we get

$$(30)\qquad z(g_1(t))\geqslant\int_{t_1}^{g_1(t)}p_1(x_1)\dots\int_{t_{n-2}}^{g_{n-2}(x_{n-3})}p_{n-2}(x_{n-2})\times\int_{t_{n-1}}^{g_{n-1}(x_{n-2})}p_{n-1}(x_{n-1})\mathrm{d}x_{n-1}\mathrm{d}x_{n-2}\dots\mathrm{d}x_1,\quad t\geqslant t_1\geqslant\gamma(t_0).$$

Interchanging the order of integration in (30) we obtain

$$(31)\qquad z(g_1(t))\geqslant\int_{t_{n-1}}^{G_{n-1}(t)}p_{n-1}(x_{n-1})\int_{g_{n-1}^{-1}(x_{n-1})}^{G_{n-2}(t)}p_{n-2}(x_{n-2})\dots\int_{g_2^{-1}(x_2)}^{G_1(t)}p_1(x_1)\mathrm{d}x_1\cdots\mathrm{d}x_{n-2}\mathrm{d}x_{n-1},\quad t\geqslant T=\gamma(t_{n-1}).$$

Then using the monotonicity of y_n, f_{n-1}, (26), (25_n) and (13), from (31) we get

$$(32)\qquad y_1(g_1(t))\geqslant(1-\beta)f_{n-1}(y_n(t))D_{n-1}^n(G_{n-1}(t),t_{n-1};p),\quad t\geqslant T.$$

Putting (32) into the n-th equation of $(\bar{S},-1)$ and using (27) we have

$$y_n'(t)\leqslant -K_1p_n(t)f_n(f_{n-1}(y(t_n)))f_n(D_{n-1}^n(G_{n-1}(t),t_{n-1};p)),$$

where $K_1=K^2f_n(1-\beta)$, $t\geqslant T$.

Multiplying the last inequality by $(f_n(f_{n-1}(y_n(t))))^{-1}$ and then integrating from T to u ($>T$) we get

$$(33)\qquad K_1\int_T^u p_n(t)f_n(D_{n-1}^n(G_{n-1}(t),t_{n-1};p))\mathrm{d}t\leqslant\int_{y_n(T)}^{y_n(u)}\frac{\mathrm{d}x}{f_n(f_{n-1}(x))}.$$

Then (28) together with (33) for $u\to\infty$ contradicts (29). Therefore $N_n^+=\emptyset$ if $\sigma=-1$.

Ib) Let $\sigma = 1$, $y \in N_n^+$, $n \geqslant 2$. Taking into account $y_1(g_1(t)) > 0$ on $[\gamma(t_0), \infty)$ we obtain from the n-th equation of $(\bar{S}, 1)$ that $y_n(t)$ is nondecreasing. Therefore there exist a $L_n > 0$ and a $t_1 \geqslant \gamma(t_0)$ such that $y_n(g_n(t)) \geqslant L_n$ on $[t_1, \infty)$. From (23_n) for $i = 1$, taking into account (C_7) and the last inequality we obtain

$$z(g_1(t)) \geqslant f_{n-1}(L_n) \int_{t_1}^{g_1(t)} p_1(x_1) \ldots \int_{t_{n-1}}^{g_{n-1}(x_{n-2})} p_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} \ldots \mathrm{d}x_1, \quad t \geqslant t_1.$$

Interchanging the order of integration in the last inequality and using (25_n) and (13) we get

$$y_1(g_1(t)) \geqslant (1-\beta) f_{n-1}(L_n) D_{n-1}^n(G_{n-1}(t), t_{n-1}; p), \ t \geqslant T \geqslant \gamma(t_{n-1}).$$

Putting the last inequality into the n-th equation of $(\bar{S}, 1)$ and then using (27) we successively obtain

$$y_n'(t) \geqslant K_2 p_n(t) f_n(D_{n-1}^n(G_{n-1}(t), t_{n-1}; p), \tag{34}$$

where $K_2 = K f_n((1-\beta) f_{n-1}(L_n))$, $t \geqslant T$. Integrating (34) from T to $u \to \infty$ and using (29) we have $\lim_{t\to\infty} y_n(t) = \infty$. Then by Lemma 2, $\lim_{t\to\infty} y_i(t) = \infty$, $i = 1, \ldots, n$.

II) Let $y \in N_l^+$, $2 \leqslant l \leqslant n-1$. Interchanging the order of integration in (24_1), then using the monotonicity of y_n, f_{n-1}, (26), (25_l), (13) we get

$$y_1(g_1(t)) \geqslant (1-\beta) f_{n-1}(|y_n(t)|) D_{n-1}^l(G_{n-1}(t), t_{n-1}; p). \tag{35}$$

Putting (35) into the n-th equation of $(\bar{S}, \sigma)$ and then proceeding in the same way as in the case Ia), we arrive at a contradiction with (29). We have proved that $N_l^+ = \emptyset$ if $2 \leqslant l \leqslant n-1$, $\sigma(-1)^{n+l} = -1$.

III) Let $y \in N_1^+$, $(\sigma(-1)^n = 1)$. Then in view of $y_1(t) > 0$, the first equation of $(\bar{S}, \sigma)$ implies that $z(t)$ (> 0) is a decreasing function for large t. Therefore $\lim_{t\to\infty} z(t) = L \geqslant 0$ exists. We suppose that $L > 0$. Then there exists a $t_1 \geqslant t_0$ such that

$$L \leqslant z(t) \leqslant 2L \quad \text{on} \quad [t_1, \infty). \tag{36}$$

(i) Let $n \geqslant 3$. Then (22_2) together with (9) gives

$$-y_2(g_2(t)) \geqslant \int_{g_2(t)}^{s_2} p_2(x_2) \ldots \int_{g_{n-2}(x_{n-3})}^{s_{n-2}} p_{n-2}(x_{n-2})$$

$$\times \int_{g_{n-1}(x_{n-2})}^{s_{n-1}} P_{n-1}(x_{n-1}) \mathrm{d}x_{n-1} \mathrm{d}x_{n-2} \ldots \mathrm{d}x_1, \quad t \geqslant \gamma(t_1) = t_2.$$

Putting the last inequality into the first equation of $(\bar{S},\sigma)$, then integrating from t_2 to $g_1(t)$ and using (36) we have

$$(37)\quad z(g_1(t)) \geqslant L \geqslant z(t_2) - z(g_1(t)) \geqslant \int_{t_2}^{g_1(t)} p_1(x_1) \int_{g_2(x_1)}^{s_2} p_2(x_2)\dots \int_{g_{n-2}(x_{n-1})}^{s_{n-2}} p_{n-2}(x_{n-2}) \times \int_{g_{n-1}(x_{n-2})}^{s_{n-1}} P_{n-1}(x_{n-1})\mathrm{d}x_{n-1}\mathrm{d}x_{n-2}\dots \mathrm{d}x_2\mathrm{d}x_1.$$

Interchanging the order of integration in (37), then using the monotonicity of y_n, f_{n-1}, (26), (25_l) and (14) we obtain

$$(38)\qquad y_1(g_1(t)) \geqslant a_0\, f_{n-1}(|y_n(t)|)D^1_{n-1}(G_{n-1}(t), t_n;p),$$

where a_0 is the constant from (14).

(ii) Let $n=2$ $(\sigma=1)$. Integrating the first equation of $(\bar{S},1)$ from t_1 to $g_1(t)$, then using the monotonicity of y_2, f_1, (26), (36) and (14) we get

$$(39)\qquad \begin{aligned} y_1(g_1(t)) \geqslant a_0 L \geqslant a_0\, f_1(|y_2(t)|) \int_{t_1}^{g_1(t)} p_1(x)\mathrm{d}x \\ = a_0\, f_1(|y_2(t)|)D^1_1(G_1(t), t_1;p). \end{aligned}$$

If we put (38) or (39) into the last equation of $(\bar{S},\sigma)$ and then we proceed in the same way as in the case Ia) we get a contradiction with (29). Therefore $L=0$, i.e. $\lim_{t\to\infty} z(t)=0$. Then by Lemma 5 we have $\lim_{t\to\infty}\inf y_1(t)=0$, $\lim_{t\to\infty} y_i(t)=0$, $i=2,\dots,n$.

IV) Let $y\in N_2^-$ $(\sigma(-1)^n=-1)$. Then by Lemma 6 $\lim_{t\to\infty} z(t)=0$, $\lim_{t\to\infty} y_i(t)=0$, $i=1,2,\dots,n$.

The proof of Theorem 1 is complete. □

Theorem 2. *Let the assumptions* (C_1)–(C_7), (27), (28) *hold and let*

$$(40)\qquad g_n(t)\leqslant t,\quad G_{n-1}(t)\geqslant t \quad \text{on}\quad [0.\infty).$$

If

$$(41)\qquad \lim_{u\to\infty}\int_T^u p_n(t)f_n(D^l_{n-1}(t,T;p)\mathrm{d}t=\infty$$

for $l = 1, 2, \ldots, n$, *where* $\sigma(-1)^{n+l+1} = 1$ *or* $l = n$, *then the conclusion of Theorem* 1 *holds.*

Proof. The proof is similar to that of Theorem 1, only we replace (26) and $D^l_{n-1}(G_{n-1(t)}, T; p)$ by (40) and $D^l_{n-1}(t, T; p)$, respectively. □

Theorem 1 (Theorem 2) improves and generalizes Theorem 1 (Theorem 2) in the paper [7].

Let the function

$$(\bar{C}_1) \qquad a(t) \text{ satisfy } (C_1), \text{ where } a(t) \text{ is not positive on } [0, \infty).$$

Remark 5. Let $(\bar{C}_1)$ be fulfilled. Then $N_2^- = \emptyset$ in view of Remark 2, and it is easy to see that the property (P_1) holds only if $\sigma(-1)^n = 1$. Then Theorem 1 (Theorem 2) with regard to Remark 4 implies the following theorems:

Theorem 3. *Let the assumptions* $(\bar{C}_1)$, (C_2)–(C_7), (26)–(29) *hold. Then the system* $(\bar{S}, -1)$ *has the property A and the system* $(\bar{S}, 1)$ *has the property B.*

Theorem 4. *Let the assumptions* $(\bar{C}_1)$, (C_2)–(C_7), (27), (28), (40), (41) *hold. Then the conclusion of Theorem* 3 *holds.*

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