Juhani Riihentaus

Removable singularities for Bloch and normal functions

Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 4, 723–741

Persistent URL: http://dml.cz/dmlcz/128430

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

REMOVABLE SINGULARITIES FOR BLOCH AND NORMAL FUNCTIONS

JUHANI RIIHENTAUS, Oulu

(Received March 10, 1992)

1. INTRODUCTION

1.1. Suppose that Ω is a domain in $\mathbf{C}^n$, $n \geqslant 1$, $E \subset \Omega$ is closed in Ω and $f\colon \Omega \setminus E \to \mathbf{C}$ is a holomorphic function. Kaufman [13, Theorem (a), p. 369] proved that f has a holomorphic extension f^* to Ω provided the $(2n-1)$-dimensional Hausdorff measure of E is zero and f is (areal) BMO in the whole of Ω. As a matter of fact, Kaufman gave a slightly more general result, but only for the case $n = 1$. However, his argument can be applied also for arbitrary n. Cima and Graham [4, Theorem 1, p. 691], [8, Theorem 2, pp. 177–178] proved that, provided E is a subvariety in Ω satisfying a certain geometric condition and f is holomorphic and (areal) BMO in $\Omega \setminus E$, then f has a holomorphic BMO extension f^* to Ω. We improve these results in section 2 in Theorems 2.10, 2.13 and 2.19. We also give there a short proof to a result announced by Poletskiĭ and Shabat [20, Theorem 2.3, p. 79] concerning negligible sets for the Kobayashi pseudodistance, see Lemma 2.2 below.

In section 3 in Theorems 3.2, 3.3 and in Corollary 3.4 we give related removability results for normal functions in the case when $n = 1$. Especially, answering to a question of Järvi [12, Remark 3, p. 1174] in the case $n = 1$, we show that if Ω is simply connected, if f is normal in $\Omega \setminus E$ and if the hyperbolic distancies between different points of E are above some positive constant, then f has a normal extension f^* to Ω. We conclude by giving in section 4 an answer to a question of Harvey and Polking [9, p. 42]. This question was repeated by Polking [21, p. 273].

1.2. Our notation is fairly standard. Nevertheless, for the convenience of the reader, we recall the following.

We identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, $n \geqslant 1$. If $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are points of $\mathbf{C}^n$, then we write $\overline{w} = (\overline{w}_1, \ldots, \overline{w}_n)$ and

$$z \cdot w = \langle z, \overline{w} \rangle = \sum_{j=1}^{n} z_j w_j.$$

If $z \in \mathbf{C}^n$ and $A, B \subset \mathbf{C}^n$, we write $d(A, B)$ for the euclidean distance between A and B. Also we write $d(z, A) = d(\{z\}, A)$. By $d(A)$ we mean the euclidean diameter of A. The notation $A \subset\subset B$ means that A is relatively compact in B. We write $B^{2n}(z_0, r)$ for the open ball in $\mathbf{C}^n$, with center z_0 and radius r. We also use the standard notation $D = B^2(0, 1)$ for the unit disk in the complex plane $\mathbf{C}$. If $n \geqslant 2$, $z_0 = (z_1^0, \ldots, z_n^0) \in \mathbf{C}^n$ and $R = (r_1, \ldots, r_n)$, $r_j > 0$, $j = 1, \ldots, n$, then we write

$$P = P(z_0, R) = B^2(z_1^0, r_1) \times \ldots \times B^2(z_n^0, r_n)$$

for the open polydisk with center z_0 and polyradius R. By $\partial_0 P$ we mean the distinguished boundary of P,

$$\partial_0 P = \partial_0 P(z_0, R) = \partial B^2(z_1^0, r_1) \times \ldots \times \partial B^2(z_n^0, r_n).$$

Moreover, if $1 \leqslant j \leqslant n$, then we write

$$A(Z_j) = \{z_j \in \mathbf{C} \colon z = (z_j, Z_j) \in A\},$$
$$A(z_j) = \{Z_j \in \mathbf{C}^{n-1} \colon z = (z_j, Z_j) \in A\}$$

for the sections of A. Here

$$z = (z_j, Z_j) \in \mathbf{C}^n, \quad Z_j = (z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n) \in \mathbf{C}^{n-1}.$$

The α-dimensional, $\alpha > 0$, Hausdorff outer measure is denoted by H^α. Recall that $H^0(A)$ gives the number of the points of the set A. The α-dimensional, $\alpha > 0$, upper (respectively lower) Minkowski content is denoted by M^α (respectively M_α). In general one has $C(\alpha, n) H^\alpha(A) \leqslant M_\alpha(A) \leqslant M^\alpha(A)$ for each $A \subset \mathbf{C}^n$. However, $C(d, n) H^d$, M_d and M^d all reduce to the d-dimensional Lebesgue measure on compact subsets of a smooth submanifold of $\mathbf{C}^n$ $(= \mathbf{R}^{2n})$ of dimension d. For the definitions, and for these and other properties see [9, p. 41] and [21, pp. 263–264]. The Lebesgue measure is denoted by m. As usual, $C = C(*)$ means a constant wich depends on the indicated quantities and which may vary from line to line.

The spherical distance in the extended complex plane $\mathbf{C}^* = \mathbf{C} \cup \{\infty\}$ is defined by

$$(1) \qquad \begin{aligned} q(a, b) &= \frac{|a - b|}{\sqrt{1 + |a|^2}\sqrt{1 + |b|^2}} \qquad \text{for } a, b \in \mathbf{C}, \\ q(a, \infty) &= \frac{1}{\sqrt{1 + |a|^2}} \qquad \text{for } a \in \mathbf{C}. \end{aligned}$$

Suppose that Ω is a domain in $\mathbf{C}^n$, $n \geqslant 1$. It is well-known that if V is a k-dimensional analytic subvariety in Ω, then $H^{2k}(V \cap K) < \infty$ for each compact set $K \subset \Omega$. Suppose that $f \not\equiv \infty$ is meromorphic in Ω. Then there are analytic subvarieties P_f, the *pole set* of f, and I_f, the *indeterminancy set* of f, in Ω such that $I_f \subset P_f$, $H^{2n-4}(I_f \cap K) < \infty$, $H^{2n-2}(P_f \cap K) < \infty$ for each compact set $K \subset \Omega$, f is holomorphic in $\Omega \setminus P_f$, f is spherically continuous in $\Omega \setminus I_f$ and $f|P_f \setminus I_f = \infty$. In the sequel we use the following convention: When we write "$\ldots f\colon \Omega \to \mathbf{C}^*$ is a meromorphic function ..." or "... a meromorphic function with values in $\mathbf{C}^*$ ..." or something equivalent, then it is always meant that f is meromorphic in Ω *and* $I_f = \emptyset$, i.e. locally in Ω either f or $1/f$ is holomorphic. Our terminology (probably nonstandard) here follows the one used by Cima and Krantz [5, p. 305]. Note also that the term "holomorphic mapping to $\mathbf{C}^*$" is used, see e.g. [12].

1.3. Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 1$. The infinitesimal form of the *Kobayashi metric* is given by $F_\Omega\colon \Omega \times \mathbf{C}^n \to \mathbf{R}$,

$$F_\Omega(z,\xi) = \inf\Big\{\alpha : \alpha > 0,\ \text{there is } f\colon D \to \Omega \text{ holomorphic such that } f(0) = z \quad\text{and}\quad (f'(0))(e_1) = \frac{\xi}{\alpha}\Big\}. \tag{2}$$

Here $f'(0) = df(0)$ and $e_1 = (1, 0, \ldots, 0) \in \mathbf{C}^n$.

The *Kobayashi pseudodistance* in Ω is given by $K_\Omega\colon \Omega \times \Omega \to \mathbf{R}$,

$$K_\Omega(z,z') = \inf \int_0^1 F_\Omega(\gamma(t), \gamma'(t))\, dt, \tag{3}$$

where the infimum is taken over all C^1 curves $\gamma\colon [0,1] \to \Omega$ with $\gamma(0) = z$, $\gamma(1) = z'$. Recall that K_Ω is continuous, satisfies the axioms for pseudodistance and is a distance, when Ω is hyperbolic. (Indeed, this is the very definition of hyperbolicity; for example, bounded domains of $\mathbf{C}^n$ are hyperbolic.)

Let Ω be a proper subdomain of $\mathbf{C}^n$, $n \geqslant 1$. The *quasihyperbolic metric* is given by $w_\Omega\colon \Omega \times \mathbf{C}^n \to \mathbf{R}$,

$$w_\Omega(z,\xi) = \frac{|\xi|}{d(z, \partial\Omega)}. \tag{4}$$

The quasihyperbolic distance k_Ω is defined analogously to (3). Note that here is used only C^1 curves, unlike in [26, 3.2, p. 33]; this is possible by [18, Corollary 4.8, p. 183].

Using (2), (4) and (3) one sees at once that

$$F_\Omega(z,\xi) \leqslant w_\Omega(z,\xi) \quad\text{and}\quad K_\Omega(z,z') \leqslant k_\Omega(z,z') \tag{5}$$

for all $z, z' \in \Omega$ and $\xi \in \mathbf{C}^n$.

If $n = 1$ and Ω is a hyperbolic domain in $\mathbf{C}$ (i.e. it has at least two boundary points in $\mathbf{C}$), then the Kobayashi metric F_Ω and distance K_Ω agree with the hyperbolic (Poincaré) metric λ_Ω and distance ϱ_Ω, respectively. If Ω is moreover simply connected, then

$$(6)\qquad \frac{1}{4}w_\Omega(z,\xi) \leqslant \lambda_\Omega(z,\xi) \leqslant w_\Omega(z,\xi) \quad\text{and}\quad \frac{1}{4}k_\Omega(z,z') \leqslant \varrho_\Omega(z,z') \leqslant k_\Omega(z,z')$$

for all $z, z' \in \Omega$ and $\xi \in \mathbf{C}$. For these and other related facts see e.g. [14, pp. 368–372], [2, p. 37], [5, p. 305], [20, pp. 73–75], [26, pp. 19–36], [12, p. 1171] and [19, pp. 102–103].

1.4. Next we recall the definitions of BMO, quasi-Bloch, Bloch, quasi-normal and normal functions.

Let G be a domain in $\mathbf{R}^n$, $n \geqslant 1$. A measurable function $f\colon G \to \mathbf{C}$ is BMO if there is a constant $C = C(f)$ such that for each ball $B \subset G$ there is a constant $c = c(B, f)$ so that

$$\int_B |f(x) - c|\,\mathrm{d}m(x) \leqslant Cm(B).$$

Let Ω be a proper domain in $\mathbf{C}^n$, $n \geqslant 1$. A holomorphic function $f\colon \Omega \to \mathbf{C}$ is *quasi-Bloch* if there is a constant $C = C(f)$ such that

$$(7)\qquad |f(z) - f(z')| \leqslant Ck_\Omega(z,z')$$

for all $z, z' \in \Omega$. One sees easily that (7) is equivalent with the existence of a constant $C' = C'(f, n)$ for which

$$|\nabla f(z)\cdot\xi| \leqslant C'\frac{|\xi|}{d(z,\partial\Omega)}$$

for all $z \in \Omega$ and $\xi \in \mathbf{C}^n$. Here

$$\nabla f(z) = \Big(\frac{\partial f}{\partial z_1}(z), \dots, \frac{\partial f}{\partial z_n}(z)\Big).$$

Instead of quasi-Bloch functions one often considers Bloch functions. Suppose that Ω is an arbitrary domain in $\mathbf{C}^n$. A holomorphic function $f\colon \Omega \to \mathbf{C}$ is *Bloch* if there is a constant $C = C(f)$ such that

$$|f(z) - f(z')| \leqslant CK_\Omega(z,z')$$

for all $z, z' \in \Omega$. Note that we use here the integrated form of the usual definition: A holomorphic function $f\colon \Omega \to \mathbf{C}$ is Bloch if there is a constant $C = C(f)$ such that

$$|\nabla f(z)\cdot\xi| \leqslant CF_\Omega(z,\xi)$$

for all $z \in \Omega$ and $\xi \in \mathbf{C}^n$. See [15, p. 146].

Because of (5) Bloch functions are quasi-Bloch. Conversely, quasi-Bloch functions are Bloch if $n = 1$ and Ω is hyperbolic, by Minda [19, Theorem 1, p. 105], and, if Ω is a ball and $n \geqslant 2$, in view of Timoney's result [25, Theorem 4.7, p. 260]. We point out below in Remark 2.15 that, for $n \geqslant 2$, Bloch functions form, in general, a proper subclass of quasi-Bloch functions. (Maybe, this is more or less known, but we have been unable to find any reference.) On the other hand, quasi-Bloch and holomorphic BMO classes are the same, by the following result:

1.5. Lemma. ([4, Basic Lemma, p. 693]) *Let Ω be a proper domain in $\mathbf{C}^n$, $n \geqslant 1$. Let f be holomorphic in Ω. Then f is quasi-Bloch if and only if f is* BMO.

1.6. We use Cima's and Krantz's definition for normal functions [5, pp. 305–306], see also [12, p. 1171], however, in the following integrated form: Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 1$. Let $f : \Omega \to \mathbf{C}^*$ be a meromorphic function. One says that f is *normal* (*in* Ω) if there is a constant $C > 0$ such that

$$q(f(z), f(z')) \leqslant C K_\Omega(z, z') \tag{8}$$

for all $z, z' \in \Omega$. The minimum of those constants C for which (8) holds true is called the *order of normality* of f, and it is denoted by C_f.

Quasinormal (*weakly normal*) meromorphic functions are defined analogously to (8), just replacing the Kobayashi distance K_Ω by the quasihyperbolic distance k_Ω.

Normal functions are quasi-normal by (5) above. There are quasinormal functions which are not normal, even in the case when $n = 1$. See [17, p. 6] and [19, pp. 102–103]. If $n = 1$ and Ω is a hyperbolic domain in $\mathbf{C}$, then the definition (8) is often written as follows: A meromorphic function $f : \Omega \to \mathbf{C}^*$ is normal if and only if there is a positive constant C such that

$$\frac{f^{\#}(z)}{\lambda_\Omega(z)} \leqslant C$$

for all $z \in \Omega$. Here $f^{\#} = \frac{|f'|}{1+|f|^2}$ is the spherical derivative of f, which is defined and continuous in all of Ω. (Note that the spherical derivatives of f and $\frac{1}{f}$ agree.) If Ω is moreover simply connected, then f is normal if and only if it is quasinormal. See e.g. [16, Theorem 3, p. 56] and [19, pp. 102–103] (and (6) above).

2.1. We begin with a short proof to a result announced by Poletskiĭ and Shabat [20, Theorem 2.3, p. 79]. They do not give any proof in their survey article, but just announce that a proof can be based on a variation of a reasoning of Campbell, Howard and Ochiai [3, Theorem 1, p. 106]. We here base our proof on the earlier argument of Campbell and Ogawa [2, Proposition 2, p. 40] (on which the argument of Campbell, Howard and Ochiai is based, too).

2.2. Lemma. *Let Ω be a domain in $\mathbb{C}^n$, $n \geqslant 1$. Let $E \subset \Omega$ be closed in Ω such that $H^{2n-2}(E) = 0$. Then*

$$K_{\Omega\setminus E}(z, z') = K_{\Omega}(z, z')$$

for all $z, z' \in \Omega \setminus E$.

Proof. By [2, Proposition 1, p. 39] it is sufficient to show that $\mathrm{Hol}(D, \Omega \setminus E)$ is dense in $\mathrm{Hol}(D, \Omega)$. We use here the standard notation

$$\mathrm{Hol}(D, \Omega) = \{f\colon D \to \Omega\colon f \text{ is holomorphic}\}.$$

Recall also that $\mathrm{Hol}(D, \Omega)$ is equipped with the usual compact-open topology.

Write

$$M = \{f \in \mathrm{Hol}(D, \Omega)\colon f(D) \subset\subset \Omega\}.$$

It is clear that M is dense in $\mathrm{Hol}(D, \Omega)$. Take $g \in M$ arbitrarily. Then the mapping $G\colon D \times \Omega \to D \times \Omega$,

$$G(t, b) = (t, g(t) - b),$$

is clearly a holomorphic injection. It follows, at least from [6, 2.10.45, p. 202], that $H^{2n}(D \times E) = 0$. But then $H^{2n}(G(D \times E)) = 0$. Since the projections do not increase Hausdorff measure, we find a sequence $c_k \to 0$, $c_k \in \mathbb{C}^n$, $k = 1, 2, \ldots$, such that $c_k \neq g(t) - b$ for all $t \in D$, $b \in E$ and $k = 1, 2, \ldots$. Define for each $k = 1, 2, \ldots$ holomorphic mappings $g_k\colon D \to \mathbb{C}^n$,

$$g_k(t) = g(t) - c_k.$$

Since $g(D) \subset\subset \Omega$, there is a positive integer N such that $g_k(D) \subset\subset \Omega$ for all $k \geqslant N$. It is clear that $g_k \to g$ in $\mathrm{Hol}(D, \Omega)$ as $k \to \infty$. Moreover, one easily sees that $g_k(D) \subset \Omega \setminus E$ for all $k = 1, 2, \ldots$. Hence the assertion follows. $\square$

2.3. Theorem. *Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 1$, and let $E \subset \Omega$ be closed in Ω such that $H^{2n-2}(E) = 0$. If $f : \Omega \setminus E \to \mathbf{C}$ is a Bloch function (respectively if $f : \Omega \setminus E \to \mathbf{C}^*$ is a normal meromorphic function), then f has a Bloch extension $f^* : \Omega \to \mathbf{C}$ (respectively a normal extension $f^* : \Omega \to \mathbf{C}^*$).*

Proof. By [24, Lemma 3 (i), p. 115] f has a holomorphic extension f^* to Ω. Using then Lemma 2.2 above, the continuity of the Kobayashi pseudodistance K_Ω and also the continuity of f^*, we see that f^* is Bloch. To prove the normal function case, take $z_0 \in E$ arbitrarily. Choose $R > 0$ such that $B^{2n}(z_0, R) \subset\subset \Omega$ and take $z^* \in B^{2n}(z_0, \frac{R}{2}) \setminus E$ arbitrarily. Then either $f|B^{2n}(z^*, \frac{R}{2}) \setminus E$ or $\frac{1}{f}|B^{2n}(z^*, \frac{R}{2}) \setminus E$ is a holomorphic function, and has thus again by [24, Lemma 3 (i), p. 115] a holomorphic extension g to $B^{2n}(z^*, \frac{R}{2})$. Thus f has a spherically continuous meromorphic extension f^* to Ω. The normality of f^* follows then again by using Lemma 2.2, the continuity of the Kobayashi pseudodistance K_Ω and the spherical continuity of f^*. □

2.4. Remark. The above result supplements the result of [22, Corollary 3.2, p. 148]. For a partial generalization, where the assumption "$H^{2n-2}(E) = 0$" is replaced by the weaker assumption "$H^{2n-2}(E \cap K) < \infty$ for each $K \subset \Omega$ compact", see Theorem 2.13 below.

2.5. For convenience of the reader, we recall first four basic results which we need in the proof of our results, Theorem 2.10 and Theorem 2.13 below.

2.6. Lemma. ([10, Corollary 1, p. 188]) *Let G be a domain in $\mathbf{C}$. Let $E \subset G$ be closed in G and let $H^1(E) = 0$. Suppose that f is holomorphic in $G \setminus E$. If for some $p > 1$*

$$\int_{G \setminus E} |f'(z)|^p \mathrm{d}m(z) < \infty,$$

then f has a holomorphic extension f^* to G.

2.7. Lemma. ([6, 2.10.25, p. 188] or [24, Corollary 4, p. 114]) *Let $A \subset \mathbf{C}^n$, $n \geqslant 2$.*

(a) *If $H^{2n-2}(A) < \infty$, then for each j, $1 \leqslant j \leqslant n$, and for H^{2n-2}-almost all $Z_j \in \mathbf{C}^{n-1}$ $H^0(A(Z_j)) < \infty$.*

(b) *If $H^{2n-1}(A) = 0$, then for each j, $1 \leqslant j \leqslant n$, and for H^{2n-2}-almost all $Z_j \in \mathbf{C}^{n-1}$ $H^1(A(Z_j)) = 0$.*

The proof of the next lemma follows from the properties of the Kobayashi metric, just as in [23, Lemma 2.2, p. 925], see [5, Proposition 1.6 and Corollary 1.7, p. 309] and [12, Lemma 2, p. 1173].

2.8. Lemma. *Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 2$, and $1 \leqslant j \leqslant n$. Suppose that $f\colon \Omega \to \mathbf{C}$ is Bloch (respectively $f\colon \Omega \to \mathbf{C}^*$ is normal). If $Z_j \in \mathbf{C}^{n-1}$ is such that $\Omega(Z_j)$ is a nonempty domain in $\mathbf{C}$, then the holomorphic function $f_{Z_j}\colon \Omega(Z_j) \to \mathbf{C}$ (respectively the meromorphic function $f_{Z_j}\colon \Omega(Z_j) \to \mathbf{C}^*$),*

$$f_{Z_j}(z_j) = f(z_j, Z_j),$$

is Bloch (respectively normal).

2.9. Lemma ([11, Lemma 3.4, p. 299]). *Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 2$. Let $E \subset \Omega$ be closed in Ω and such that for each j, $1 \leqslant j \leqslant n$, and for H^{2n-2}-almost all $Z_j \in \mathbf{C}^{n-1}$ the section $E(Z_j)$ is totally disconnected. Let $f\colon \Omega \setminus E \to \mathbf{C}$ be a holomorphic function. If for each j, $1 \leqslant j \leqslant n$, and for H^{2n-2}-almost all $Z_j \in \mathbf{C}^{n-1}$ the function $f_{Z_j}\colon (\Omega \setminus E)(Z_j) \to \mathbf{C}$,*

$$f_{Z_j}(z_j) = f(z_j, Z_j),$$

has a holomorphic (respectively a meromorphic) extension $f^*_{Z_j}$ to $\Omega(Z_j)$, then f has a holomorphic (respectively a meromorphic) extension f^* to Ω.

Next we give our result for quasi-Bloch functions:

2.10. Theorem. *Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 1$. Let $E \subset \Omega$ be nonempty, closed in Ω and such that $M^{2n-1-p}(E) < \infty$ for some p, $0 < p \leqslant 2n-1$. If f is quasi-Bloch in $\Omega \setminus E$, then f has a holomorphic extension f^* to Ω.*

P r o o f. We first show that for each p', $1 < p' < 1+p$, $|\nabla f| \in \mathcal{L}^{p'}_{\mathrm{loc}}(\Omega)$. Since $M^{2n-1-p}(E) < \infty$, there is $M_1 < \infty$ and $\varepsilon_0 > 0$ such that $m(E_\varepsilon) \leqslant M_1 \varepsilon^{1+p}$ for all ε, $0 < \varepsilon \leqslant \varepsilon_0$. Here

$$E_\varepsilon = \{z \in \mathbf{C}^n : d(z, E) < \varepsilon\}.$$

Proceeding as in [9, p. 42], we write for $j = 0, 1, 2, \ldots$

$$K_j = \{z \in \mathbf{C}^n : d(z, E) < \varepsilon_0 2^{-j}\},$$

and get

$$E_{\varepsilon_0} = \bigcup_{j=0}^{\infty} (K_j \setminus K_{j+1}).$$

Choose then p', $1 < p' < 1+p$, arbitrarily. Then

$$\begin{aligned}\int_{E_{\varepsilon_0}} \frac{1}{d(z,E)^{p'}} \mathrm{d}m(z) &\leqslant \sum_{j=0}^{\infty} \int_{K_j \setminus K_{j+1}} \frac{1}{d(z,E)^{p'}} \mathrm{d}m(z) \\ &\leqslant \sum_{j=0}^{\infty} [\varepsilon_0 2^{-(j+1)}]^{-p'} m(K_j) \\ &\leqslant \sum_{j=0}^{\infty} \varepsilon_0{}^{-p'} 2^{(j+1)p'} M_1 (\varepsilon_0 2^{-j})^{1+p} \\ &= 2^{p'} M_1 \varepsilon_0{}^{1+p-p'} \sum_{j=0}^{\infty} 2^{(p'-1-p)j} < \infty.\end{aligned}$$

Take $z_0 \in E$ arbitrarily, and choose $r_0 > 0$ so small that $d(z, \partial(\Omega \setminus E)) = d(z,E)$ for all $z \in B^{2n}(z_0, r_0)$. Then we get

$$\int_{E_{\varepsilon_0} \cap B^{2n}(z_0,r_0)} |\nabla f(z)|^{p'} \mathrm{d}m(z) \leqslant C \int_{E_{\varepsilon_0}} \frac{1}{d(z,E)^{p'}} \mathrm{d}m(z) < \infty.$$

Thus $|\nabla f| \in \mathcal{L}^{p'}_{\mathrm{loc}}(\Omega)$. Using then Fubini's theorem, the fact that $H^{2n-1}(E) = 0$ (since $M^{2n-1-p}(E) < \infty$), Lemma 2.7 (b), Lemma 2.6 and Lemma 2.9, we see that f has a holomorphic extension f^* to Ω, concluding the proof. □

2.11. Remark. Because of Shiffman's result [24, Lemma 3 (i), p. 115], the above result is of interest only for $0 < p \leqslant 1$. In this case our theorem gives, in view of Lemma 1.5, a partial improvement to the cited result of Kaufman. Using Theorem 2.10 we give below in Theorem 2.19 an improvement to the aforementioned result of Cima and Graham.

2.12. Remark. We give an example which shows that if $0 < p \leqslant 1$ then the condition "$M^{2n-1-p}(E) < \infty$" in the above theorem cannot be replaced by the condition "$H^{2n-1-p}(E) < \infty$", except of course when $n = p = 1$. As a matter of fact, our example shows that not even the condition "$H^{2n-2}(E) < \infty$" is sufficient, except when $n = 1$. Our example is a modification of Timoney's, and Cima's and Graham's example [4, pp. 696, 699].

We first construct our exceptional set in $\mathbf{C}^n$. For later need in section 4, the construction is given in a slightly more general form than is actually necessary here. Suppose first that $n \geqslant 2$ and $0 < q \leqslant 1$. Write for $k = 2, 3, \ldots$

$$A_k = D^{n-1} \times \partial B^2\Big(0, \frac{1}{k^q}\Big).$$

Choose for each $k = 2, 3, \dots$ a finite number of points $a_1^k, a_2^k, \dots, a_{N(k)}^k \in A_k$ such that for each $z \in A_k$

$$|z - a_j^k| < \frac{1}{k^q} - \frac{1}{(k+1)^q},$$

for some $j \in \{1, 2, \dots, N(k)\}$. This is possible, since $\overline{A_k}$ is compact. Set then

$$E = \bigcup_{k=2}^{\infty} \{a_1^k, a_2^k, \dots, a_{N(k)}^k\} \cup (D^{n-1} \times \{0\}).$$

Then $E \subset D^n$ is closed in D^n. Clearly now $H^{2n-2}(E) < \infty$. Define f,

$$f(z) = f(z_1, \dots, z_{n-1}, z_n) = \frac{1}{z_n}.$$

Then f is holomorphic in $D^n \setminus E$, but has no holomorphic extension to D^n. Choose then $q = 1$. It remains to show that f is quasi-Bloch in $D^n \setminus E$. It is sufficient to show that $d(z, E)|\nabla f(z)|$ is bounded in $D^n \setminus E$. Take $z = (z_1, \dots, z_{n-1}, z_n) \in D^n \setminus E$ arbitrarily and suppose that

$$\frac{1}{k+1} \leqslant |z_n| < \frac{1}{k}$$

for some $k \in \mathbf{N}$. Then

$$d(z, E) \leqslant 2\left(\frac{1}{k} - \frac{1}{k+1}\right),$$

and thus

$$d(z, E)|\nabla f(z)| \leqslant \frac{2(k+1)^2}{k(k+1)} \leqslant 4.$$

Suppose then that $n = 1$ and $0 < p < 1$. Proceeding as above, we get a countable set E (for which then $H^{1-p}(E) = 0$) and a quasi-Bloch function f in $D \setminus E$, which has no holomorphic extension to D.

On the other hand, the assumption "$H^{2n-2}(E) < \infty$" is sufficient for Bloch functions (and for normal functions, too):

2.13. Theorem. *Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 1$. Let $E \subset \Omega$ be closed in Ω and such that $H^{2n-2}(E \cap K) < \infty$ for all $K \subset \Omega$ compact. If f is Bloch (respectively normal) in $\Omega \setminus E$, then f has a holomorphic (respectively meromorphic) extension f^* to Ω.*

Proof. Because of Lemma 2.7 (a), Lemma 2.8 and Lemma 2.9 it is sufficient to give the proof for $n = 1$. But then the exceptional set E is locally finite in Ω. Since Bloch functions are quasi-Bloch, the result follows from Theorem 2.10. The normal function case is proved similarly, see [23, Theorem 3.5, p. 927]. □

2.14. Remark. Timoney's example [4, p. 696] shows that the holomorphic extension f^* need not be Bloch. Similarly, the meromorphic extension f^* need not be a meromorphic *function*, see [12, Remark 2, p. 1174] and also [23, 3.6, p. 927].

2.15. Remark. Because of Theorem 2.13 the quasi-Bloch function f given in Remark 2.12 above cannot be Bloch. Thus for $n \geqslant 2$ the Bloch class is, in general, a proper subclass of the quasi-Bloch class, unlike for the case $n = 1$. See [19, Theorem 1, p. 105] and [25, Theorem 4.7, p. 260].

2.16. In order to improve Cima's and Graham's result [4, Theorems 1 and 1', pp. 691, 697] we recall for convenience here their covering condition A. (In fact, we rewrite their condition in a slightly more compact form.)

Let Ω be a domain in $\mathbf{C}^n$, $n \geqslant 1$. Let $E \subset \Omega$ be closed in Ω. Then E satisfies the *A covering condition* if the following conditions are satisfied:

(a) There exist polydisks P_α, $\alpha \in \Lambda$, such that $\overline{P_\alpha} \subset \Omega$, $\alpha \in \Lambda$, and

$$E \subset \cup_{\alpha \in \Lambda} P_\alpha .$$

(b) There exists a positive constant c_1 such that

$$d(P_\alpha) \leqslant c_1 d(P_\alpha, \partial\Omega)$$

for all $\alpha \in \Lambda$.

(c) There exists a positive constant c_2 such that

$$d(\partial_0 P_\alpha, E) \geqslant c_2 d(P_\alpha, \partial\Omega)$$

for all $\alpha \in \Lambda$.

(d) There is a positive constant c_3 such that for any $z \in E$ there exists a polydisk P_{α_0}, $\alpha_0 \in \Lambda$, with $z \in P_{\alpha_0}$ and

$$d(z, \partial P_{\alpha_0}) \geqslant c_3 d(P_{\alpha_0}, \partial\Omega).$$

When $n \geqslant 2$ examples of sets satisfying the A covering condition are subvarieties of bounded domains which extend across the boundary of the domain and are smooth near the boundary. See [4, pp. 697, 701].

When $n = 1$ examples of such sets are sets E which are *δ-separated* (*sparse*) *in* Ω in the following sense: There is a positive constant δ such that

$$|a - b| \geqslant \delta \cdot d(a, \partial\Omega)$$

for all $a, b \in E$, $a \neq b$.

Note that if $A \subset \Omega$ is separated in Ω, then Ω is necessarily a proper open subset of $\mathbf{C}$. When $\Omega = D$ the above separatedness condition can be expressed both using the pseudohyperbolic distance ψ_D,

$$\psi_D(a,b) = \frac{|a-b|}{|1-\bar{a}b|}, \quad a, b \in D,$$

and using the hyperbolic distance ϱ_D, as follows:

2.17. Proposition. *If $A \subset D$, then the following conditions are equivalent:*
(i) *There is $\delta > 0$ such that A is δ-separated in D.*
(ii) *There is $\delta' > 0$ ($\delta' = \delta'(\delta)$) such that $\psi_D(a,b) \geqslant \delta'$ for all $a, b \in A$, $a \neq b$.*
(iii) *There is $\delta'' > 0$ ($\delta'' = \delta''(\delta)$) such that $\varrho_D(a,b) \geqslant \delta''$ for all $a, b \in A$, $a \neq b$.*

The equivalence of (i) and (ii) follows from [4, (1), p. 691, and (6), p. 694]. The equivalence of (ii) and (iii) follows from the fact that

$$\varrho_D(a,b) = \frac{1}{2}\log\frac{1+\psi_D(a,b)}{1-\psi_D(a,b)}.$$

See also [8, p. 177] and [7, (1.4), p. 286 and pp. 2, 5].

2.18. Remark. It is easy to see that the A covering condition and the Minkowski dimension condition "$M^{2n-1-p}(E) < \infty$ for some p, $0 < p \leqslant 1$" are mutually independent. In fact, one finds easily a set E which satisfies the A covering condition but for wich $M^{2n-1-p}(E) = \infty$ for some p, $0 < p \leqslant 1$. On the other hand, choosing $n = 1$, $\Omega = D$ and

$$E = \left\{1 - \frac{1}{k} : k \in \mathbf{N}\right\}$$

one gets a set for wich $M^{1-p}(E) = 0$ when $0 < p < \frac{1}{2}$, but which does not satisfy the A covering condition. We leave the details to the reader.

The next theorem improves the cited result of Cima and Graham:

2.19. Theorem. *Let Ω be a proper domain in $\mathbf{C}^n$, $n \geqslant 1$. Let $E \subset \Omega$ be closed in Ω and such that $M^{2n-1-p}(E) < \infty$ for some p, $0 < p \leqslant 2n-1$. Suppose further that E satisfies the A covering condition. If f is quasi-Bloch in $\Omega \setminus E$, then f has a quasi-Bloch extension f^* to Ω.*

Proof. By Theorem 2.10 f has a holomorphic extension f^* to Ω. We show by a short argument that f^* is quasi-Bloch in Ω. (Cima's and Graham's argument

[4, Lemmas 2 and 3, pp. 698–699] can certainly also be applied in this more general situation.)

We know that there is a positive constant C such that

$$|\nabla f(z)| \leqslant C \frac{1}{d(z, \partial\Omega')} \tag{9}$$

for all $z \in \Omega \setminus E$. Here $\Omega' = \Omega \setminus E$. We must show that for some positive constant C'

$$|\nabla f^*(z)| \leqslant C' \frac{1}{d(z, \partial\Omega)} \tag{10}$$

for all $z \in \Omega$. For this purpose take $z \in \Omega$ arbitrarily.

Suppose first that $z \in P_{\alpha_0}$ for some $\alpha_0 \in \Lambda$. Using the maximum principle, the condition (c) above and (9), we find a $z_0 \in \partial_0 P_{\alpha_0}$ such that

$$|\nabla f^*(z)| \leqslant |\nabla f(z_0)| \leqslant C \frac{1}{d(z_0, \partial\Omega')}. \tag{11}$$

By the conditions (b) and (c) we have clearly

$$\begin{aligned} d(z, \partial\Omega) &\leqslant d(P_{\alpha_0}) + d(P_{\alpha_0}, \partial\Omega) \leqslant c_1 d(P_{\alpha_0}, \partial\Omega) + d(P_{\alpha_0}, \partial\Omega) \\ &\leqslant (c_1 + 1) \cdot \frac{1}{c_2} d(z_0, E). \end{aligned}$$

Similarly, we get with the aid of (b)

$$\begin{aligned} d(z, \partial\Omega) &\leqslant d(P_{\alpha_0}) + d(z_0, \partial\Omega) \leqslant c_1 d(P_{\alpha_0}, \partial\Omega) + d(z_0, \partial\Omega) \\ &\leqslant (c_1 + 1) d(z_0, \partial\Omega). \end{aligned}$$

Thus (10) follows from (11) when $z \in \cup_{\alpha \in \Lambda} P_\alpha$.

Consider then the case when $z \notin \cup_{\alpha \in \Lambda} P_\alpha$. Take $z^* \in E$ such that $d(z, E) = |z - z^*|$. It is clearly sufficient to consider the case when

$$d(z, E) < \frac{d(z, \partial\Omega)}{2}. \tag{12}$$

By the condition (d) we find a polydisk P_{β_0}, $\beta_0 \in \Lambda$, such that $z^* \in P_{\beta_0}$ and

$$d(z, E) = |z - z^*| \geqslant d(z^*, \partial P_{\beta_0}) \geqslant c_3 d(P_{\beta_0}, \partial\Omega). \tag{13}$$

Then it follows from (12) and (b) that

$$\begin{aligned} d(z, \partial\Omega) &\leqslant d(z, E) + d(P_{\beta_0}) + d(P_{\beta_0}, \partial\Omega) \\ &\leqslant d(z, E) + c_1 d(P_{\beta_0}, \partial\Omega) + d(P_{\beta_0}, \partial\Omega) \\ &\leqslant \frac{d(z, \partial\Omega)}{2} + (c_1 + 1) \cdot d(P_{\beta_0}, \partial\Omega). \end{aligned}$$

Thus we get from (13)

$$d(z, \partial\Omega) \leqslant 2(c_1 + 1) d(P_{\beta_0}, \partial\Omega) \leqslant \frac{2}{c_3}(c_1 + 1) d(z, E).$$

Therefore (10) follows from (9), concluding the proof. □

3. Quasinormal and normal functions on planar domains

3.1. We begin by giving a removability result for normal functions. Our proof for this case is based partly on a variant of the arguments in [12, proof of Lemma 1, p. 1172] and [16, proof of Theorem 9, p. 63], see also [20, p. 75].

3.2. Theorem. *Let Ω be a hyperbolic domain in $\mathbf{C}$. Suppose that $E \subset \Omega$ and E is δ-separated in Ω, where $\delta > 0$. Let $f: \Omega \setminus E \to \mathbf{C}^*$ be a meromorphic function. If f is normal, then f has a quasinormal meromorphic extension $f^*: \Omega \to \mathbf{C}^*$.*

Proof. By [16, Theorem 9, p. 62] f has a meromorphic extension $g = f^*: \Omega \to \mathbf{C}^*$.

Write $\Omega' = \Omega \setminus E$. Let C be a positive constant such that

$$f^{\#}(z) \leqslant C \frac{1}{d(z, \partial\Omega')}$$

for all $z \in \Omega'$.

It is sufficient to show that there is a positive constant C^* such that for all $z \in \Omega$

$$g^{\#}(z) \leqslant C^* \frac{1}{d(z, \partial\Omega)}.$$

We may suppose that g is nonconstant and that $0 < \delta < 1$. We first consider the situation near the exceptional set E. Take $b \in E$ arbitrarily and let $r_0 = \delta d(b, \partial\Omega)$. Since E is δ-separated in Ω, we know that $B^2(b, r_0) \setminus \{b\} \subset \Omega'$.

Suppose first that $|g(b)| \leqslant 1$. We claim that $|g(z) - g(b)| < 1$ for all $z \in B^2(b, r_b)$, where $r_b = r_0 e^{-40C}$. Suppose, on the contrary, that $|g(z) - g(b)| \geqslant 1$ for some

$z \in B^2(b, r_b)$. Choose $z_0 \in B^2(b, r_b)$ such that $|g(z_0) - g(b)| = 1$ and $|g(z) - g(b)| < 1$ for all $z \in B^2(b, |z_0 - b|)$. Write $r = |z_0 - b|$. First of all, note that $|g(z)| \leqslant 2$ for all $z \in \overline{B}^2(b, r)$. Thus we have for $z, z' \in \overline{B}^2(b, r)$

$$\frac{1}{5}|g(z') - g(z)| \leqslant q(g(z'), g(z))$$

and therefore also

$$\frac{1}{5}|g'(z)| \leqslant g^{\#}(z)$$

for all $z \in \overline{B}^2(b, r)$. But then we have for $z', z'' \in \partial B^2(b, r)$

$$\begin{aligned}
|g(z'') - g(z')| &\leqslant 5 \int_{\partial B^2(b,r)} g^{\#}(z)|\mathrm{d}z| = 5 \int_{\partial B^2(b,r)} \frac{|g'(z)|}{1 + |g(z)|^2}|\mathrm{d}z| \\
&\leqslant 5C \int_{\partial B^2(b,r)} \frac{1}{2|z - b| \log \frac{r_\Omega}{|z-b|}}|\mathrm{d}z| = \frac{5C}{2} \int_0^{2\pi} \frac{r}{r \log \frac{r_\Omega}{r}} \mathrm{d}\theta \\
&\leqslant \frac{5\pi C}{\log \frac{r_\Omega}{r_b}} = \frac{5\pi C}{\log \mathrm{e}^{40C}} = \frac{\pi}{8}.
\end{aligned}$$

Since g is nonconstant and holomorphic in a neighborhood of $\overline{B}^2(b, r)$, we thus have for $z', z'' \in \overline{B}^2(b, r)$

$$|g(z'') - g(z')| \leqslant \frac{\pi}{8}.$$

But this is a contradiction with our assumption that $|g(z_0) - g(b)| = 1$.

Now we know that $|g(z)| \leqslant 2$ for all $z \in B^2(b, r_b)$. By a standard application of the Cauchy integral formula one sees that for $z \in B^2(b, \frac{r_b}{2})$

$$\begin{aligned}
g^{\#}(z) \leqslant |g'(z)| &\leqslant \frac{8}{r_b} = \frac{8\mathrm{e}^{40C}}{\delta} \frac{1}{d(b, \partial\Omega)} \leqslant \frac{8\mathrm{e}^{40C}}{\delta} \frac{1}{\frac{d(z,\partial\Omega)}{1+\frac{\delta}{2}\mathrm{e}^{-40C}}} \\
&\leqslant \frac{8\mathrm{e}^{40C} + 4\delta}{\delta} \frac{1}{d(z, \partial\Omega)}.
\end{aligned}$$

Suppose then that $|g(b)| \geqslant 1$. If we write $h = \frac{1}{g}$, then h is a normal meromorphic function for which $C_h = C_g$. Since also $h^{\#}(z) = g^{\#}(z)$ for all $z \in \Omega$, we get, proceeding as above, that

$$g^{\#}(z) = h^{\#}(z) \leqslant |h'(z)| \leqslant \frac{8\mathrm{e}^{40C} + 4\delta}{\delta} \frac{1}{d(z, \partial\Omega)}$$

for all $z \in B^2(b, \frac{r_b}{2})$.

Until now we know that

$$g^{\#}(z) \leqslant \frac{8e^{40C} + 4\delta}{\delta} \frac{1}{d(z, \partial\Omega)} \tag{14}$$

for all $z \in U$, where

$$U = \bigcup_{b \in E} B^2\left(b, \frac{r_b}{2}\right).$$

We then consider the case when $z \in \Omega \setminus U$. Suppose first that $d(z, \partial\Omega') > \frac{d(z,\partial\Omega)}{3}$. Then it follows from the normality of f and from [19, p. 102] (see also (6) above) that

$$g^{\#}(z) \leqslant C \frac{1}{d(z, \partial\Omega')} \leqslant 3C \frac{1}{d(z, \partial\Omega)}. \tag{15}$$

Suppose then that $d(z, \partial\Omega') = |z - a| \leqslant \frac{d(z,\partial\Omega)}{3}$, where $a \in \partial\Omega'$. Since $z \in \Omega \setminus U$, we know that

$$\frac{\delta}{2} e^{-40C} d(a, \partial\Omega) \leqslant |z - a| \leqslant \frac{d(z, \partial\Omega)}{3}.$$

Hence

$$|z - a| \geqslant d(z, \partial\Omega) - d(a, \partial\Omega) \geqslant d(z, \partial\Omega) - \frac{2}{\delta} e^{40C} |z - a|,$$

thus

$$\left(1 + \frac{2}{\delta} e^{40C}\right) |z - a| \geqslant d(z, \partial\Omega).$$

But then

$$\begin{aligned} g^{\#}(z) &\leqslant \frac{C}{|z - a|} \leqslant \frac{C}{\frac{d(z,\partial\Omega)}{1 + \frac{2}{\delta} e^{40C}}} \\ &\leqslant \frac{C(2e^{40C} + \delta)}{\delta} \frac{1}{d(z, \partial\Omega)}. \end{aligned} \tag{16}$$

From (14), (15) and (16) it follows that $g = f^*$ is quasinormal in Ω, concluding the proof. □

If Ω is simply connected, we thus get, using [19, p. 102] (see (6) above), the following new extension result for normal functions:

3.3. Theorem. *Let Ω be a simply connected hyperbolic domain in $\mathbf{C}$. Suppose that $E \subset \Omega$ and E is δ-separated in Ω, where $\delta > 0$. Let $f \colon \Omega \setminus E \to \mathbf{C}^*$ be a meromorphic function. If f is normal, then f has a normal meromorphic extension $f^* \colon \Omega \to \mathbf{C}^*$.*

Using the conformal invariencies of the hyperbolic distance and the normality and also Proposition 2.17 above, our result, Theorem 3.3, can be rewritten also to the following neat form:

3.4. Corollary. *Let Ω be a simply connected hyperbolic domain in $\mathbf{C}$. Suppose that $E \subset \Omega$ and that there is $\delta > 0$ such that $\varrho_\Omega(a,b) \geqslant \delta$ for all $a, b \in E$, $a \neq b$. Let $f : \Omega \setminus E \to \mathbf{C}^*$ be a meromorphic function. If f is normal, then f has a normal meromorphic extension $f^* : \Omega \to \mathbf{C}^*$.*

A similar result can, of course, be obtained also for Bloch functions:

3.5. Corollary. *Let Ω be a simply connected hyperbolic domain in $\mathbf{C}$. Suppose that $E \subset \Omega$ and that there is $\delta > 0$ such that $\varrho_\Omega(a,b) \geqslant \delta$ for all $a, b \in E$, $a \neq b$. If f is Bloch in $\Omega \setminus E$, then f has a Bloch extension f^* to Ω.*

Observe, however, that Theorem 2.19 in the case $n = 1$ and Ω simply connected, gives a much stronger result for Bloch functions.

4. A remark concerning a question of Harvey and Polking

4.1. We conclude by answering to a question of Harvey and Polking, in the case of holomorphic functions.

Bochner [1], see Harvey and Polking [9, Theorem 2.5, p. 42], proved the following result. Let G be an open set in $\mathbf{R}^n$, $n \geqslant 2$, and let $P(x, D)$ be a linear partial differential operator of order m on G. Suppose further that $E \subset G$ is closed in G, $M^{n-m-p}(E \cap K) < \infty$ for all compact sets $K \subset G$, $f \in \mathcal{L}^1_{\mathrm{loc}}(G)$, $f(x) = o(d(x, E)^{-p})$ uniformly for x on compact subsets of G and $P(x, D)f = 0$ in $G \setminus E$. Then $P(x, D)f = 0$ in G. Polking [21, 2.3.5, 2.4.12, pp. 267, 271–272] showed that in the above result the upper Minkowski content can be replaced by the lower one. These results apply, with $m = 1$, and are of interest also for holomorphic functions, see [21, p. 273].

Harvey and Polking [9, p. 42] and also later Polking [21, p. 273] asked whether in the above result the Minkowski upper (lower) content could be replaced by the Hausdorff measure. The following example shows that this is not possible for any p, $0 < p \leqslant 1$, in the case of holomorphic functions, except of course, when $n = p = 1$.

Suppose first that $n \geqslant 2$ and $0 < p \leqslant 1$. Let E and f be as in the example given in Remark 2.12 above. Choose $q = p$ in the definition of E. Then $H^{2n-2}(E) < \infty$, and it remains only to show that

$$f(z) = o(d(z, E)^{-p}) \tag{17}$$

uniformly for z on compact subsets of D^n. Suppose that $z = (z_1, \dots, z_{n-1}, z_n) \in D^n \setminus E$ and that

$$\frac{1}{(k+1)^p} \leqslant |z_n| < \frac{1}{k^p}$$

for some $k \in \mathbf{N}$. Then

$$d(z, E) \leqslant 2\Big[\frac{1}{k^p} - \frac{1}{(k+1)^p}\Big],$$

and thus

$$d(z,E)^p|f(z)| \leqslant 2^p\Big[\frac{1}{k^p} - \frac{1}{(k+1)^p}\Big]^p (k+1)^p \leqslant 2^{p+1}\Big[\Big(1+\frac{1}{k}\Big)^p - 1\Big]^p k^{p-p^2}, \tag{18}$$

since $0 < p \leqslant 1$. Using again the fact that $0 < p \leqslant 1$, we see at once that

$$0 \leqslant \Big[\Big(1+\frac{1}{k}\Big)^p - 1\Big]^p k^{p-p^2} \leqslant k^{-p^2} \longrightarrow 0$$

as $k \to \infty$. From this and from (18) it follows easily that (17) holds.

The case $n = 1$ and $0 < p < 1$ can be treated similarly.

Additional remarks. After this manuscript was completed, P. Koskela (Department of Mathematics, University of Jyväskylä, SF-40351 Jyväskylä, Finland), Removable singularities for analytic functions, University of Jyväskylä, Preprint 137, January 1992, gave a similar result to Theorem 2.10 in the case when $n = 1$, in the connection of his related results concerning certain Lipschitz classes.

We have referred above in Abstract, in 1.1 and in Remark 2.11 to a result of Kaufman [13, Theorem (a), p. 369], in the special case when the exceptional set E is closed in the domain Ω. After submitting our manuscript we have, however, found that this result has in fact been obtained already by J. Král, Singularites non essentielles des solutions des equations aux derivees partielles, Séminaire de Théorie du Potentiel Paris 1972–1974, LNM 518 (1976) (eds. F. Hirsch and G. Mokobodzki), pp. 95–106; Corollaire 1, p. 99, as a part of his results for partial differential equations.

References

[1] *S. Bochner*: Weak solutions of linear partial differential equations, J. Math. Pures Appl. *35* (1956), 193–202.

[2] *L.A. Campbell and R.H. Ogawa*: On preserving the Kobayashi pseudodistance, Nagoya Math. J. *57* (1975), 37–47.

[3] *L.A. Campbell, A. Howard and T. Ochiai*: Moving holomorphic disks off analytic subsets, Proc. Amer. Math. Soc. *60* (1976), 106–108.

[4] *J.A. Cima and I.R. Graham*: Removable singularities for Bloch and BMO functions, Ill. J. Math. *27* (1983), 691–703.

[5] *J.A. Cima and S.G. Krantz*: The Lindelöf principle and normal functions of several complex variables, Duke Math. J. *50* (1983), 303–328.
[6] *H. Federer*: Geometric measure theory, Springer, Berlin, 1969.
[7] *J.B. Garnett*: Bounded analytic functions, Academic Press, New York, 1981.
[8] *I.R. Graham*: Removable singularities for holomorphic functions which satisfy the area-BMO condition, Several Complex Variables, Proc. Hangzhou Conf. 1981 (J.J. Kohn, Q.-k. Lu, R. Remmert and Y.T. Siu, eds.), Birkhäuser Boston, Inc., Boston, Mass., 1984, pp. 175–180.
[9] *R. Harvey and J.C. Polking*: Removable singularities of solutions of linear partial differential equations, Acta Math. *125* (1970), 39–56.
[10] *L.I. Hedberg*: Removable singularities and condenser capacities, Ark. Mat. *12* (1974), 181–201.
[11] *J. Hyvönen and J. Riihentaus*: Removable singularities for holomorphic functions with locally finite Riesz mass, J. London Math. Soc. (2) *35* (1987), 296–302.
[12] *P. Järvi*: An extension theorem for normal functions, Proc. Amer. Math. Soc. *103* (1988), 1171–1174.
[13] *R. Kaufman*: Hausdorff measure, BMO, and analytic functions, Pac. J. Math. *102* (1982), 369–371.
[14] *S.G. Krantz*: Function theory of several complex variables, John Wiley, New York, 1982.
[15] *S.G. Krantz and D. Ma*: Bloch functions on strongly pseudoconvex domains, Ind. Univ. Math. J. *37* (1988), 145–163.
[16] *O. Lehto and K.I. Virtanen*: Boundary behaviour and normal meromorphic functions, Acta Math. *97* (1957), 47–65.
[17] *O. Lehto and K.I. Virtanen*: On the behaviour of meromorphic functions in the neighborhood of an isolated singularity, Ann. Acad. Sci. Fenn. A I Math. *240* (1957), 1–9.
[18] *G.J. Martin*: Quasiconformal and bi-lipschitz homeomorphisms, uniform domains and the quasihyperbolic metric, Trans. Amer. Math. Soc. *292* (1985), 169–191.
[19] *D. Minda*: Bloch and normal functions on general planar regions, Holomorphic functions and moduli, Vol. I, Proc. Workshop Berkeley, CA, 1986, Math. Sci. Res. Inst. Publ. 10 (D. Drasin, C.J. Earle, F.W. Gehring, I. Kra and A. Marden, eds.), Springer, New York, 1988, pp. 101–110.
[20] *E.A. Poletskiĭ and B.V. Shabat*: Invariant metrics, Encyclopaedia of Mathematical Sciences, Vol. 9, Several complex variables III (G.M. Khenkin, ed.), Springer, Berlin, 1989, pp. 63–111.
[21] *J.C. Polking*: A survey of removable singularities, Seminar on Nonlinear Partial Differential Equations, Berkeley, CA, 1983, Math. Sci. Res. Inst. Publ. 2 (S.S. Chern, ed.), Springer, New York, 1984, pp. 261–292.
[22] *J. Riihentaus*: An extension theorem for meromorphic functions of several variables, Ann. Acad. Sci. Fenn. A I Math. *4* (1978/1979), 145–149.
[23] *J. Riihentaus*: A nullset for normal functions in several variables, Proc. Amer. Math. Soc. *110* (1990), 923–933.
[24] *B. Shiffman*: On the removal of singularities of analytic sets, Michigan Math. J. *15* (1968), 111–120.
[25] *R.M. Timoney*: Bloch functions in several complex variables I, Bull. London Math. Soc. *12* (1980), 241–267.
[26] *M. Vuorinen*: Conformal geometry and quasiregular mappings, Springer, LNM 1319, Berlin, 1988.

Author's address: Department of Mathematics, University of Oulu, SF-90570 Oulu, Finland.