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WEAK AND EXTRA-WEAK TYPE INEQUALITIES FOR THE MAXIMAL OPERATOR AND THE HILBERT TRANSFORM

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1. INTRODUCTION

Let Φ be a nondecreasing finite function on $[0,\infty)$, not vanishing identically and satisfying $\Phi(0)=0$, let σ, ϱ be appropriate measures in $\mathbf{R}^n$, and let T be a homogeneous operator. The usual two-weight weak type inequality in L^p,

$$\varrho(\{|Tf|>\lambda\}) \leqslant C\lambda^{-p} \int_{\mathbf{R}^n} |f(x)|^p \, d\sigma,$$

where C is independent of f and $\lambda>0$, and $\{|Tf|>\lambda\}$ stands for $\{x\in\mathbf{R}^n;\ |Tf(x)|>\lambda\}$, has at least two different analogues when replacing t^p by $\Phi(t)$:

$$\varrho(\{|Tf|>\lambda\})\cdot\Phi(\lambda) \leqslant C\int_{\mathbf{R}^n} \Phi(C\,|f(x)|)\, d\sigma, \tag{1}$$

"*weak type inequality*", and

$$\varrho(\{|Tf|>\lambda\}) \leqslant C\int_{\mathbf{R}^n} \Phi(C\,|f(x)|/\lambda)\, d\sigma, \tag{2}$$

"*extra-weak type inequality*" (this terminology goes back to [18], for justification see Remark 1 and Remark 2).

We start with proving some simple preliminary results concerning Φ and related functions (Section 2), and use them in Section 3 to give a characterization of the couples of measures (σ,ϱ) for which (1) or (2) hold with $T=M_\mu$, where M_μ is the Hardy-Littlewood maximal operator related to a doubling measure μ (cf. [6], [1], [2], [15], [17] and [18]). This characterization is slightly more general than that in [18], where Φ is assumed to be a Young function. We also give a new direct proof of necessity of the condition for the extra-weak type inequality.

As a consequence we obtain in Section 4 a new general characterization for the A_∞ condition, of independent interest, which sheds light onto the relationship between two conditions proved earlier by Hruščev [11] and Fujii [5].

The main results are the theorems in Section 5, which give necessary and sufficient conditions on a weight w for the inequalities

$$w(\{H^*f > \lambda\}) \cdot \Phi(\lambda) \leqslant C \int_{-\infty}^{\infty} \Phi(C|f|)w,$$

and

$$w(\{H^*f > \lambda\}) \leqslant C \int_{-\infty}^{\infty} \Phi(C|f|\lambda^{-1})w$$

to hold, where H^* is the maximal Hilbert transform. In the latter case Φ is assumed to satisfy the Δ_2 condition near zero.

Positive constants independent of the appropriate quantities are always denoted with C and need not keep their value from line to line. Throughout we take $0 \cdot \infty$ to be zero.

2. The functions Φ, $\tilde{\Phi}$, R_Φ and S_Φ

We define the *complementary function* to Φ by

$$\tilde{\Phi}(t) = \sup_{s \geqslant 0}(st - \Phi(s)).$$

Clearly, $\tilde{\Phi}(0) = 0$ and $\tilde{\Phi}$ is nondecreasing. The subadditivity of supremum easily implies that $\tilde{\Phi}$ is always convex. For any Φ we have $(\tilde{\Phi})\tilde{} \leqslant \Phi$, equality holds if Φ itself is convex. If $\Phi_1 \leqslant \Phi_2$, then $\tilde{\Phi}_2 \leqslant \tilde{\Phi}_1$, and if $\Phi_1(t) = a\Phi(bt)$, $a, b > 0$, then

$$\tilde{\Phi}_1(t) = a\tilde{\Phi}(t/ab).$$

Moreover, the Young inequality $st \leqslant \Phi(s) + \tilde{\Phi}(t)$ holds.

We say that $\Phi \in \Delta_2$ if $\Phi(2t) \leqslant C\Phi(t)$ for $t \geqslant 0$.

It is also worth to notice that unlike Φ, the function $\tilde{\Phi}$ may jump to infinity at some point $t > 0$. For example, if $\Phi(t) = t$, then $\tilde{\Phi}(t) = \infty \cdot \chi_{(1,\infty)}(t)$. It can even be $\tilde{\Phi} \equiv \infty$ everywhere on $(0, \infty)$ (put e.g. $\Phi(t) = \sqrt{t}$). We say that Φ is *reasonable* if there exists $t > 0$ such that $\tilde{\Phi}(t) < \infty$.

We put

$$R_\Phi(t) = \frac{\Phi(t)}{t} \qquad \text{and} \qquad S_\Phi(t) = \frac{\tilde\Phi(t)}{t}, \qquad t \geqslant 0.$$

Lemma 1. *The following statements are equivalent.*

(i) *The function Φ is reasonable;*

(ii) *there exists $\varepsilon > 0$ such that S_Φ is bounded on $[0, \varepsilon)$;*

(iii) *there exist $C, T > 0$ such that $R_\Phi(t) \geqslant C$ for $t \geqslant T$.*

Proof. (i)⇒(iii). Suppose that (iii) is not true, i.e., there is a sequence $t_n \to \infty$ such that $R_\Phi(t_n) < 1/n$. Then for any $t > 0$

$$\tilde\Phi(t) \geqslant \sup_{n\in\mathbb{N}} t_n(t - R_\Phi(t_n)) \geqslant \sup_{n\in\mathbb{N}} t_n(t - \frac{1}{n}) = \infty,$$

whence Φ is not reasonable.

(iii)⇒(ii). Assume that (ii) is not valid; then there is a sequence $t_n \to 0_+$ such that $S_\Phi(t_n) > n$, $n \in \mathbb{N}$. So, there exists another sequence, s_n, such that $nt_n < s_n(t_n - R_\Phi(s_n))$. Obviously it must be $s_n > n$ and $R_\Phi(s_n) < t_n$, which contradicts (iii).

The remaining implication is obvious. □

The equivalence of (i) and (ii) says that once $\tilde\Phi$ is finite near zero, it is bounded by a linear function near zero, which might seem to be somewhat surprising. But it naturally corresponds to the fact that $\tilde\Phi(0) = 0$ and $\tilde\Phi$ is convex.

We say that Φ is *quasiconvex* if there exists a convex function Φ_0 such that $\Phi(t) \leqslant \Phi_0(t) \leqslant C\Phi(Ct)$, $t \geqslant 0$.

Lemma 2. *([10]) The following statements are equivalent.*

(i) *Φ is quasiconvex;*

(ii) *there exists $C > 0$ such that for $s \leqslant t$*

$$\frac{\Phi(s)}{s} \leqslant C\frac{\Phi(Ct)}{t};$$

(iii) *there exists $C > 0$ such that for any cube Q and function f*

$$\Phi\Big(\frac{1}{\mu(Q)}\int_Q |f(x)|\,\mathrm{d}\mu(x)\Big) \leqslant C\,\frac{1}{\mu(Q)}\int_Q \Phi(C|f(x)|)\,\mathrm{d}\mu(x);$$

(iv) *there exists $C > 0$ such that for all $s, t > 0$ and $\alpha \in (0,1)$ we have*

$$\Phi\,(\alpha s + (1-\alpha)t) \leqslant C[\alpha\Phi(Cs) + (1-\alpha)\Phi(Ct)].$$

Let us recall that if Φ itself is convex, then all the statements of Lemma 2 hold with $C = 1$. In particular, S_Φ is always nondecreasing.

Analogously to quasiconvexity we can define quasiconcavity; then the corresponding counterpart lemma holds. We omit the details.

Corollary 1. *A quasiconvex function is reasonable.*

P r o o f. Let Φ be quasiconvex. Then it follows from Lemma 2, (ii), that

$$R_\Phi(t) \geqslant C^{-1}T^{-1}\Phi(C^{-1}T)$$

for any $0 \leqslant T \leqslant t$. Taking a T so that $\Phi(C^{-1}T) > 0$, we get from Lemma 1 that Φ is reasonable. □

Let Φ be quasiconvex. We say that Φ is a *Young's function* if $\lim_{t\to 0+} R_\Phi(t) = 0$ and $\lim_{t\to\infty} R_\Phi(t) = \infty$. If $R_\Phi(t) \leqslant C$, $t \geqslant 0$, we say that Φ is of *bounded type near* ∞ ($\Phi \in B_\infty$). If $R_\Phi(t) \geqslant C^{-1}$, $t > 0$, we say that Φ is of *bounded type near 0* ($\Phi \in B_0$).

Lemma 3. *Let Φ be convex. Then $\Phi \in B_0$ if, and only if, $\tilde{\Phi} \equiv 0$ near 0, and $\Phi \in B_\infty$ if, and only if, $\tilde{\Phi} \equiv \infty$ near ∞.*

P r o o f. Assume that $\Phi \in B_\infty$, that is, $R_\Phi(t) \leqslant C$. Then, clearly, for $t > C$,

$$\tilde{\Phi}(t) = \sup_{s>0} s(t - R_\Phi(s)) = \infty.$$

If $\Phi \in B_0$, that is, $R_\Phi \geqslant C^{-1}$, then for $t \leqslant C^{-1}$

$$\tilde{\Phi}(t) = \sup_{s>0} s(t - R_\Phi(s)) = 0,$$

since the expression in the brackets is negative.

Conversely, let $\tilde{\Phi} \equiv 0$ on $[0, \varepsilon]$. Note that as Φ is convex, we have $(\tilde{\Phi})\tilde{} = \Phi$. Therefore,

$$\Phi(t) = \max\left\{\sup_{s\leqslant\varepsilon} ts;\ \sup_{s>\varepsilon} (ts - \tilde{\Phi}(s))\right\} \geqslant \varepsilon t, \qquad t \geqslant 0.$$

If $\tilde{\Phi} \equiv \infty$ on $[T, \infty)$, then

$$\Phi(t) = \sup_{s>0} (st - \tilde{\Phi}(s)) = \sup_{s\leqslant T} (st - \tilde{\Phi}(s)) \leqslant Tt\ , \qquad t \geqslant 0.$$

□

Lemma 4. *If Φ is convex, then*

$$\Phi(\lambda S_\Phi(t)) \leqslant C\lambda\tilde{\Phi}(t), \qquad t \geqslant 0,\ \lambda \in [0,1]. \tag{3}$$

P r o o f. Since $\Phi(0) = 0$ and Φ is convex, it will suffice to prove

$$\Phi(S_\Phi(t)) \leqslant C\tilde{\Phi}(t), \qquad t \geqslant 0. \tag{4}$$

First, if Φ is a Young function, then (4) holds with $C = 1$ (see [18]). In this case the Young inequality implies $t \leqslant \Phi^{-1}(t)\tilde{\Phi}^{-1}(t)$, and it thus suffices to substitute $t \to \tilde{\Phi}(t)$.

Next, keeping in mind that Φ and $\tilde{\Phi}$ are convex, we can observe using Lemma 2, (ii), that for $t \in [\varepsilon, T]$, $\varepsilon, T > 0$, it is

$$\Phi(S_\Phi(t)) = R_\Phi(S_\Phi(t))S_\Phi(t) \leqslant \varepsilon^{-1}R_\Phi(S_\Phi(T))\tilde{\Phi}(T).$$

Hence, it will suffice to prove that (4) holds near 0 and near ∞.

Let $\Phi \in B_0 \cap B_\infty$. Then by Lemma 3, (4) holds trivially for $t \in [0,\varepsilon] \cup [T,\infty]$.

If $\Phi \in B_\infty \setminus B_0$, then (4) holds trivially for $t \in [T,\infty)$. Moreover, there exists a Young function Ψ such that $\Psi(t) = \Phi(t)$ for $t \in [0,\varepsilon]$. Let $t \in (0, R_\Psi(\varepsilon))$, and $\tau = R_\Psi^{-1}(t)$. Then

$$\tilde{\Psi}(t) = \sup_{0<s<\tau} s(t - R_\Psi(s)) = \sup_{0<s<\tau} s(t - R_\Phi(s)) = \tilde{\Phi}(t),$$

that is, $\tilde{\Phi}$ near zero is determined only by the behaviour of Φ near zero. As Ψ is Young's, (4) holds for Ψ, and hence also for Φ and small values of t.

Finally, if $\Phi \in B_0 \setminus B_\infty$, then (4) holds trivially for $t \in [0,\varepsilon]$, and there exists a Young function Ψ such that $\Psi(t) = \Phi(t)$ for $t \geqslant T$. It is not hard to verify that $\tilde{\Phi}$ and $\tilde{\Psi}$ coincide for large values of t (cf. [14], Theorem I.2.1). As Ψ is Young's, (4) holds for Ψ, and hence also for Φ and large values of t. □

Corollary 2. (cf. [18]). *If Φ is convex, then for all $t \geqslant 0$*

$$R_\Phi(S_\Phi(t)) \leqslant Ct. \tag{5}$$

P r o o f. Multiply (4) by $1/S_\Phi(t)$. □

Let μ be a complete σ-finite Borel measure, satisfying the doubling condition $\mu(2Q) \leqslant C\mu(Q)$, where $2Q$ is the cube concentric with Q and with sides twice as long. Let ϱ and σ be measures absolutely continuous with respect to μ and vice versa, that is, there exist measurable functions $\frac{d\mu}{d\varrho}$, $\frac{d\varrho}{d\mu}$, $\frac{d\mu}{d\sigma}$, and $\frac{d\sigma}{d\mu}$.

For a μ-measurable function h and a μ-measurable set E we shall write $h(E) = \int_E h\,d\mu$ and $h_E = (\mu(E)^{-1})h(E)$.

In this section we shall be concerned with the inequalities

$$\varrho(\{M_\mu f > \lambda\}) \cdot \Phi(\lambda) \leqslant C \int_{\mathbf{R}^n} \Phi(C\,|f(x)|)\,d\sigma, \tag{6}$$

and

$$\varrho(\{M_\mu f > \lambda\}) \leqslant C \int_{\mathbf{R}^n} \Phi(C\,|f(x)|/\lambda)\,d\sigma, \tag{7}$$

where the Hardy-Littlewood maximal operator related to μ is given by

$$M_\mu f(x) = \sup_{Q \ni x} \frac{1}{\mu(Q)} \int_Q |f(y)|\,d\mu(y).$$

Lemma 5. (i) *Let the weak type inequality* (6) *hold. Then* Φ *is quasiconvex.*
(ii) *Let the extra-weak type inequality* (7) *hold. Then* Φ *is reasonable.*

Proof. (i) Take K such that the set $E = \{\frac{d\varrho}{d\mu}(x) \geqslant K^{-1};\ \frac{d\sigma}{d\mu}(x) \leqslant K\}$ has positive measure and let Q be a cube such that $\mu(Q \cap E) > \mu(Q)/2$. By (6),

$$\Phi\left(2^{-1}\,|f|_{Q\cap E}\right) \leqslant CK^2\,\Phi(C|f|)_{Q\cap E}\ . \tag{8}$$

Let $s, t > 0$ and $\alpha \in (0,1)$. Write $Q \cap E$ as $F \cup F'$, where $\mu(F) = \alpha \cdot \mu(Q \cap E)$, and define $f(x) = s \cdot \chi_F(x) + t \cdot \chi_{F'}(x)$. Then (8) turns to

$$\Phi(2^{-1}(\alpha s + (1-\alpha)t)) \leqslant CK^2(\alpha\Phi(Cs) + (1-\alpha)\Phi(Ct)),$$

which is by Lemma 2 equivalent to the quasiconvexity of Φ.

(ii) Assume that Φ is not reasonable. Then, by Lemma 1, there is a sequence $\{t_n\}$, $t_n \nearrow \infty$, such that $\Phi(t_n) < n^{-1}t_n$. Taking arbitrary cube Q and its subsets

E_n in order that $\mu(Q) = C^{-1} t_n \mu(E_n)$ where C is from (7), and putting $f = \chi_{E_n}$ and $\lambda = \frac{\mu(E_n)}{\mu(Q)}$ in (7) we get

$$\varrho(Q) \leqslant C\,\Phi\left(C \cdot \frac{\mu(Q)}{\mu(E_n)}\right) \sigma(E_n) < C \cdot \frac{\mu(Q)}{\mu(E_n)} \cdot \frac{\sigma(E_n)}{n},$$

which yields $\varrho_Q \leqslant \frac{C}{n} \cdot \sigma_{E_n}$. Letting shrink E_n to a density point of $\{0 < \sigma(x) < \infty\}$, we get $\varrho = 0$ almost everywhere on the set where σ is finite. However, this contradicts the mutual absolute continuity of the measures ϱ and σ.

We have seen that the weak type inequalities turn out to be strong enough to guarantee quasiconvexity of Φ, while the extra-weak type ones imply merely reasonability of Φ. This is caused by the fact that (7), unlike (6), provides some control of the growth of Φ only from one side.

From now on we shall assume for simplicity sake that Φ itself is convex.

The pair (σ, ϱ) is said to satisfy the *$A_\Phi(\mu)$ condition* ($(\sigma, \varrho) \in A_\Phi(\mu)$) if either Φ is Young's and there exist C, ε such that

$$\sup_{\alpha>0} \sup_{Q} \alpha \frac{\varrho(Q)}{\mu(Q)} R_\Phi\left(\frac{\varepsilon}{\mu(Q)} \int_Q S_\Phi\left(\alpha^{-1} \frac{d\mu}{d\sigma}\right) d\mu\right) \leqslant C, \tag{9}$$

or $\Phi \in B_0 \cup B_\infty$ and there is C such that for all Q and almost every $x \in Q$

$$\frac{\varrho(Q)}{\mu(Q)} \leqslant C \frac{d\sigma}{d\mu}(x). \tag{10}$$

The pair (σ, ϱ) is said to satisfy the *$E_\Phi(\mu)$ condition* if there are C, $\varepsilon > 0$ such that

$$\sup_Q \frac{1}{\mu(Q)} \int_Q S_\Phi\left(\varepsilon \cdot \frac{d\mu}{d\sigma}(x) \cdot \frac{\varrho(Q)}{\mu(Q)}\right) d\mu \leqslant C. \tag{11}$$

We shall prove that the pairs (σ, ϱ) satisfying $A_\Phi(\mu)$, or $E_\Phi(\mu)$, are good for weak, or extra-weak, resp., type inequalities involving the operator M_μ.

The conditions (9) and (11) take their origin in the well-known Muckenhoupt's A_p *condition* for couples of weights (w, u) (see [16])

$$\sup_Q \left(\frac{1}{|Q|} \int_Q u(x)\, dx\right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}\, dx\right)^{p-1} \leqslant C,$$

and its simple reformulation

$$\sup_Q \frac{1}{|Q|} \int_Q \left(\frac{u_Q}{w(x)}\right)^{p'-1} dx \leqslant C,$$

respectively, where $|Q| = \int_Q \mathrm{d}x$, $u_Q = |Q|^{-1} \int_Q u$, and $p' = p/(p-1)$. The inequality (10) is known as the A_1 *condition* ([16]). The $A_\Phi(\mu)$ condition in the form similar to (9) was introduced in [18], but the key discovery is due to Kerman and Torchinsky [12], see also [6]. Clearly, if $\Phi(t) = t^p$, then $A_\Phi(\mu) = E_\Phi(\mu) = A_p(\mu)$. □

Theorem 1. *The following statements are equivalent.*

(i) *There exists* $C > 0$ *such that for all* f *and* λ *the inequality* (6) *holds;*

(ii) *there exists* $C > 0$ *such that for all* f *and* Q*,*

$$\varrho(Q) \cdot \Phi(|f|_Q) \leqslant C \int_Q \Phi(C\,|f(x)|)\,\mathrm{d}\sigma; \tag{12}$$

(iii) $(\sigma, \varrho) \in A_\Phi(\mu)$.

Theorem 2. *The following statements are equivalent.*

(i) *There exists* $C > 0$ *such that for all* f *and* $\lambda > 0$ *the inequality* (7) *holds;*

(ii) *there exists* $C > 0$ *such that for all* f *and* Q*,*

$$\varrho(Q) \leqslant C \int_Q \Phi(C\,|f(x)|/|f|_Q)\,\mathrm{d}\sigma; \tag{13}$$

(iii) $(\sigma, \varrho) \in E_\Phi(\mu)$.

The next remark sheds light on the connection between the statements of both theorems and justifies our terminology "weak" and "extra-weak".

Remark 1. Each statement of Theorem 1 implies its counterpart in Theorem 2.

Indeed, inserting $\lambda = 1$ in (6) we get

$$\varrho(\{M_\mu f > 1\}) \leqslant C \int_{\mathbb{R}^n} \Phi(C\,|f(x)|)\,\mathrm{d}\sigma,$$

which is by homogeneity of M_μ equivalent to (7). Similarly, taking $(|f|_Q)^{-1} \cdot f$ instead of f in (12) we get (13). Lastly, to see that $A_\Phi(\mu) \subset E_\Phi(\mu)$, simply put $\alpha = \frac{\mu(Q)}{\varrho(Q)}$ in (9) in case Φ is Young's, or use (10) in case $\Phi \in B_0 \cup B_\infty$.

Lemma 6. *Assume that* $(\sigma, \varrho) \in A_1(\mu)$ *(that is,* (10) *holds). Then the weak-type inequality* (6) *holds for any* Φ.

Proof. As μ is doubling, standard covering argument yields

$$\lambda \cdot \varrho(\{M_\mu f > \lambda\}) \leqslant C \int |f(x)| \, \mathrm{d}\sigma.$$

Moreover, the convexity of Φ gives via Lemma 2, (iii), that $\Phi(M_\mu f) \leqslant M_\mu(\Phi(f))$. Hence

$$\begin{aligned} \varrho(\{M_\mu f > \lambda\}) \cdot \Phi(\lambda) &= \varrho(\{\Phi(M_\mu f) > \Phi(\lambda)\}) \cdot \Phi(\lambda) \\ \leqslant \varrho(\{M_\mu[\Phi(f)] > \Phi(\lambda)\}) \cdot \Phi(\lambda) &\leqslant C \int_{\mathbb{R}^n} \Phi(|f(x)|) \, \mathrm{d}\sigma. \end{aligned}$$

□

Lemma 7. *If $\Phi \in B_0 \cup B_\infty$ and the estimate* (12) *holds, then $(\sigma, \varrho) \in A_1(\mu)$.*

Proof. Let $\Phi \in B_0$. Then inserting $f = \chi_E$, $E \subset Q$, in (12), we get

$$\frac{\mu(E)}{\mu(Q)} \leqslant C \, \Phi\left(\frac{\mu(E)}{\mu(Q)}\right) \leqslant C \, \Phi(C) \, \frac{\sigma(E)}{\varrho(Q)},$$

which yields $(\sigma, \varrho) \in A_1(\mu)$. Now let $\Phi \in B_\infty$. As already observed (Remark 1), (12) suffices for (13). Putting $f = \chi_E$ this time in (13) we obtain

$$\varrho(Q) \leqslant C \, \sigma(E) \, \Phi\left(C \frac{\mu(Q)}{\mu(E)}\right) \leqslant C \, \sigma(E) \, \frac{\mu(Q)}{\mu(E)},$$

which is $A_1(\mu)$, again. □

Proof of Theorem 1. If Φ is a Young function, the proof can be done as in [18] with trivial changes. Assume that $\Phi \in B_0 \cup B_\infty$; then the implications (ii)$\Rightarrow$(iii)$\Rightarrow$(i) follow from Lemmas 7 and 6, and the implication (i)$\Rightarrow$(ii) is a consequence of the obvious inclusion $Q \subset \{M_\mu f > |f|_Q/2\}$. □

Proof of Theorem 2. That (i) implies (ii) follows again from the inclusion $Q \subset \{M_\mu f > |f|_Q/2\}$.

The implication (iii)$\Rightarrow$(i) can be proved following the lines of the proof in [18].

The proof of (ii)$\Rightarrow$(iii) in [18] requires somewhat complicated theory of norms in Orlicz spaces and saturation of the Hölder inequality. We give here a much simpler direct proof, applicable to a general Φ.

Let Q be a fixed cube. If $\varrho(Q) = 0$, there is nothing to prove. Let $0 < \varrho(Q) < \infty$.

Assume first that $\Phi \notin B_\infty$. Then $\tilde{\Phi}$, and hence also S_Φ, is finite on $(0,\infty)$. Given $k \in \mathbf{N}$, put $Q_k = \{x \in Q;\ \frac{\mathrm{d}\sigma}{\mathrm{d}\mu}(x) > 1/k\}$ and

$$g(x) = g_k(x) = S_\Phi\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \chi_{Q_k}(x)$$

with ε to be specified later. It follows from (ii) that

$$\int\limits_{Q_k} \tilde{\Phi}\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \mathrm{d}\sigma = \varepsilon\, g_Q\, \varrho(Q) \leqslant C\varepsilon\varrho(Q) + I_Q,$$

where I_Q is defined as follows: $I_Q = 0$ if $g_Q \leqslant C$ (C is the bigger of the constants from (13) and (3)), and

$$I_Q = C\varepsilon g_Q \int\limits_Q \Phi\left(\frac{C}{g_Q} g(x)\right) \mathrm{d}\sigma \qquad \text{if} \quad g_Q > C.$$

Hence, using (3) with $\lambda = C/g_Q$,

$$\begin{aligned} I_Q &= C\varepsilon g_Q \int\limits_{Q_k} \Phi\left(\frac{C}{g_Q} S_\Phi\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right)\right) \mathrm{d}\sigma \\ &\leqslant C^3\varepsilon \int\limits_{Q_k} \tilde{\Phi}\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \mathrm{d}\sigma, \end{aligned}$$

which yields

$$\int\limits_{Q_k} \tilde{\Phi}\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \mathrm{d}\sigma \leqslant C\varepsilon\varrho(Q) + C^3\varepsilon \int\limits_{Q_k} \tilde{\Phi}\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \mathrm{d}\sigma. \tag{14}$$

Now (remember that S_Φ is nondecreasing),

$$\begin{aligned} \int\limits_{Q_k} \tilde{\Phi}\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \mathrm{d}\sigma &= \varepsilon \frac{\varrho(Q)}{\mu(Q)} \int\limits_{Q_k} S_\Phi\left(\varepsilon \cdot \frac{\varrho(Q)}{\mu(Q)} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \mathrm{d}\mu \\ &\leqslant \varepsilon\varrho(Q) \cdot S_\Phi\left(k\varepsilon \frac{\varrho(Q)}{\mu(Q)}\right) < \infty, \end{aligned}$$

whence we can take ε sufficiently small ($\varepsilon < C^{-3}$) and subtract in (14) to get thereby

$$\int\limits_{Q_k} \tilde{\Phi}\left(\varepsilon \frac{\varrho(Q)}{\mu(Q)} \frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(x)\right) \mathrm{d}\sigma \leqslant \frac{C\varepsilon}{1 - C^3\varepsilon}\, \varrho(Q).$$

Since $\mu(Q \setminus \bigcup Q_k) = 0$ and the constant at the right does not depend on k, (iii) follows.

The situation is much simpler if $\Phi \in B_\infty$, since then $R_\Phi(t) \leqslant C$, and inserting $f = \chi_E$, $E \subset Q$, into (ii) gives

$$\varrho(Q) \leqslant C \frac{\mu(Q)}{\mu(E)} \sigma(E).$$

So, (σ, ϱ) belongs to $A_1(\mu)$. It follows easily from (5) that (10) always implies (9), and therefore $A_1(\mu) \subset A_\Phi(\mu)$ for every Φ. As $A_\Phi(\mu) \subset E_\Phi(\mu)$ for every Φ (Remark 1), we are done. □

Corollary 3. *If $\Phi \in B_\infty$, then $A_\Phi(\mu) = E_\Phi(\mu) = A_1(\mu)$.*

P r o o f. The proof of Lemma 7 shows that if $\Phi \in B_\infty$, then $E_\Phi(\mu) \subset A_1(\mu)$. The remaining inclusions have been already established. □

4. The condition A_∞

In this section we assume that $\sigma \equiv \varrho$. Recall that Φ is convex.

We say that $\varrho \in A_\infty(\mu)$ if there exist $\delta, \varepsilon \in (0,1)$ such that $E \subset Q$ and $\mu(E) < \delta\mu(Q)$ imply $\varrho(E) < \varepsilon\varrho(Q)$.

Both the endpoints of the A_p scale, the classes A_1 and A_∞, are of exceptional meaning. Between A_1 and all other A_p's there is a significant gap. For example, putting $\Phi(t) = t(1 + \log^+ t)^K$, we get $A_1(\mu) \subset E_\Phi(\mu) \subset \bigcap_{p>1} A_p(\mu)$, where both the inclusions are proper (see [2], [15], [17]). A different situation can be found near A_∞; it is known (e.g. [4]) that $A_\infty = \bigcup_{p>1} A_p$. This fact will allow us to obtain new characterizations of A_∞.

The idea is simple: First, it is easy to prove that $E_\Phi(\mu) \subset A_\infty(\mu)$ in any case of Φ. Further, we know that $A_\Phi(\mu) \subset E_\Phi(\mu)$ (Remark 1). Therefore, it will suffice to take Φ with sufficiently rapid growth so that $A_p(\mu) \subset A_\Phi(\mu)$ for all p, and then it must be $A_\Phi(\mu) = E_\Phi(\mu) = A_\infty(\mu)$.

The condition A_∞ has been intesively studied and a lot of equivalent statements have been proved ([4], [7], [11], [5] etc.). In the particular (weighted) case $\mathrm{d}\mu = \mathrm{d}x$ and $\mathrm{d}\varrho = w(x)\mathrm{d}x$, Hruščev ([11]) proved that $w \in A_\infty$ (we write $w \in A_\infty$ instead of $\varrho \in A_\infty(\mu)$), if, and only if,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x\right) \exp\left(\frac{1}{|Q|}\int_Q \log\frac{1}{w(x)}\mathrm{d}x\right) \leqslant C. \tag{15}$$

(An independent proof of this result was given by García-Cuerva and Rubio de Francía in [7]). By a different argument, Fujii ([5]) obtained (among others) another characterization of A_∞,

$$\sup_Q \int_Q \log^+ \left(\frac{w(x)}{w_Q}\right) w(x)\,\mathrm{d}x \leqslant C\, w(Q). \tag{16}$$

We shall prove a new general characterization of A_∞ expressed in terms of $E_\Phi(\mu)$ conditions, which covers (15) and (16) as particular cases and clarifies their mutual relationship.

Theorem 3. *Let Φ be such that $S_\Phi(t^\alpha)$ is quasiconcave on $(0,\infty)$ for any $\alpha \geqslant \alpha_0$ and some α_0. Then $A_\Phi(\mu) = E_\Phi(\mu) = A_\infty(\mu)$.*

Proof. First, let $\varrho \in E_\Phi(\mu)$. Then, inserting $\varrho = \sigma$ and $f = \chi_E$, $E \subset Q$, in Theorem 2, (ii), we get

$$\frac{\varrho(Q)}{\varrho(E)} \leqslant C\,\Phi\left(C\,\frac{\mu(Q)}{\mu(E)}\right).$$

Therefore, if $E' = Q \setminus E$ and $\mu(E') < \delta\mu(Q)$, we have $\varrho(E') < \varepsilon\varrho(Q)$, where $(1-\varepsilon)^{-1} = C\Phi(C/(1-\delta))$. In other words, $\varrho \in A_\infty(\mu)$. Note that this inclusion, $E_\Phi(\mu) \subset A_\infty(\mu)$, holds for any Φ.

Now, let $\varrho \in A_\infty(\mu)$. Then there is $p > \alpha_0 + 1$ such that $\varrho \in A_p(\mu)$ (see e.g. [4]). By our assumption, the function $F(t) = S_\Phi(t^{p-1})$ is quasiconcave. Taking ε small enough ($\varepsilon C \leqslant 1$) and using Jensen's inequality and (5), we get

$$\begin{aligned}
&\frac{\alpha\varrho(Q)}{\mu(Q)}\cdot R_\Phi\left(\frac{\varepsilon}{\mu(Q)}\int_Q S_\Phi\left(\frac{1}{\alpha}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\right)\mathrm{d}\mu\right)\\
&= \frac{\alpha\varrho(Q)}{\mu(Q)}\cdot R_\Phi\left(\frac{\varepsilon}{\mu(Q)}\int_Q F\left(\left(\frac{1}{\alpha}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\right)^{p'-1}\right)\mathrm{d}\mu\right)\\
&\leqslant \frac{\alpha\varrho(Q)}{\mu(Q)}\cdot R_\Phi\left(C\varepsilon F\left(\frac{C}{\mu(Q)}\int_Q\left(\frac{1}{\alpha}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\right)^{p'-1}\mathrm{d}\mu\right)\right)\\
&\leqslant C\cdot\frac{\alpha\varrho(Q)}{\mu(Q)}\cdot\left(\frac{1}{\mu(Q)}\int_Q\left(\frac{1}{\alpha}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\right)^{p'-1}\mathrm{d}\mu\right)^{p-1}
\end{aligned}$$

Hence, $A_\infty(\mu) \subset A_\Phi(\mu)$. Since $A_\Phi(\mu) \subset E_\Phi(\mu)$ always, the proof is complete. □

Remark 2. As we know, if $\Phi(t) = t^p$ or $\Phi \in B_\infty$, then $A_\Phi = E_\Phi$. Now, Theorem 3 describes another class of functions Φ with this property. However, the inclusion $A_\Phi \subset E_\Phi$ is proper in general. The following two examples are essentially due to Bagby [1]:

If $\Phi(t) = t^p$ for $t \in [0,1)$ and $\Phi(t) = t^q$ for $t \in [1,\infty)$, where $p < q$, then $A_\Phi = A_p$ but $E_\Phi = A_q$.

If $\Phi(t) = t^p(\log_+ t+1)^{-q}$, $p > 1$, $q > 0$, μ is Lebesgue measure, $d\varrho = d\sigma = x^{p-1}\mathrm{d}x$, then $(\sigma,\varrho) \in E_\Phi$, but $(\sigma,\varrho) \notin A_\Phi = A_p$.

Theorem 4. *The following statements are equivalent.*

(i) $\varrho \in A_\infty(\mu)$;

(ii) *there is* C *such that for every* Q

$$\sup_Q \frac{1}{\mu(Q)} \int_Q \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\cdot\frac{\varrho(Q)}{\mu(Q)}\right)\mathrm{d}\mu \leqslant C; \tag{17}$$

(iii) *there is* C *such that for every* Q

$$\sup_Q \frac{1}{\varrho(Q)} \int_Q \log\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\mu}(x)\cdot\frac{\mu(Q)}{\varrho(Q)}\right)\mathrm{d}\varrho \leqslant C. \tag{18}$$

Proof. To prove that (i)$\Leftrightarrow$(ii), put $\tilde{\Phi}(t) = t(1+\log^+ t)$. Then $\tilde{\Phi}$ is convex and so it is indeed a complementary function (e.g. to the function $(\tilde{\Phi})^{\tilde{}}$). On the other hand, $S_\Phi(t^\alpha) = 1 + \alpha\log^+ t$ is evidently quasiconcave for any $\alpha > 0$. Theorem 3 therefore implies that $\varrho \in A_\infty(\mu)$ if, and only if, $\varrho \in E_\Phi(\mu)$, or

$$\sup_Q \frac{1}{\mu(Q)} \int_Q \log^+\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\cdot\frac{\varrho(Q)}{\mu(Q)}\right)\mathrm{d}\mu \leqslant C.$$

This inequality obviously implies (17), but in fact they are equivalent. This will be seen once we prove that for any Q

$$\frac{1}{\mu(Q)} \int_Q \log^+\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\cdot\frac{\varrho(Q)}{\mu(Q)}\right)\mathrm{d}\mu \leqslant \frac{1}{\mu(Q)} \int_Q \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\cdot\frac{\varrho(Q)}{\mu(Q)}\right)\mathrm{d}\mu + \frac{1}{e}, \tag{19}$$

cf. [11], Lemma 1.

To prove (19), put $E = \left\{x \in Q\,;\ \frac{\varrho(Q)}{\mu(Q)} \leqslant \frac{\mathrm{d}\varrho}{\mathrm{d}\mu}(x)\right\}$. Then, by the Jensen inequality, applied to the convex function $-\log$,

$$\begin{aligned}
&\frac{1}{\mu(Q)}\int\limits_Q \log\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\cdot\frac{\varrho(Q)}{\mu(Q)}\right)\mathrm{d}\mu - \frac{1}{\mu(Q)}\int\limits_Q \log^+\left(\frac{\mathrm{d}\mu}{\mathrm{d}\varrho}(x)\cdot\frac{\varrho(Q)}{\mu(Q)}\right)\mathrm{d}\mu\\
&= \frac{\mu(E)}{\mu(Q)}\cdot\frac{1}{\mu(E)}\int\limits_E\left(-\log\frac{\mathrm{d}\varrho}{\mathrm{d}\mu}(x)\right)\mathrm{d}\mu + \frac{1}{\mu(Q)}\int\limits_E \log\frac{\varrho(Q)}{\mu(Q)}\mathrm{d}\mu\\
&\geqslant -\frac{\mu(E)}{\mu(Q)}\log\left(\frac{1}{\mu(E)}\int\limits_E\frac{\mathrm{d}\varrho}{\mathrm{d}\mu}(x)\mathrm{d}\mu\right) + \frac{\mu(E)}{\mu(Q)}\cdot\log\frac{\varrho(Q)}{\mu(Q)}\\
&= \frac{\mu(E)}{\mu(Q)}\cdot\log\left(\frac{\varrho(Q)}{\mu(Q)}\cdot\frac{\mu(E)}{\varrho(E)}\right) \geqslant \frac{\mu(E)}{\mu(Q)}\cdot\log\left(\frac{\mu(E)}{\mu(Q)}\right) \geqslant -\frac{1}{e},
\end{aligned}$$

since $\min\limits_{t\in(0,1)} t\log t = -1/\mathrm{e}$.

The equivalence of (ii) and (iii) follows from the equivalence of $\varrho \in A_\infty(\mu)$ and $\mu \in A_\infty(\varrho)$, which was proved by Coifman and Fefferman [4] provided that both μ and ϱ were doubling. In our case μ is assumed to be doubling from the very beginning and $\varrho \in A_\infty(\mu)$ easily yields that also ϱ is doubling. The proof is thus complete. □

To round off this section, put finally $\mathrm{d}\mu(x) = \mathrm{d}x$ and $\mathrm{d}\varrho(x) = w(x)\mathrm{d}x$. Then (17) turns to

$$\sup_Q \frac{1}{|Q|}\int\limits_Q \log\frac{w_Q}{w(x)}\mathrm{d}x \leqslant C,$$

exactly what we obtain after taking log of the left hand side of (15). In view of this, (17) is equivalent to the Hruščev condition (15). Similarly, (18) turns to

$$\sup_Q \frac{1}{w(Q)}\int\limits_Q \log\left(\frac{w(x)}{w_Q}\right) w(x)\,\mathrm{d}x \leqslant C,$$

which is the Fujii condition (16) (even a slightly better one, as the "+" sign is removed).

5. The Hilbert transform

In the sequel we assume that $n = 1$. Recall that Φ is still convex. The symbol I will always stand for an open interval on the real line and if $I = (a, b)$, we denote $I' = [b, 2b - a)$. We shall also restrict ourselves to the case $\mathrm{d}\mu(x) = \mathrm{d}x$ (the Lebesgue measure), and $\mathrm{d}\varrho(x) = d\sigma(x) = w(x)\mathrm{d}x$, where w is a positive measurable function (*weight*). Thus, $w \in A_\Phi$ if either Φ is Young's and

$$\sup_{\alpha, I} \alpha w_I \cdot R_\Phi\left(\frac{\varepsilon}{|I|} \int\limits_I S_\Phi\left(\frac{1}{\alpha w(x)}\right) \mathrm{d}x\right) \leqslant C,$$

or $\Phi \in B_0 \cup B_\infty$ and $w \in A_1$, that is,

$$w_I \leqslant C \cdot \operatorname{ess\,inf}\{w(x);\, x \in I\}.$$

Similarly, $w \in E_\Phi$ if

$$\sup_I \frac{1}{|I|} \int\limits_I S_\Phi\left(\varepsilon \frac{w_I}{w(x)}\right) \mathrm{d}x \leqslant C.$$

The maximal operator M treated in this section is defined by

$$Mf(x) = \sup\{|f|_I;\, I \ni x\}.$$

The Hilbert transform is given for any function f satisfying

$$\int\limits_{-\infty}^{\infty} |f(x)|\, (1 + |x|)^{-1}\, \mathrm{d}x < \infty$$

by the Cauchy principal value integral

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0+} \int\limits_{\mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)} \frac{f(y)}{x - y}\, \mathrm{d}y.$$

Similarly we define the maximal Hilbert transform

$$H^* f(x) = \frac{1}{\pi} \sup_{\varepsilon > 0} \left| \int\limits_{\mathbb{R} \setminus (x-\varepsilon, x+\varepsilon)} \frac{f(y)}{x - y}\, \mathrm{d}y \right|.$$

We shall prove the following theorems.

Theorem 5. *The following statements are equivalent.*

(i) *There exists* $C > 0$ *such that for all* f *for which* H^*f *is defined and all* λ

$$w(\{H^*f > \lambda\}) \cdot \Phi(\lambda) \leqslant C \int_{-\infty}^{\infty} \Phi(C\,|f(x)|)\, w(x)\, dx; \tag{20}$$

(ii) $\Phi \in \Delta_2$, *and there exists* $C > 0$ *such that for all* f *and* λ

$$w(\{Mf > \lambda\}) \cdot \Phi(\lambda) \leqslant C \int_{-\infty}^{\infty} \Phi(C\,|f(x)|)\, w(x)\, dx; \tag{21}$$

(iii) $\Phi \in \Delta_2$ *and* $w \in A_\Phi$.

Theorem 6. *Let* $\Phi \in \Delta_2^0$, *that is,* $\Phi(2t) \leqslant C\Phi(t)$ *for* $t \in (0,1)$. *Then the following statements are equivalent.*

(i) *There exists* $C > 0$ *such that for all* f *for which* H^*f *is defined and all* $\lambda > 0$

$$w(\{H^*f > \lambda\}) \leqslant C \int_{-\infty}^{\infty} \Phi(C\,|f(x)|/\lambda)\, w(x)\, dx; \tag{22}$$

(ii) *there exists* $C > 0$ *such that for all* f *and* $\lambda > 0$

$$w(\{Mf > \lambda\}) \leqslant C \int_{-\infty}^{\infty} \Phi(C\,|f(x)|/\lambda)\, w(x)\, dx; \tag{23}$$

(iii) $w \in E_\Phi$.

Remark 3. For any interval $I = (a,b)$, $f \geqslant 0$ and $x \in I$ we have

$$H^*(\chi_{I'}f)(x) \geqslant |H(\chi_{I'}f)(x)| \geqslant (2\pi)^{-1} f_{I'}. \tag{24}$$

Similarly, for $x \in I'$ we have

$$H^*(\chi_I f)(x) \geqslant |H(\chi_I f)(x)| \geqslant (2\pi)^{-1} f_I. \tag{25}$$

Now, (24) and (20), applied to $f = \chi_{I'}$ and $\lambda < (2\pi)^{-1}$, lead to

$$w(I) \leqslant C\, w(I'). \tag{26}$$

Note that (22) together with (24) implies (26), too. Given $f \geqslant 0$ and $\lambda > 0$, put $\Omega = \{Mf > \lambda\}$ and let F be any compact subset of Ω. Then

$$F \subset \bigcup_{j=1}^{N} I_j, \qquad \text{where} \qquad f_{I_j} > \lambda.$$

By [8], Lemma 4.4, Chap. I, §4, there is a disjoint subfamily $\{J_j\}$ of $\{I_j\}$ such that $w(\bigcup I_j) \leqslant 2\sum w(J_j)$. Thus, by (26), (24), and (25)

$$\begin{aligned} w(F) &\leqslant w(\bigcup I_j) \leqslant 2\sum w(J_j) \\ &\leqslant C\sum w(J_j{}') \\ &\leqslant C\sum w\left(\{|H(f\chi_{J_j})| > (2\pi)^{-1}\lambda\}\right) \end{aligned}$$

As F was arbitrary, this inequality shows that

$$w(\{Mf > \lambda\}) \leqslant Cw(\{|Hf| > \lambda\}),$$

and therefore in both the above theorems the implication (i)⇒(ii) holds. Moreover, it is clear that we can replace H^*f by $|Hf|$ in Theorems 5 and 6.

P r o o f o f T h e o r e m 5. Coifman [3] proved that if $w \in A_\infty$ and $\Phi \in \Delta_2$, then

$$\sup_\lambda \Phi(\lambda) \cdot w\left(\{H^*f > \lambda\}\right) \leqslant C \sup_\lambda \Phi(\lambda) \cdot w\left(\{Mf > \lambda\}\right).$$

This proves (ii)⇒(i). It remains to prove that (i) suffices for $\Phi \in \Delta_2$, the rest follows from Theorem 1. We shall use the idea from [9]. Given $\lambda > 0$ we put $f(x) = (2C)^{-1}\lambda\chi_{(0,1)}(x)$. Then, by (i),

$$\Phi(\lambda) \leqslant C\frac{w(0,1)}{w(\{H^*\chi_{(0,1)} > 2C\})} \cdot \Phi(\lambda/2),$$

and we are done. □

P r o o f o f T h e o r e m 6. The implications (i)⇒(ii)⇒(iii) follow from Remark 3 and Theorem 2. We shall prove (iii)⇒(i). Given a function f and $\lambda > 0$, put $\Omega = \{Mf > \lambda\}$, $F = \mathbf{R} \setminus \Omega$. Then $\Omega = \bigcup I_j$, where I_j are closed intervals with disjoint interiors such that $\operatorname{dist}(F, I_j) = |I_j|$ (the *Whitney decomposition*—cf. [8]). Since $4I_j$ always meets F, it must be $|f|_{I_j} \leqslant 4\lambda$. As usual, we split f into the "good" and the "bad" parts, namely,

$$\begin{aligned} g(x) &= f(x)\chi_F(x) + \sum_j f_{I_j} \cdot \chi_{I_j}(x), \\ b(x) &= f(x) - g(x) = \sum_j \left(f(x) - f_{I_j}\right)\chi_{I_j}(x) = \sum_j b_j(x). \end{aligned}$$

To estimate the "good" part is easy. Our assumption $\Phi \in \Delta_2^0$ guarantees that $\Phi(\lambda) \geqslant C\lambda^p$ for $\lambda \in (0,1]$ and all p bigger than some p_0. As observed in the proof of Theorem 3, (iii) implies that $w \in A_\infty$, hence $w \in A_p$ for p bigger than some p_1. Therefore, for such p, H^* is bounded on L_p, w ([7], Chap. IV, Theorem 3.6), and we have for $p \geqslant \max(p_0, p_1)$ (recall that $|f| \leqslant \lambda$ almost everywhere on F)

$$
\begin{aligned}
(27) \qquad w(\{H^*g > \lambda\}) &\leqslant C \int_{-\infty}^{\infty} \left(\frac{g(x)}{\lambda}\right)^p w(x)\,\mathrm{d}x \\
&\leqslant C \int_F \left(\frac{|f(x)|}{\lambda}\right)^p w(x)\,\mathrm{d}x + C\,w(\Omega) \\
&\leqslant C \int_F \Phi\left(\frac{|f(x)|}{\lambda}\right) w(x)\,\mathrm{d}x + Cw(\Omega).
\end{aligned}
$$

Now let us deal with the "bad" part. As known ([19], Chap. II, 4.6.2), for $x \in F$ we have

$$
H^*b(x) \leqslant C \sum_j \int_{I_j} \left| \frac{1}{|x-t|} - \frac{1}{|x-t_j|} \right| \cdot |b_j(t)|\,\mathrm{d}t + C_0\,Mb(x),
$$

where t_j is the center of I_j. Note that $|x - t_j|$ is comparable to $|x-t|$ for every $t \in I_j$ and $x \in F$. Hence, making use of the definition of b_j, the estimate $|f|_{I_j} \leqslant 4\lambda$, and the estimate

$$
\frac{|I_j|}{|x-t_j|} \leqslant C\,M(\chi_{I_j})(x), \qquad x \in F,
$$

we obtain

$$
\begin{aligned}
(28) \qquad H^*b(x) &\leqslant C \sum_j \frac{|I_j|}{|x-t_j|^2} \int_{I_j} |b_j(t)|\,\mathrm{d}t + C_0\,Mb(x) \\
&\leqslant C \sum_j \left(\frac{|I_j|}{|x-t_j|}\right)^2 |f|_{I_j} + C_0\,Mb(x) \\
&\leqslant C\lambda \sum_j M^2(\chi_{I_j})(x) + C_0\,Mb(x).
\end{aligned}
$$

As already mentioned, $w \in A_p$ for some $p > 2$. Put $r = p/2$, then $r > 1$ and we can invoke the vector-valued weighted strong-type inequality ([13], Theorem 1, or

[7], Chap. 5, Theorem 6.4 and Remark 6.5 a) to obtain thereby

$$
\begin{aligned}
(29)\quad w(\{x \in F;\, C\lambda \sum_j M^2(\chi_{I_j})(x) > \lambda\}) &\leqslant C \int_F \Big[\sum_j M^2(\chi_{I_j})(x)\Big]^r w(x)\,\mathrm{d}x \\
&\leqslant C \int_\Omega \sum_j \chi_{I_j}(x)\, w(x)\,\mathrm{d}x \\
&\leqslant C \sum_j w(I_j) = C\, w(\Omega),
\end{aligned}
$$

as I_j's have disjoint interiors. Since $|b(x)| \leqslant |f(x)| + 4\lambda$, it is

$$
(30)\qquad \{x \in F;\, C_0 Mb(x) > 5C_0\lambda\} \subset \{x \in F;\, Mf(x) > \lambda\} = \emptyset.
$$

Now, (28), (29) and (30) give

$$
(31)\qquad w(\{x \in F;\, H^*b(x) > (5C_0+1)\lambda\}) \leqslant Cw(\Omega).
$$

It follows from Theorem 2 that

$$
w(\Omega) \leqslant C \int_{-\infty}^{\infty} \Phi\left(C\frac{|f(x)|}{\lambda}\right) w(x)\,\mathrm{d}x.
$$

Combined with (27) and (31) this leads to

$$
\begin{aligned}
&w(\{H^*f > (5C_0+2)\lambda\}) \\
&\leqslant w(\{H^*g > \lambda\}) + w(\{x \in F;\, H^*b > (5C_0+1)\lambda\}) + w(\Omega) \\
&\leqslant C \int_{-\infty}^{\infty} \Phi\left(C\frac{|f(x)|}{\lambda}\right) w(x)\,\mathrm{d}x,
\end{aligned}
$$

which easily yields the desired estimate. □

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