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A WEAKER FORM OF BAER'S SPLITTING PROBLEM FOR TORSION THEORIES

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1. INTRODUCTION

In this paper, all rings R have an identity element 1 and all modules are unital left R-modules unless it is specifically indicated to the contrary. Additionally, τ will always denote a nontrivial torsion theory of left R-modules with associated filter $\mathscr{L}_\tau$ of left ideals and localization Q_τ of R. For any module M, we let $\tau(M)$ denote the τ-torsion submodule of M. If $\tau(R) = 0$, the canonical map $R \to Q_\tau$ is a monomorphism. As usual, a torsion theory τ is called perfect if the τ-localization of each module is $Q_\tau \otimes M$. A module M is τ-injective if $\operatorname{Ext}_R(T, M) = 0$ for every τ-torsion T. We let $E(M)$ denote the injective hull of a module M; then $E_\tau(M) = \{e \in E(M) \mid Ie \subseteq M \text{ for some } I \in \mathscr{L}_\tau\}$ is τ-injective. For these definitions and more information on torsion theories, see [9] or [17].

A τ-torsion module T is said to have τ-bounded order if T can be embedded in a module that has a set of generators annihilated by some $I \in \mathscr{L}_\tau$. (τ-bounded order is also called uniformly negligible in some papers.) In case $\mathscr{L}_\tau$ has a cofinal subset of two-side ideals, then T has τ-bounded order if and only if $IT = 0$ for some $I \in \mathscr{L}_\tau$. Modules with τ-bounded order appear many places in the literature; for example, see [1], [3], [7], [10], and [13].

There have been a number of definition of divisibility relative to τ proposed in the literature (e.g., see [9], [12], [17], and [18].) The success of these definitions usually depends on the context in which they are used. Here we define a module D to be τ-divisible if D is a homomorphic image of a direct sum of τ-injective modules. Our class of divisible modules agrees with the usual divisible modules when τ is the usual torsion theory for a Dedekind domain. Since Q_τ is τ-injective, then every Q_τ-module is τ-divisible. As with the usual class of divisible modules over an integral domain, our class of τ-divisible modules is closed under injective hulls, τ-injective hulls, homomorphic images, and direct sums. If $\tau(R) = 0$, then the class of τ-divisible modules is closed under direct products. While the class of τ-divisible modules may not be closed under extensions, we note that if $\mathrm{Ext}(M, D) = 0$ for each τ-divisible module D and if

$$0 \to D_1 \to X \to D_2 \to 0$$

is exact with D_1, D_2 τ-divisible, then $\mathrm{Ext}(M, X) = 0$. This fact will give the effect of extension closure for some of our work with τ-divisible modules.

Following the notation of [6], we say that a module B is a B^*-module if $\mathrm{Ext}_R(B, X) = 0$ for each τ-divisible X and each X with τ-bounded order. In [6], B^*-modules were studied for the usual torsion theory over a valuation domain. The motivation for studying B^*-modules in [6] comes from the study of Baer modules over commutative integral domains. The purpose of this paper is to initiate the study of B^*-modules for torsion theories over more general rings. This general study has an interesting relationship with (1) the study of τ-injective modules, (2) the Bounded Splitting Problem for torsion theories (see [1], [3], [7], and [10]), and (3) the Baer problem for torsion theories (see [8]).

In Section two we present some basic propositions that are useful for studying B^*-modules. Since B^*-modules are defined in terms of two distinct classes of modules, we separate these properties to facilitate their use. We call a module M a D^*-module if $\mathrm{Ext}_R(M, D) = 0$ for every τ-divisible module D. We characterize D^*-modules in Theorem 3.1 under the mild assumption that $\tau(R) = 0$. In Theorem 4.1 we characterize the modules M such that $\mathrm{Ext}(M, T) = 0$ for all T with τ-bounded order, provided that $\tau(R) = 0$ and $\mathscr{L}_\tau$ has a cofinal subset of two-sided ideals. We then use Theorem 4.1 to obtain a generalization of some results ([7, Theorem 2.2] and [1, Theorem 2.3]) on the Bounded Splitting Problem for torsion theories. Finally, in Section Five we combine our results to give some applications for B^*-modules. For example, finitely generated B^*-modules over local rings are free, and Q_τ-modules that are B^*-modules are characterized.

We will use $\mathrm{pd}(M)$ and $\mathrm{wd}(M)$ to denote the projective and weak dimensions, respectively, of a module M. Other terminology from homological algebra can be found in [2] or [16].

2. Basic Lemmas

In this section we give some basic results that will be useful in the study of B^*-modules. These results show that some of the basic properties of B^*-modules for the usual torsion theory over a valuation domain extend to arbitrary torsion theories over much more general rings. Due to the definition of B^*-modules, these basic properties are mostly homological in nature. To facilitate the use of these basic results, we also separate out the hypothesis that B is a D^*-module whenever possible.

We begin with the restriction on the homological dimension of a D^*-module.

Lemma 2.1. $\operatorname{pd} B \leqslant 1$ *for every* D^**-module* B.

Proof. Since $E(M)/M$ is τ-divisible for every module M, we have the exact sequence:

$$0 = \operatorname{Ext}\big(B, E(M)/M\big) \to \operatorname{Ext}^2(B, M) \to \operatorname{Ext}^2\big(B, E(M)\big) = 0$$

□

Lemma 2.2. *Let* M *be a right* Q_τ*-module. If* B *is a* D^**-module, then* $\operatorname{Tor}^R(M, B) = 0$.

Proof. Let ${}_ZC$ be injective, and let B be a D^*-module. Since M is a right Q_τ-module, then $\operatorname{Hom}_Z(M, C)$ is τ-divisible. So by hypothesis and [2, VI. 5.1], we have

$$0 = \operatorname{Ext}_R\big(B, \operatorname{Hom}_Z(M, C)\big) \cong \operatorname{Hom}_Z\big(\operatorname{Tor}^R(M, B), C\big).$$

Since ${}_ZC$ can be any injective, we must have $\operatorname{Tor}^R(M, B) = 0$. □

Kaplansky's basic idea [14] (see also [7] and [10]) gives us more information about Tor.

Lemma 2.3. *Let* R *be a commutative ring. If* $\operatorname{Ext}_R(B, T) = 0$ *for all* T *with* τ*-bounded order, then* $\operatorname{Tor}(B, R/I) = 0$ *for all* $I \in \mathscr{L}_\tau$.

Proof. Since $I \in \mathscr{L}_\tau$, then $\operatorname{Hom}_Z(R/I, E)$ has τ-bounded order for any injective ${}_ZE$. By hypothesis and [2, VI. 5.1]

$$0 = \operatorname{Ext}\big(B, \operatorname{Hom}_Z(R/I, E)\big) \cong \operatorname{Hom}_Z\big(\operatorname{Tor}(B, R/I), E\big).$$

Since ${}_ZE$ can be any injective, then $\operatorname{Tor}(B, R/I) = 0$. □

In case τ is the usual torsion theory over a commutative domain, then every nonzero ideal is in $\mathscr{L}_\tau$; so Lemma 2.3 gives ${}_RB$ flat. However, in the general case, very few ideals may be in $\mathscr{L}_\tau$; so we need to do a little more work.

Proposition 2.4. *If* R *is a commutative ring, then every* B^**-module is flat.*

Proof. Let B be a B^*-module. Using Lemma 2.3, we obtain $\operatorname{Tor}(B, T) = 0$ for all τ-torsion T by a standard transfinite induction argument.

Since $0 \to \tau(M) \to M \to M/\tau(M) \to 0$ is exact for any module ${}_RM$, it is now sufficient to show that $\operatorname{Tor}^R(B, F) = 0$ for any τ-torsionfree F. Since $\operatorname{wd} B \leqslant \operatorname{pd} B \leqslant 1$ by Lemma 2.1, the natural inclusion $F \to E_\tau(F)$ gives an exact sequence:

$$0 = \operatorname{Tor}_2\big(B, E_\tau(F)/F\big) \to \operatorname{Tor}_1(B, F) \to \operatorname{Tor}_1\big(B, E_\tau(F)\big).$$

But $E_\tau(F)$ is always a Q_τ-module; so $\mathrm{Tor}_1\big(B, E_\tau(F)\big) = 0$ by Lemma 2.2, and the result follows from the exact sequence. □

We can also consider some other basic relationships of D^*-modules with $\otimes$.

Lemma 2.5. *If B is a D^*-module, then $Q_\tau \otimes_R B$ is a projective Q_τ-module.*

Proof. Let B be a D^*-module. Since $\mathrm{Tor}^R(Q_\tau, B) = 0$ by Lemma 2.2, then the hypothesis and [2, VI.4.1.3] yield

$$\mathrm{Ext}_{Q_\tau}(Q_\tau \otimes B, D) \cong \mathrm{Ext}_R(B, D) = 0$$

for each Q_τ-module D. □

Proposition 2.6. *If Q_τ is a D^*-module, then the multiplication map $\mu\colon Q_\tau \otimes_R Q_\tau \to Q_\tau$ is an isomorphism; i.e., the canonical map $R \to Q_\tau$ is an epimorphism in the category of rings.*

Proof. Note that

$$0 \to \ker\mu \to Q_\tau \otimes_R Q_\tau \xrightarrow{\mu} Q_\tau \to 0$$

splits as an exact sequence of Q_τ-modules and that $\ker\mu \cong \tau(Q_\tau \otimes_R Q_\tau)$. But $Q_\tau \otimes Q_\tau$ is a projective Q_τ-module by Lemma 2.5. Thus

$$\tau(Q_\tau \otimes_R Q_\tau) \subseteq \tau\big(\bigoplus Q_\tau\big) = \bigoplus \tau(Q_\tau) = 0,$$

so that $\ker\mu = 0$. □

In case τ is the usual torsion theory over a domain, the flatness of a B^*-module makes if τ-torsionfree. In the general commutative case, we must modify this conclusion.

Proposition 2.7. *Let R be a commutative ring, and let B be a B^*-module. Then $\tau(B) = \tau(R)B$.*

Proof. By Proposition 2.4, B is flat. Hence

$$0 = \mathrm{Tor}^R(Q_\tau/\overline{R}, B) \to \overline{R} \otimes_R B \to Q_\tau \otimes_R B$$

is exact, where $\overline{R} \cong R/\tau(R)$. From this sequence and Lemma 2.5, we obtain the exact sequence

$$0 \to B/\tau(R)B \xrightarrow{\alpha} \bigoplus Q_\tau.$$

Since $\bigoplus Q_\tau$ is τ-torsionfree, we must have $\tau(B)/\tau(R)B \subseteq \ker\alpha = 0$, and hence $\tau(B) = \tau(R)B$. □

We also note that in the noncommutative case, B^*-modules may be far from torsionfree and that conclusion of Proposition 2.7 may not hold. For example, if R is the ring of differential polynomials over a universal differential field, then R is well-known [4] to be a principal left and right ideal domain with the property that each (usual) torsion module is injective. Since each divisible module is also injective for this ring R, then every R-module is a B^*-module. Hence there are non-flat B^*-modules in this case (cf. Proposition 2.4.)

However, Proposition 2.7 suggests that the theory of B^*-modules can be expected to be smoother if τ is a faithful torsion theory (i.e., if $\tau(R) = 0$). This will be true even in the noncommutative case, as we will see in subsequent sections.

3. D^*-MODULES

In studying B^*-modules, Fuchs and Viljoen [6] effectively separate out the D^*-modules for the usual torsion theory over a valuation domain as those modules B with $\text{pd}_R B \leqslant 1$. In this section we give a general characterization of D^*-modules for arbitrary torsion theories over any ring with $\tau(R) = 0$. This characterization bears some relationship to the results of Section 4 of [18], where a different form of divisibility is studied. It also lays the groundwork for studying the structure of B^*-modules and simplifies the study of rings in which certain classes of modules are D^*-modules (e.g., see Corollaries 3.2 and 3.3.)

We begin with our characterization of D^*-modules for faithful torsion theories.

Theorem 3.1. *Let $\tau(R) = 0$. Then following statements are equivalent for a module B.*

(1) *B is a D^*-module.*

(2) $\text{pd}\, B \leqslant 1$, $\text{Tor}_1^R(Q_\tau, B) = 0$, *and $Q_\tau \otimes_R B$ is a projective Q_τ-module.*

Proof. (1) $\Longrightarrow$ (2) is immediate from Lemmas 2.1, 2.2, and 2.5.

(2) $\Longrightarrow$ (1). Let D be τ-divisible, and let $\bigoplus E_\alpha \to D$ be an epimorphism, where each E_α is τ-injective. Let F_α be a free R-module with an epimorphism $F_\alpha \to E_\alpha$. Since $\tau(R) = 0$, $F_\alpha \subseteq \bigoplus Q_\tau$; so the τ-injectivity of each E_α gives rise to an epimorphism $\bigoplus_\alpha(\bigoplus Q_\tau) \to \bigoplus E_\alpha \to D$. Since $\text{Tor}(Q_\tau, B) = 0$, [2, VI.4.1.3] yields

$$\text{Ext}_R\,(B, \bigoplus Q_\tau) \cong \text{Ext}_{Q_\tau}(Q_\tau \otimes_R B, \bigoplus Q_\tau) = 0,$$

as $Q_\tau \otimes_R B$ is Q_τ-projective. Since $\text{pd}\, B \leqslant 1$, we have an exact sequence

$$\text{Ext}_R(B, \bigoplus Q_\tau) \to \text{Ext}_R(B, D) \to 0,$$

and hence $\text{Ext}_R(B, D) = 0$ by exactness. □

Fuchs and Viljoen [6, Lemma 1.6] observe that the only ideals of a commutative valuation ring that are B^*-modules for the usual torsion theory are the principal ideals. Similarly, Grimaldi [11, Theorem 3] examines when every ideal of an integral domain is a Baer module. Our next two corollaries provide this type of information.

Corollary 3.2. *The following statements are equivalent when* $\tau(R) = 0$.

(1) *Every finitely generated left ideal of R is a* D^**-module.*

(2) *For each finitely generated left ideal I,* $\operatorname{pd} I \leqslant 1$ *and* $Q_\tau \otimes_R I$ *is a projective* Q_τ*-module, and* $\operatorname{wd}(Q_\tau)_R \leqslant 1$.

Corollary 3.3. *Let* τ *be perfect and let* $\tau(R) = 0$. *Then the following statements are equivalent.*

(1) *Every left ideal of R is a* D^**-module.*

(2) $\ell \cdot g\ell \cdot \dim R \leqslant 2$ *and* Q_τ *is a left hereditary ring.*

(3) *Every submodule of a free left R-module is a* D^**-module.*

Proof. (1) $\Longrightarrow$ (2). Since τ is perfect, each left ideal of Q_τ has the form $Q_\tau \otimes_R I$ for some left ideal I of R. Hence the result follows easily from Theorem 3.1.

(2) $\Longrightarrow$ (3). Let ${}_RA \subseteq \bigoplus R$. Since τ is perfect, $(Q_\tau)_R$ is flat and

$$Q_\tau \otimes_R A \subseteq Q_\tau \otimes_R (\bigoplus R) \cong \bigoplus Q_\tau .$$

Since Q_τ is left hereditary then $Q_\tau \otimes A$ must be projective as a Q_τ-module. So the result follows from Theorem 3.1.

(3) $\Longrightarrow$ (1). Trivial. □

4. Bounded Splitting

In this section we examine the other half of the definition of B^*-modules, namely the modules B for which $\operatorname{Ext}(B,T) = 0$ for all T with τ-bounded order.

The determination of such B is closely related to the Bounded Splitting Problem for torsion theories, which asks when all τ-torsionfree B satisfy $\operatorname{Ext}_R(B,T) = 0$ for all T with τ-bounded order. Various aspects of the Bounded Splitting Problem have been examined by many authors (e.g., see [1], [3], [7], [10], and [13].) We are able to use our characterization in Theorem 4.1 to give a generalization of [1, Theorem 2.3] and [7, Theorem 2.2]. We note that Theorem 4.1 also has a relationship to the study of Baer modules (also called UF-modules); these are the modules B for which $\operatorname{Ext}_R(B,T) = 0$ for all τ-torsion T (e.g., see [5], [6], [8], [11], and [14].)

The proof of our next result is inspired by work on BSP.

Theorem 4.1. *Let $\tau(R) = 0$ and assume that $\mathcal{L}_\tau$ has a cofinal subset of two-sided ideals. Then the following statements are equivalent for a module ${}_RB$.*

(1) $\mathrm{Ext}(B,T) = 0$ for all T with τ-bounded order.

(2) $\mathrm{Tor}^R(R/K,B) = 0$ and B/KB is a projective R/K-module for each two-sided ideal $K \in \mathcal{L}_\tau$.

Proof. (1) $\Longrightarrow$ (2). Let K be a two-sided ideal in $\mathcal{L}_\tau$. Then $\mathrm{Hom}_R(R/K,C)$ has τ-bounded order for any ${}_RC$. If ${}_RC$ is injective, then [2, VI.5.1] and (1) yield

$$\mathrm{Hom}_R\left(\mathrm{Tor}^R(R/K,B),C\right) \cong \mathrm{Ext}_R\left(B,\mathrm{Hom}_R(R/K,C)\right) = 0.$$

Since ${}_RC$ can be any injective, we must have $\mathrm{Tor}^R(R/K,B) = 0$.

Let an exact sequence

$$(*) \qquad 0 \to M \to N \xrightarrow{g} B/KB \to 0$$

of R/K-modules be given. We wish to show that $(*)$ splits. Since $\mathrm{Ext}_R(B,M) = 0$ by (1), then there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & H & \underset{f}{\overset{k}{\rightleftarrows}} & B & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \xrightarrow{g} & B/KB & \longrightarrow & 0 \end{array}$$

where p is the natural map, H is formed by a pull-back, and $kf = 1_B$. Thus $ghf = pkf = p1_B = p$ and $hf(KB) = Khf(B) \subseteq KN = 0$. Hence hf induces a homomorphism $(hf)'\colon B/KB \to N$ such that $g(hf)' = 1_{B/KB}$. Therefore, $(*)$ splits.

(2) $\Longrightarrow$ (1). Let ${}_RT$ satisfy $KT = 0$ for some two-sided ideal $K \in \mathcal{L}_\tau$. For any exact sequence

$$0 \longrightarrow T \longrightarrow X \longrightarrow B \longrightarrow 0,$$

(2) gives a diagram with split second row:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & X & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 = \mathrm{Tor}(R/K,B) & \longrightarrow & T & \rightleftarrows & X/KX & \longrightarrow & B/KB & \longrightarrow & 0 \end{array}$$

We readily see that the composition $X \to X/KX \to T$ gives a map to split the first row of the diagram. □

Remark. Since $\mathrm{Hom}_Z(R/I,C)$ has τ-bounded order for any right ideal I such that $I \supseteq {}_RK_R \in \mathcal{L}_\tau$, then the argument in the first paragraph of the proof of Theorem 4.1 also shows that $\mathrm{Tor}^R(R/I,B) = 0$.

If R has a lot of cyclic flat modules (e.g., if R is a von Neumann regular ring), then Theorem 4.1 gives nicer results when applied to all left ideals.

Corollary 4.2. *Let $\tau(R) = 0$ and assume that $\mathscr{L}_\tau$ has a cofinal subset of two-sided ideals K such that $(R/K)_R$ is flat. Then the following statements are equivalent.*

(1) $\mathrm{Ext}_R(I, T) = 0$ *for all left ideals I of R and all T with τ-bounded order.*

(2) *R/K is a left hereditary ring for all two-sided ideals $K \in \mathscr{L}_\tau$ such that $(R/K)_R$ is flat.*

(3) $\mathrm{Ext}_R(A, T) = 0$ *for every submodule ${}_RA$ of a free module and every T with τ-bounded order.*

Proof. (1) $\Longrightarrow$ (2). Let K be a two-sided ideal in $\mathscr{L}_\tau$ with $(R/K)_R$ flat. Then $0 = \mathrm{Tor}^R(R/K, K) \cong K/K^2$ and hence $K^2 = K$. Let $K \subseteq {}_RI \subseteq R$. Now $I/K \cong I/KI$ is a projective R/K-module by Theorem 4.1. Therefore R/K is left hereditary.

(2) $\Longrightarrow$ (3). Let ${}_RA \subseteq \bigoplus R$ and let K be a two-sided ideal in $\mathscr{L}_\tau$ with $(R/K)_R$ flat. Then

$$0 \to R/K \otimes_R A \to R/K \otimes_R (\bigoplus R)$$

is exact, and hence A/KA is isomorphic to a submodule of $\bigoplus R/K$. Since R/K is left hereditary, then A/KA is a projective R/K-module, and the result follows from Theorem 4.1.

(3) $\Longrightarrow$ (1). Trivial. □

Minor modifications of this proof yield the following similar result.

Corollary 4.3. *Let $\tau(R) = 0$ and assume that $\mathscr{L}_\tau$ has a cofinal subset of two-sided ideals K such that $(R/K)_R$ is flat. Then the following statements are equivalent.*

(1) $\mathrm{Ext}_R(I, T) = 0$ *for all finitely generated left ideals $I \in \mathscr{L}_\tau$ and all T with τ-bounded order.*

(2) *R/K is a left semihereditary ring for all two-sided ideals $K \in \mathscr{L}_\tau$ such that $(R/K)_R$ is flat.*

(3) $\mathrm{Ext}_R(A, T) = 0$ *for all finitely generated submodules ${}_RA$ of a free module and all T with τ-bounded order.*

A torsion theory τ is said to have the *bounded splitting property* (BSP) if each module M, for which $\tau(M)$ has τ-bounded order, has $\tau(M)$ as a direct summand. The study of BSP was initiated by Kaplansky [13] and has been pursued by many other authors (e.g., see [1], [8], [10], and their references). It is easy to see that τ has BSP if and only if $\mathrm{Ext}(F, T) = 0$ for each τ-torsionfree F and each T with τ-bounded order.

The following two results generalize [7, Theorem 2.2] and [1, Theorem 2.3], which give information about BSP for torsion theories over commutative rings.

Theorem 4.4. *Let $\tau(R) = 0$ and assume that $\mathscr{L}_\tau$ has a cofinal subset of two-sided ideals. Then the following statements are equivalent.*

(1) *τ has BSP.*

(2) *For each two-sided ideal $K \in \mathscr{L}_\tau$, R/K is a left perfect ring and $\operatorname{Tor}_1^R(R/I, B) = 0$ for each τ-torsionfree B and each right ideal I such that $I \supseteq K$.*

Proof. (1) $\Longrightarrow$ (2). Let K be a two-sided ideal in $\mathscr{L}_\tau$. Theorem 4.1 and its following Remark show that $\operatorname{Tor}_1^R(R/I, B) = 0$ for all τ-torsionfree B and all $I_R \supseteq K$. By Theorem 4.1 we also have $(\Pi R)/K(\Pi R)$ projective as an R/K-module; so $(\Pi R)/K(\Pi R)$ is direct summand of $\bigoplus R/K$. Hence [10, Theorem 5.1] implies that R/K is left perfect.

(2) $\Longrightarrow$ (1). Let K be a two-sided ideal in $\mathscr{L}_\tau$ and let B be τ-torsionfree. Since $\operatorname{Tor}_1^R(R/I, B) = 0$ for each $I_R \supseteq K$, and $\tau(R) = 0$, an easy induction (similar to the proof of [7, Lemma 2.1]) shows that $\operatorname{Tor}_n^R(R/I, B) = 0$ for all $n \geqslant 1$. Hence [2, VI.4.1.2] yields

$$0 = \operatorname{Tor}^R(R/I, B) \cong \operatorname{Tor}^{R/K}(R/I, B/KB) \cong \operatorname{Tor}^{R/K}\left((R/K)/(I/K), B/KB\right).$$

Thus B/KB is a flat R/K-module. Since R/K is left perfect, B/KB must be a projective R/K-module. Therefore, τ must have BSP via Theorem 4.1. □

Corollary 4.5. *Let R be a commutative ring with $\tau(R) = 0$. Then the following statements are equivalent.*

(1) *τ has BSP.*

(2) *For each $K \in \mathscr{L}_\tau$, R/K is a perfect ring and $\operatorname{Tor}_1^R(R/K, B) = 0$ for each τ-torsionfree B.*

5. B^*-MODULES

In this section we combine our previous results to obtain some information about B^*-modules.

We begin with an example that further illustrates the differences between the general case and the classical commutative domain case.

Example 5.1. Let P be the ring of differential polynomials over a universal differential field [4], and let M be a maximal left ideal of P. Let $R = \{r \in P \mid Mr \subseteq M\}$ be the idealizer of M in P. Then R is a left and right hereditary, left and right noetherian domain with unique nontrivial two-sided ideal M [15]. Then $\mathscr{L} = \{R, M\}$ forms a filter for a torsion theory τ of left R-modules (as $M^2 = M$ and R/M is a division ring). Now τ is perfect, $\mathscr{L} = \mathscr{L}_\tau$ has a cofinal subset of two-sided ideals, each τ-torsion module is isomorphic to $\bigoplus R/M$, and $\tau(R) = 0$. We make the following observations about B^*-modules for R.

(1) R/M is a D^*-module. (Since R is left noetherian, then $\bigoplus Q_\tau$ is τ-injective [9, 41.1]; since R is left hereditary, homomorphic images of $\bigoplus Q_\tau$ must be τ-injective [17, p. 212].)

(2) Since each τ-torsion module is semisimple, $\mathrm{Ext}(R/M, \bigoplus R/M) = 0$.

(3) By (1) and (2), R/M is a τ-torsion B^*-module.

(4) In view of (3), Proposition 2.7 cannot be extended to the case in which $\mathscr{L}_\tau$ has a cofinal subset of two-sided ideals.

(5) Every submodule of a free R-module is a B^*-module.

(6) Let S be a faithful simple R-module. Then $\mathrm{Ext}(S, R/M) \neq 0$ [15, Theorem 1.3]. Hence S is not a D^*-module even though $\mathrm{pd}\, S \leqslant 1$ and Q_τ is flat (cf. Theorem 3.1), and R does not have BSP for τ (cf. Theorem 4.4).

Combining Theorems 3.1 and 4.1, we have the following result.

Theorem 5.2. *Let $\tau(R) = 0$ and assume that $\mathscr{L}_\tau$ has a cofinal subset of two-sided ideals. Then an R-module B is a B^*-module if and only if the following conditions hold:*

(1) $\mathrm{pd}_R\, B \leqslant 1$.

(2) $\mathrm{Tor}_1^R(Q_\tau, B) = 0$.

(3) *$Q_\tau \otimes_R B$ is a projective Q_τ-module.*

(4) *For each two-sided ideal $K \in \mathscr{L}_\tau$, $\mathrm{Tor}_1^R(R/K, B) = 0$ and B/KB is a projective R/K-module.*

Throughout [6] Q_τ plays a special role in examining B^*-modules. Our next two results indicate that this role carries over to a much more general setting than R being a valuation domain.

Proposition 5.3. *Let R be a commutative ring, let τ be perfect, and let $\tau(R) = 0$. Then a left Q_τ-module B is a D^*-module if and only if $\mathrm{pd}_R\, B \leqslant 1$ and ${}_{Q_\tau}B$ is projective.*

Proof. ($\Longrightarrow$) Theorem 3.1 gives the result since $Q_\tau \otimes_R B \cong B$ in this case.

($\Longleftarrow$). Since τ is perfect, Theorem 3.1 implies that B is a D^*-module. Let $KT = 0$ for some $K \in \mathscr{L}_\tau$. Clearly $K\,\mathrm{Ext}_R(B,T) = 0$. So we only need to show that $K\,\mathrm{Ext}_R(B,T) = \mathrm{Ext}_R(B,T)$. From the exact sequence

$$\mathrm{Hom}_R\left(B, E(T)\right) \to \mathrm{Hom}_R\left(B, E(T)/T\right) \to \mathrm{Ext}_R(B, M) \to 0,$$

we see that it is sufficient to show that $K\,\mathrm{Hom}_R\left(B, E(T)/T\right) = \mathrm{Hom}_R\left(B, E(T)/T\right)$. Let $f \in \mathrm{Hom}_R\left(B, E(T)/T\right)$ and let $1 = \sum_{i=1}^{n} q_i x_i$ with $q_i \in Q_\tau$ and $x_i \in K$ (as τ is perfect). For any $b \in B$ we have

$$f(b) = \left(\sum_{i=1}^{n} q_i x_i\right) f(b) = \sum_{i=1}^{n} f(bq_i x_i) = \sum_{i=1}^{n} x_i f(bq_i) = \sum_{i=1}^{n} x_i (q_i f)(b)$$

since R is commutative. Thus $f = \sum_{i=1}^{n} x_i(q_i f) \in K \operatorname{Hom}_R\left(B, E(T)/T\right)$ as desired. □

Corollary 5.4. *Let R be a commutative ring, let τ be perfect, and let $\tau(R) = 0$. Then Q_τ is a B^*-module if and only if* $\operatorname{pd}_R Q_\tau \leqslant 1$.

For the usual torsion theory over a valuation domain, any finitely generated B^*-module is free [6]. We generalize this to torsion theories over commutative local rings. (R has a unique maximal ideal, but no chain conditions are assumed.)

Proposition 5.5. *Let R be a commutative local ring R with $\tau(R) = 0$. Then every finitely generated B^*-module is free.*

Proof. Let B be a finitely generated B^*-module and consider an exact sequence

$$0 \to K \to F \to B \to 0$$

with ${}_RF$ finitely generated and free. Since $\operatorname{pd}_R B \leqslant 1$ by Lemma 2.1, then K is projective and hence free (as R is local). Write $K \cong \bigoplus R$ and choose $L \subseteq K$ with $L \cong \bigoplus M$, where M is the maximal ideal of R. Since $M \in \mathcal{L}_\tau$, then $K/L \cong \bigoplus(R/M)$ has τ-bounded order. Since B is a B^*-module, the sequence

$$0 \to K/L \to F/L \to B \to 0$$

must split. Hence K/L is finitely generated. By our construction, this forces K to be finitely generated. But B is flat by Proposition 2.4. Since any finitely related flat module is projective and since R is local, we now have that B is free. □

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