J. Blair Roll

Locally partially ordered groups

Czechoslovak Mathematical Journal, Vol. 43 (1993), No. 3, 467–481

Persistent URL: http://dml.cz/dmlcz/128411

Czechoslovak Mathematical Journal, 43 (118) 1993, Praha

LOCALLY PARTIALLY ORDERED GROUPS

J. BLAIR ROLL, Findlay

(Received October 10, 1991)

1. INTRODUCTION

The additive group of real numbers together with its natural order and the subgroups of this group with their inherited orders are generic examples of ordered groups. The study of these groups has led to many generalizations and a vast amount of literature since the early 1900's [F]. More recently, the closely related circle group, C, has been investigated [L, M, N1, N2]. In this paper,[1] C is viewed as a "locally totally ordered" group, and some generalizations are investigated.

Definition 1.1 (L. S. Rieger [R]). A cyclically ordered group is a group, K on which there is defined a ternary relation (a,b,c) which satisfies the following axioms:

(i) if a, b and c are distinct elements of K, then precisely one of (a,b,c) and (a,c,b) holds;

(ii) (a,b,c) implies (b,c,a);

(iii) (a,b,c) and (a,c,d) imply (a,b,d);

(iv) (a,b,c) implies (xa,xb,xc) and (ax,bx,cx) for all x in K.

Recall that an ordered group, or o-group, is a group which is totally ordered by a relation which is preserved by the group operation. Throughout this paper, groups will be written multiplicatively with identity element e.

Rieger characterized cyclically ordered groups as quotients of o-groups as follows.

Theorem 1.1 (Rieger [R]). *Suppose that G is an o-group with an element z in the center such that $e \leqslant z$ and the powers of z are unbounded. Define a ternary relation on $G/\langle z\rangle$ by $(\bar a,\bar b,\bar c)$ if and only if $r_a < r_b < r_c$ or $r_b < r_c < r_a$ or $r_c < r_a < r_b$ where r_x denotes the unique elements of $\bar x$ satisfying $e \leqslant r_x < z$. Then $G/\langle z\rangle$ is a*

[1] The basis of this paper is the author's Ph.D. thesis, directed by W. Charles Holland at Bowling State University, Bowling Green, Ohio [B]

cyclically ordered group. Conversely, if K is a cyclically ordered group, there exists an o-group G and an element z in the center of G such that their powers of z are unbounded and $K \simeq G/\langle z\rangle$ with cyclical ordering described above.

Rieger's axioms are not easily generalized. For example, there is no obvious ternary relation which describes the torus, $C \times C$, as a "partially cyclically ordered" group analogue of the partially ordered group $\mathbf{R} \times \mathbf{R}$. In the first part of this paper, we investigate the natural relationship between locally totally ordered sets and cyclically ordered sets and prove that cyclically ordered sets are images of totally ordered sets.

Vítězslav Novák [N1] investigated properties of cyclically ordered sets in the early 1980's. Novák defined a complete cyclic order on a set as a ternary relation satisfying conditions equivalent to axioms (i), (ii) and (iii) of definition 1.1.

Definition 1.2. A cyclically ordered set is a set, S, on which there is defined a complete cyclic order.

Cyclically ordered sets are locally totally ordered in the following sense.

Definition 1.3. Suppose that C is a cyclically ordered set with distinct elements a and b. Define $C_{a,b}$ and C_a by:

$$C_{a,b} = \{z \in C \mid (a,z,b)\} \quad \text{and} \quad C_a = \{z \in C \mid z \neq a\}.$$

i.e., $C_{a,b}$ is the subset of C consisting of points between a and b and C_a is the set C with one point removed. We note that both $C_{a,b}$ and C_a are totally ordered by the relation $\leqslant_a$ defined by $x \leqslant_a y$ if and only if $x = y$ or (a,x,y). ([N1])

A natural relationship between total orders and cyclical orders on a given set is described in the following theorem, proved by Novák [N1].

Theorem 1.2. *Suppose that the set, S, is totally ordered by the relation $\leqslant$. Define (a,b,c) on S by (a,b,c) iff a, b, and c are distinct and $a < b < c$ or $b < c < a$ or $c < a < b$. Then (a,b,c) is a cyclical order of S, which we will call the associated cyclical order of $(S,\leqslant)$. Furthermore, if C is a cyclically ordered set, there exists a total order on C such that the cyclical ordering of C is the associated cyclical order.*

Note that given a cyclically ordered set, C, the total order whose associated cyclical order is the cyclical order of C is not unique.

For an arbitrary fixed element, a, of a cyclically ordered set S, we will call the order $\leqslant_a$ defined in Definition 1.3 the *associated total order on C determined by a.*

Before we prove an analog of Rieger's theorem for cyclically ordered sets, we define a type of mapping to correspond to the natural map in Rieger's theorem.

Definition 1.4. If S is totally ordered set and C is a cyclically ordered set, a map $\varphi\colon S \to C$ will be called a covering map if φ satisfies the following conditions:

(i) for every s in S there exist $x, y \in S$ s.t. $x < s < y$ and φ maps (x, y) onto $C_{x\varphi, y\varphi}$ in a one to one and order preserving manner;

(ii) if $a, b \in C$ and $x, y \in C_{a,b}$ and $x \leqslant_a y$, then there exist $s, t \in S$ s.t. $s < t$, $s\varphi = x$ and $t\varphi = y$.

In Rieger's theorem, the natural map from G onto $G/\langle z\rangle$ is in fact a covering map. We prove the following analog of Rieger's theorem for cyclical ordered sets.

Theorem 1.3. *If C is a cyclically ordered set, there exists a totally ordered set, S, and a covering map $\varphi\colon S \to C$.*

Proof. Suppose that $a \in C$. Let $S = C \times Z$ and order S lexicographically from the right. That is, define $\leqslant$ on s by $(c_1, m) \leqslant (c_2, n)$ iff $m < n$ or $m = n$ and $c_1 \leqslant_a c_2$. Let $\varphi\colon S \to C$ be the projection map onto C, i.e. $(c, n)\varphi = c$. We will show that φ is a covering map. It is clear that φ satisfies condition (i). For condition (ii) suppose $c, d \in C$ and $x, y \in C_{c,d}$ satisfy $x \leqslant_c y$. We find elements s and t in S with $s < t$ that map to x and y respectively. If $a = c$, $a = x$, or $a = y$ then the elements $s = (c, 1)$ and $t = (d, 1)$ satisfy the required condition. The other possibilities are that a, c, x and y are distinct and (a, x, y) or a, c, x and y are distinct and (a, y, x). In either of these cases $s = (c, 0)$ and $t = (d, 1)$ will do.

In addition to the preceding discussion of cyclically ordered sets, we are led to consider cyclically ordered groups as locally totally ordered groups by the following example. □

Example 1.1. Let C denote the unit circle group in the complex plane with cyclical ordering induced by counterclockwise orientation. Let $T = C \times C$ with componentwise multiplication. (See figure 1.1.) C is clearly a cyclically ordered group, but T is not cyclically ordered by the product of the cyclical order of C with itself. Worse yet, the restriction of this product to the factor C is not even the original cyclicall order of C, but is in fact, the empty relation.

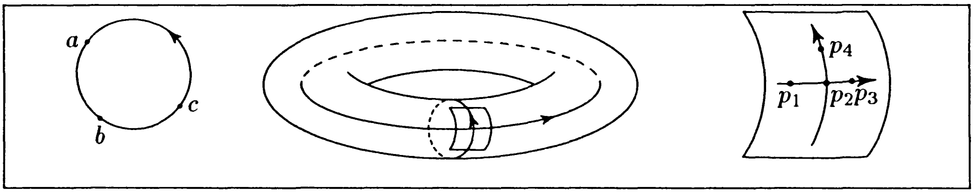

Figure 1.1

There is, however, a natural orientation of T, which is the product of the counterclockwise orientation with itself. If we consider the total orders induced on the proper arcs of C, rather than the entire cyclical order, then the products of these ordered arcs are lattice ordered sets and the restriction of these lattice orders to the original arcs coincides with the original orders induced by the orientation.

LOCALLY PARTIALLY ORDERED SETS

Before investigating locally partially ordered groups we define and explore properties of locally partially ordered sets.

Jimmie D. Lawson [L] defined a locally partially ordered manifold as a C^{∞} manifold with a collection of partially ordered open sets, $\{U_x \mid x \in M\}$, such that if $z \in U_x \cap U_y$, then the restrictions of $<_x$ and $<_y$ agree on some neighborhood of z. We define a locally partially ordered set in a similar fashion.

Definition 2.1. A locally partially ordered set, or lo.p.o. set, is a set, S, together with a collection, A, of partially ordered subsets, $(C, \leqslant_C)$, of S shich satisfies the conditions: (i) A covers S and (ii) if D, $E \in A$ and $p \in D \cap E$, there exists C in A with $p \in C \subset D \cap E$ such that the restrictions of $\leqslant_D$ and $\leqslant_E$ agree on C, i.e. the members of A are pairwise compatible. The collection, A, will be called an atlas on S and an element, C, of A will be called a chart.

If (S, A) is a lo.p.o. set, then the collection $T_A = \{U \subseteq S \mid U$ is a union of charts in $A\}$ is a topology on S with basis A. By an open subset of S, we mean a subset which is open in T_A.

Note that the unit circle, C, with atlas the collection of proper open arcs of C together with the order induced by counterclockwise orientation is a lo.p.o. set, each chart of which is totally ordered. Furthermore, the torus, $C \times C$, with atlas consisting of products of charts described above ordered by $(a, b) < (c, d)$ if and only if $a < c$ and $b < d$ is a lo.p.o. set each chart of which is a lattice.

Notation. Suppose that S is a p.o. set and $a, b \in S$ with $a \neq b$. Then we will use common interval notation as follows:

$$(a, b)_S = \{x \mid a <_S x <_S b\},$$
$$[a, b)_S = \{x \mid a \leqslant_S x <_S b\},$$

etc. Also, let $\bar{S} = S \cup \{\infty\}$ and $\underline{S} = S \cup \{-\infty\}$.

Observe that totally ordered sets and cyclically ordered sets are lo.p.o. sets with atlases defined as follows. For a totally ordered set, S, we define an atlas on S, which we call the standard atlas on S, as

$$\{((a, b)_S, \leqslant_S) \mid a \in \underline{S}, b \in \bar{S}, \text{ and } a \neq b\}.$$

For a cyclically ordered set, S, we define the standard atlas, A, consisting of all sets $C_{a,b}$ defined as follows:

$$C_{a,b} = \begin{cases} \{z \in S \mid (a,z,b)\} & \text{if } a \neq b, \\ \{z \in S \mid z \neq a\} & \text{if } a = b, \end{cases}$$

ordered by $x < y$ on $C_{a,b}$ if and only if (a,x,y).

An example of a lo.p.o. set with lattice ordered charts is a product of lo.p.o. sets with totally ordered charts. Suppose that $\{(S_\lambda, A_\lambda) \mid \lambda \in \Lambda\}$ is such a collection of lo.p.o. sets, and let $S = \prod_{\lambda \in \Lambda} S_\lambda$ and $A = \{(C, \leqslant_C) \mid C = \prod_{\lambda \in \Lambda} C_\lambda,\ C_\lambda \in A_\lambda\, \forall \lambda \in \Lambda\}$ with $\langle x_\lambda \rangle \leqslant_C \langle y_\lambda \rangle$ if and only if $x_\lambda \leqslant_\lambda y_\lambda\, \forall \lambda \in \Lambda$ and $\leqslant_\lambda$ is the partial ordered associated with C_λ, then (S, A) is a lo.p.o. set each chart of which is a lattice.

To define equivalent atlases and the completion of an atlas on a set, we need two preliminary definitions.

Definition 2.2. If (S, A) is a lo.p.o. set and if $(a, b) \in S \times S$, we define a string from a to b in A as a finite collection $\{(a_i, C_i)\}_{i=1}^n$ from $S \times A$ such that (i) $a_1 = a$ and $a_n = b$; (ii) $a_i, a_{i+1} \in C_i$ and $a_i <_i a_{i+1}$ for $1 \leqslant i \leqslant n-1$ where $\leqslant_i$ the partial order associated with C_i; and (iii) $[a_i, a_{i+1})_i$ and $[a_j, a_{j+1})_j$ are disjoint for $i \neq j$ and $1 \leqslant i, j \leqslant n-1$. A loop is a string from an element to itself.

Definition 2.3. If (S, A) is a lo.p.o. set and $(B, \leqslant_B)$ is a partially ordered open subset of S, then B is compatible with A if B satisfies:

(i) $x \in B$ implies there exists C in A such that $x \in C \subseteq B$, and $\leqslant_C\ =\ \leqslant_{B|C}$;
(ii) $x, y \in B$ and $x <_B y$ imply that there exists a string

$$\{(a_i, C_i)\}_{i=1}^n \text{ from } x \text{ to } y \text{ in } A \text{ s.t. } C_i \subseteq B,\ 1 \leqslant i \leqslant n.$$

That is, for compatible subsets of S, being connected is consistent.

Definition 2.4. Let A and B be atlases on a set S. Then A is equivalent to B, denoted A Eq B, if and only if $T_A = T_B$, every chart of A is compatible with B, and every chart of B is compatible with A.

Definition 2.5. The completion of an atlas A on S, denoted $C(A)$, is $\{(B, \leqslant_B) \mid B \subseteq S \text{ and } B \text{ is compatible with } A\}$.

Theorem 2.1. *Equivalence of atlases on a given set S is an equivalence relation.*

The proof of this theorem is routine. There are two useful corollaries.

Corollary 2.2. *If A and B are atlases on a set S, and $(C, \leqslant_C)$ is compatible with A, and A Eq B then $(C, \leqslant_C)$ is also compatible with B.*

Corollary 2.3. *If A and B are atlases on a set S and $T_A = T_B$ then: $A \operatorname{Eq} C(A)$; $C(C(A)) = C(A)$; $A \subseteq B$ implies that $C(A) \subseteq C(B)$; and $A \operatorname{Eq} B$ if and only if $C(A) = C(B)$.*

Theorem 2.4. *If (S, A) is a lo.p.o. set, then $C(A)$ is an atlas on S and $C(A)$ is the unique maximal atlas containing A which is equivalent to A.*

Proof. Since $A \subseteq C(A)$, $C(A)$ covers S. If B, $C \in C(A)$ and $x \in B \cap C$, then there are charts D and E in A such that $x \in D \subseteq B$ with $\leqslant_D = \leqslant_{B|D}$ and $x \in E \subseteq C$ with $\leqslant_E = \leqslant_{C|E}$. Since $x \in D \cap E$ there is G in A such that $x \in G \subseteq D \cap E \subseteq B \cap C$ and $\leqslant_G$ agrees with $\leqslant_D$ and $\leqslant_E$ so $\leqslant_G$ agrees with $\leqslant_B$ and $\leqslant_C$. Thus $C(A)$ is an atlas on S. If B is another atlas on S such that $A \subseteq B$ and if A Eq B then by Corollary 2.3, $C(A) = C(B)$ and so $B \subseteq C(A)$. Thus $C(A)$ is maximal with respect to containing A and being equivalent to A. □

We will characterize totally ordered sets and cyclically ordered sets as lo.p.o. sets. To do so, we need a few more definitions.

Definition 2.6. A lo.p.o. set, (S, A) is connected if whenever $s, t \in S$, $s \neq t$, there exists a string from s to t or there exists a string from t to s.

Definition 2.7. A lo.p.o. set, (S, A) is simply connected if (S, A) is connected and contains no loop.

Definition 2.8. If s, t are distinct element of lo.p.o. set (S, A) and if there is a string $\{(a_i, C_i)\}_{i=1}^n$ from s to t that satisfies the condition that for every string $\{(b_j, D_j)\}_{j=1}^n$ from s to t, $\bigcup_{i=1}^{n-1} [a_i, a_{i+1}]_i = \bigcup_{j=1}^{m-1} [b_j, b_{j+1}]_j$, then we will call the set $\bigcup_{i=1}^{n-1} [a_i, a_{i+1}]_i$ the path from s to t.

Notation. If the path from s to t exists we will denote it by $\overline{st}$. In this case we will also use the following notation:

$$\overset{\circ\!-\!\circ}{st} = \overline{st} - \{s, t\},$$
$$\vec{s} = \{y \in S \mid y \in \overline{sv} \text{ for some } v \in S\},$$
$$\overset{\circ\!\to}{s} = \vec{s} - \{s\},$$
$$\overleftarrow{s} = \{y \in S \mid y \in \overline{vs} \text{ for some } v \in S\} \text{ and}$$
$$\overset{\leftarrow\!\circ}{s} = \overleftarrow{s} - \{s\}.$$

Definition 2.9. If (S, A) is a lo.p.o. set then (S, A) is totally connected if for each pair s, t of distinct elements of S, $\overline{st}$ exists.

Totally ordered sets correspond to simply connected lo.p.o. sets that satisfy five conditions as given in the following theorem.

Theorem 2.5. *Suppose that (S, A) is a lo.p.o. set that satisfies the following five conditions:*

(i) *(S, A) is simply connected;*

(ii) *every chart in A is totally ordered;*

(iii) *$a, b \in S$, $a \neq b$ implies that either $\overline{ab}$ or $\overline{ba}$ exists;*

(iv) *if C is a chart in A and $a, b \in S$ then if $\overset{\circ\!-\!\circ}{ab}$, resp. $\overset{\circ\!\rightarrow}{a}$, resp. $\overset{\leftarrow\!\circ}{a}$ is a subset of C then $(\overset{\circ\!-\!\circ}{ab}, \leqslant_C)$, resp. $(\overset{\circ\!\rightarrow}{a}, \leqslant_C)$, resp. $(\overset{\leftarrow\!\circ}{a}, \leqslant_C)$ is a chart in A;*

(v) *$x \in C \in A$ implies there exists $a, b \in S$ with $x \in \overset{\circ\!-\!\circ}{ab} \subseteq C$, or $a \in S$ with $x \in \overset{\circ\!\rightarrow}{a} \subseteq C$, or $a \in S$ with $x \in \overset{\leftarrow\!\circ}{a} \subseteq C$.*

Then there exists a unique total order $\leqslant$ on S such that the standard atlas on $(S, \leqslant)$ is equivalent to A. Conversely, if $(S, \leqslant)$ is a totally ordered set with standard atlas A, then (S, A) is a lo.p.o. set satisfying conditions (i) *through* (v) *above.*

Proof. Define $<$ on S by $a < b$ if and only if $\overline{ab}$ exists. Then $\leqslant$ is antisymmetric and transitive since (S, A) is simply connected. Condition (ii) guarantees that for all a, $b \in S$, precisely one of $a = b$, $a < b$, or $b < a$ holds. Condition (iv) yields that A is equivalent to the standard atlas on $(S, \leqslant)$. Uniqueness follows from the fact that if two atlases on S are equivalent and a, $b \in S$, there is a string from a to b in each atlas. The proof of the converse is routine observing that $\overset{\circ\!-\!\circ}{ab} = (a, b)_S$, $\overset{\circ\!\rightarrow}{a} = (a, \infty)_S$ and $\overset{\leftarrow\!\circ}{a} = (-\infty, a)_S$.

Cyclically ordered sets correspond to totally connected lo.p.o. sets that satisfy similar conditions to those of theorem 2.5. □

Theorem 2.6. *Suppose that (S, A) is a lo.p.o. set that satisfies the following five conditions:*

(i) *(S, A) is totally connected;*

(ii) *every chart in A is a totally ordered set and is strictly contained in S;*

(iii) *if a and b are distinct elements of S, then $\overline{ab} \cup \overline{ba} = S$ and $\overline{ab} \cap \overline{ba} = \{a, b\}$;*

(iv) *C is a chart and $\overline{ab} \subseteq C$ imply $(\overset{\circ\!-\!\circ}{ab}, \leqslant_{C|\overset{\circ\!-\!\circ}{ab}})$ is also a chart;*

(v) *$x \in C \in A$ implies that there exists $a, b \in S$ with $x \in \overset{\circ\!-\!\circ}{ab} \subseteq C$ or $a \in S$ with $\overset{\circ\!\rightarrow}{a} = C$.*

Then there exists a unique cyclical order R on S such that the standard atlas on S induced by R is equivalent to A. Conversely, if S is a cyclically ordered set with *standard atlas A, then (S, A) is a lo.p.o. set satisfying conditions* (i) *through* (v) *above.*

Proof. Define a cyclical order on S by (a, b, c) if and only if a, b, and c are distinct elements of S and $a \in \overline{cb}$. Then by condition (ii) we know that for distinct elements a, b, and c of S precisely one of (a, b, c) and (a, c, b) holds and that (a, b, c) implies (b, c, a). Conditions (i) and (ii) together with being totally connected guarantee that if (a, b, c) and (a, c, d) then (a, b, d) and that the standard atlas on S induced by this order is equivalent to A. Uniqueness follows from condition (iii) and transitivity of the equivalence of atlases. The proof of the converse follows from the observation that $\overset{\circ\!-\!\circ}{ab} = C_{a,b}$ and $\overset{\circ\!\rightarrow}{a} = C_{a,a}$. □

3. Structure preserving maps

In this section, we will define mappings that preserve the structure of lo.p.o. sets and then define isomorphic lo.p.o. sets.

Definition 3.1. Suppose that (S, A) and (R, B) are lo.p.o. sets and that φ: $S \longrightarrow R$. Then, φ is *structure preserving* if φ satisfies the conditions:

(i) if $s\varphi \in D \in B$ then there exists C in A such that $s \in C$, $C\varphi \subseteq D$, and $a \leqslant_C b$ implies $a\varphi \leqslant_D b\varphi$,

(ii) a, $b \in C \in A$ and $a \leqslant_C b$, implies there is a string, $\{(b_j, D_j)\}_{j=1}^{n}$ from $a\varphi$ to $b\varphi$ in (R, B) such that $D_j \subseteq A\varphi$; $1 \leqslant j \leqslant n$

and φ is *structure reversing* if φ satisfies the conditions:

(i) if $s\varphi \in D \in B$ then there exists C in A such that $s \in C$, $C\varphi \subseteq D$, and $a \leqslant_C b$ implies $b\varphi \leqslant_D a\varphi$,

(ii) a, $b \in C \in A$ and $a \leqslant_C b$, implies there is a string, $\{(b_j, D_j)\}_{j=1}^{n}$ from $b\varphi$ to $a\varphi$ in (R, B) such that $D_j \subseteq A\varphi$; $1 \leqslant j \leqslant n$.

A straightforward argument verifies that the composition of structure preserving maps in structure preserving.

If a lo.p.o. set consists of a cyclically ordered set, C, together with its standard atlas, then structure preserving permutations of that lo.p.o. set coincide with permutations of C that preserve its cyclical order. Similarly, structure preserving permutations of a lo.p.o. set consisting of a totally ordered set together with its standard atlas coincide with permutations of T that preserve its total order.

If $(S, \leqslant_S)$ and $(R, \leqslant_R)$ are totally ordered sets and A and B are the standard atlases on S and R, respectively, then a mapping, f, from S to R which is structure preserving is order preserving. The converse does not hold as in demonstrated by the following example.

Example 3.1. Let Q denote the set of rational numbers, $S = \big((-\infty, 1) \cup \{2\} \cup (3, \infty)\big) \cap Q$, and $i\colon S \to Q$ be the injection map. Then i is clearly order preserving, but i is not structure preserving since $2i = 2 \in \left(\frac{3}{2}, \frac{5}{2}\right)$ but there is no open interval in S about 2 which maps into $(\frac{3}{2}, \frac{5}{2})$.

Definition 3.2. Suppose that (S, A) and (R, B) are lo.p.o. sets and that φ is a one to one mapping of S onto R. We will say that φ is an *sp*-isomorphism, and that (S, A) and (R, B) are *sp*-isomorphic, if and only if both φ and φ^{-1} are structure preserving.

Note that two atlases, A and B, on a set, S are equivalent if and only if the lo.p.o. sets (S, A) and (S, B) are *sp*-isomorphic, since A and B are equivalent if and only if the identity map is both structure preserving from (S, A) to (S, B) and from (S, B) to (S, A).

We conclude this section with two examples.

Example 3.2. Let C denote the unit circle in the complex plane with counter-clockwise orientation and let A denote the standard atlas on C. Let B be the atlas on C consisting of the union of the collection of all proper open arcs of C with the collection of disjoint unions of proper open arcs of C. Then (S, A) is *sp*-isomorphic to (S, B) since the union of disjoint proper open arcs of C is contained in some proper open arc of C.

Example 3.3. Suppose $\mathbf{R}$ is the set of real numbers and $S = \mathbf{R} \setminus (0, 1)$. Let $\leqslant$ denote the usual ordering on S and define $\preceq$ by $a \preceq b$ if and only if (i) $a = b$ or (ii) $a < b \leqslant 0$ or (iii) $1 \leqslant a < b$ or (iv) $b \leqslant 0$ and $a \geqslant 1$. Let A and B be the standard atlases on $(S, \leqslant)$ and $(S, \preceq)$, respectively. Then $(S, \leqslant)$ and $(S, \preceq)$, respectively. Then $(S, \leqslant)$ and $(S, \preceq)$ are not *sp*-isomorphic since S is an element of A and $0 \leqslant_S 1$ but there is no string in B from 0 to 1.

4. Locally partially ordered groups

The concepts of locally partially ordered sets and structure preserving maps are now used to define locally partially ordered groups.

Definition 4.1. Suppose that G is a group and that A is an atlas on the set G. Then (G, A) is a locally partially ordered group, or. lo.p.o. group, if and only if (G, A) satisfies the following conditions:

(i) the product mapping $(x, y) \to yx$ from $G \times G$ to G is continuous in T_A;
(ii) the inverse mapping $x \to x^{-1}$ from G to G is structure reversing;
(iii) left and right translations are structure preserving.

Example 4.1. If G is any p.o. group and B is the trivial atlas, i.e. $B = \{(G, \leqslant)\}$, then (G, B) is a lo.p.o. group.

Example 4.2. Suppose that G is an o-group and that A is the standard atlas on the totally ordered set G. Then (G, A) is a lo.p.o. group.

Example 4.3. Suppose that G is a cyclically ordered group and that A is the standard atlas on the cyclically ordered set G. Then (G, A) is a lo.p.o. group.

Example 4.4. Suppose that for each λ in Λ, (G_λ, A_λ) is a lo.p.o. group. Let $G = \prod_{\lambda \in \Lambda} G_\lambda$ and let $A = \prod_{\lambda \in \Lambda} A_\lambda$ with $\leqslant_C$ defined for $C = \prod_{\lambda \in \Lambda} C_\lambda$ in A by $(a_\lambda) \leqslant_C (b_\lambda)$ if and only if $a_\lambda <_\lambda b_\lambda$ (where $<_\lambda$ denotes the partial order on C_λ) for all λ in Λ. Then (G, A) is a lo.p.o. group.

Given a lo.p.o. group (G, A) and a chart C in A, we would like to know that C^{-1} is also in A and, that $a \leqslant_C b$ if and only if $b^{-1} \leqslant_{C^{-1}} a^{-1}$. Theorem 4.1 establishes that inverses of charts may be included in an atlas.

Theorem 4.1. *Suppose that (G, A) is a lo.p.o. group. For each $C \in A$, define a partial order on C^{-1} by $a \leqslant_{C^{-1}} b$ if and only if $b^{-1} \leqslant_C a^{-1}$. Let $B = A \cup \{(C^{-1}, \leqslant_{C^{-1}}) \mid C \in A\}$. Then B is an atlas on G, B Eq A and (G, B) is a lo.p.o. group.*

Proof. B is an atlas on the set G since A is an atlas on the set G and the mapping $x \to x^{-1}$ is structure reversing. Clearly, $T_A = T_B$. To show B Eq A, it suffices to show that if D is in $B \setminus A$ then D is compatible with A. If $x \in D \in B \setminus A$, then $D = C^{-1}$ for some C in A and, since the inverse map is structure reversing, there exists E in A such that $x \in E$ and $E^{-1} \subseteq C$. Since D is open in T_A, there exist F in A such that $x \in F \cap D$. Now, $x \in E \cap F$ so there is a chart, G, in A such that $x \in G \subseteq E \cap F$ and $\leqslant_G \ = \ \leqslant_{E|G}$. But then $<_D \ = \ <_{D|G}$ since $a <_G b$ implies $a <_E b$ which implies $b^{-1} <_C a^{-1}$ and this implies $a <_D b$. Thus, condition (i) is satisfied. Condition (ii) follows directly from the fact that the inverse map is structure reversing.

Finally, we must show that (G, B) is a lo.p.o. group. Since $A \subseteq B$ and A Eq B, the mapping $(x, y) \to xy$ is continuous in T_B. The mapping $x \to x^{-1}$ is structure reversing and the mappings $x \to xg$ and $x \to gx$ are structure preserving since if $D \in B$ then either D or D^{-1} is in A and (G, A) is a lo.p.o. group.

Henceforth we will assume that the atlas of a lo.p.o. group contains the inverse of each of its charts. The fact that for a lo.p.o. group, translations are structure preserving yields their following theorem. □

Theorem 4.2. *Suppose that (G, A) is a lo.p.o. group and that $(N, g) \in A \times G$. Define a partial order, $\leqslant_{Ng}$ on Ng by $n_1 g \leqslant_{Ng} n_2 g$ if and only if $n_1 \leqslant_N n_2$. Similarly define $\leqslant_{gN}$ on gN. Let $B = \{(Ng, \leqslant_{Ng}) \mid (N, g) \in A \times G\} \cup \{(gN, \leqslant_{gN}) \mid (N, g) \in A \times G\}$. Then (G, B) is a lo.p.o. group and B Eq A.*

Note that if N_A and B are defined by $N_A = \{C \in A \mid e \in C\}$, and $B = \{(Ng, \leqslant_{Ng}) \mid (N, g) \in N_A \times G\} \cup (gN, \leqslant_{gN}) \mid (N, g) \in N_A \times G$ then $B = A$.

Definition 4.2. Suppose that (G, A) is a lo.p.o. group. Then $N_A = \{C \in A \mid e \in C\}$ will be called the family of nuclei of (G, A) and each member of N_A will be called a nucleus.

The following theorem provides necessary and sufficient conditions for a collection of subsets of a group G, each of which contains the identity, e, of G, to be a family of nuclei for some atlas A on G for which (G, A) is a lo.p.o. group. For a similar result which is fundamental to the theory of topological groups the reader is referred to Husain [Hu] and Cohn [C].

Theorem 4.3. *Suppose that (G, A) is a lo.p.o. group and that N_A is the family of nuclei of (G, A). Then N_A satisfies the following conditions:*

(i) B, $D \in N_A$ implies that there exists C in N_A such that the restrictions of $\leqslant_B$ and $\leqslant_D$ agree on C;

(ii) $C \in N_A$ implies $C^{-1} \in N_A$ and $c_1 \leqslant_C c_2$ if and only if $c_2^{-1} \leqslant_{C^{-1}} c_1^{-1}$;

(iii) $C \in N_A$ implies there exists B in N_A such that $BB^{-1} \subseteq C$ and $\leqslant_B$ = the restriction of $\leqslant_C$ to B;

(iv) $C \in N_A$ and $g \in G$ imply that there exists $B \in N_A$ such that $gBg^{-1} \subseteq C$ and $b_1 \leqslant_B b_2$ iff $gb_1g^{-1} \leqslant_C gb_2g^{-1}$;

(v) $x \in C \in N_A$ implies that there exists B, $D \in N_A$ such that $Bx \subseteq C$, $xD \subseteq C$; $b_1 \leqslant_B b_2$ iff $b_1 x \leqslant_C b_2 x$; and $d_1 \leqslant_D d_2$ iff $xd_1 \leqslant_C xd_2$.

Conversely, if G is a group and N is a non-empty collection of partially ordered subsets of G, each of which contains the identity, e, and if N satisfies conditions (i) through (v) above, then $B = \big\{(xNy, \leqslant_{xNy}) \mid x, y \in G \text{ and } N \in N\big\}$ is an atlas on G and (G, B) is a lo.p.o. group.

P r o o f. It is clear that for a lo.p.o. group, (G, A), N_A satisfies conditions (i) through (v) of the theorem. For the converse, note that since N is non-empty, B convers G. The fact that for N and M in N and x, y, z, and w in G the charts xNy and zMw are compatible follows from conditions (i), (iv) and (v).

Continuity of the product mapping, condition (i) of the definition of structure reversing for the inverse mapping, and condition (i) of the definition of structure preserving for left and right translations follow from conditions (ii) through (v). To complete the proof that the inverse mapping is structure reversing, observe that if

$x \leqslant_{gNh} y$, then the string $\{(y^{-1}, h^{-1}N^{-1}g^{-1}), (x^{-1}, h^{-1}N^{-1}g^{-1})\}$ from y^{-1} to x^{-1} is of the required type. Similarly, to complete the proof that translations are structure preserving, note that if $x <_{zNw} y$, then the strings $\{(xg, zNwg), (yg, zNwg)\}$ and $\{(gx, gzNw), (gy, gzNw)\}$ are of the required type. $\square$

We close this section with an example of a lo.p.o. group and a family of nuclei for that group.

Example 4.5. Let C be the unit circle in the complex plane together with the cyclical order induced by counterclockwise orientation. Denote by $d(a,b)$ the minimum arc length between the points a and b on C; denote by r_σ the counterclockwise rotation satisfying $d(x, xr_\sigma) = \sigma$ for all x in C (σ a real number, $0 < \sigma \leqslant \frac{1}{2}$); and denote by x' the point of $xr_{1/2}$. Let G be the group of all permutations of C which preserve the cyclical order. For each x in C and for each positive real number ε, let $B(x,\varepsilon) = \{y \in C \mid d(x,y) < \varepsilon\}$ and let $N_\varepsilon = \{g \in G \mid \forall x, xg \in B(x,\varepsilon)\}$. Partially order N_ε by $f <_\varepsilon g$ if for all $x \in C$ and for all $w \in C$ with $w \notin B(x,\varepsilon)$, (w, xf, xg). Then, if we choose f and g in an N_ε with ε small enough that $B(x, 2\varepsilon)$ and $B(x', 2\varepsilon)$ do not intersect, say $\varepsilon = \frac{1}{12}$, the collection $\{N_\varepsilon \mid 0 < \varepsilon < \frac{1}{12}\}$ is a family of nuclei for an atlas A on G, (G, A) is a lo.p.o. group and each chart of A is a lattice.

5. On Rieger's theorem

In this final section, we characterize ordered groups and cyclically ordered groups as lo.p.o. groups and restate Rieger's theorem in terms of lo.p.o. groups.

In section 2, necessary and sufficient conditions for a lo.p.o. set to be equivalent to an ordered set with standard atlas, or to a cyclically ordered set with standard atlas, were given. In the following two theorem, these results are extended to ordered and cyclically ordered groups.

Theorem 5.1. *Suppose that G is an o-group and that A is the standard atlas on G. Then (G, A) is a lo.p.o. group and satisfies conditions* (i) *through* (v) *of Theorem* 2.5. *Conversely, if (G, A) is a lo.p.o. group that satisfies conditions* (i) *through* (v) *of Theorem* 2.5, *then there is a unique total order $\leqslant$ on G such that $(G, \leqslant)$ is an o-group whose standard atlas is equivalent to A.*

Proof. It suffices to show that for a lo.p.o. group satisfying conditions (i) through (v) of Theorem 2.5, the total order $\leqslant$ defined in the proof of Theorem 2.5 is compatible with the group operations. Suppose that $a < b$ and $g \in G$. Then there is a string from a to b, say $\{(a_i, C_i)\}_{i=1}^{n}$ such that $a_i <_i a_{i+1}$ for $1 \leqslant i \leqslant n-1$. Now, since $x \to xg$ is structure preserving, for each i between 1 and $n-1$, there is a string from $a_i g$ to $a_{i+1} g$. The union of these strings forms a string from ag to bg and so $ag < bg$. Similarly $bg < ag$.

A similar argument verifies the following theorem. □

Theorem 5.2. *Suppose that G is a cyclically ordered group and that A is the standard atlas on G. Then (G, A) is a lo.p.o. group and satisfies conditions* (i) *through* (v) *of Theorem* 2.6. *Conversely, if (G, A) is a lo.p.o. group that satisfies conditions* (i) *through* (v) *of Theorem* 2.6, *then there is a unique cyclical order* $(\ , \ , \ )$ *on G such that $(G(\ , \ ,))$ is a cyclically ordered group whose standard atlas is equivalent to A.*

Theorem 5.3 is a restatement of Rieger's theorem in terms of lo.p.o. groups. We use the term "o-group conditions" to refer to the five conditions of Theorem 2.5 while the term "cyclically ordered group conditions" refers to the five conditions of Theorem 2.6.

Theorem 5.3. (Rieger). *Suppose that (G, A) is a lo.p.o. group which satisfies the o-group conditions and there is an element z in the center of G such that for each $g \in G$, $g \in \overline{z^n z^{n+1}}$ for some integer n. Let $\pi : G \to g/\langle z \rangle$ be the natural mapping and let $B = \{(D\pi, \leqslant_{D\pi}) \mid D \in A, D \subseteq \overset{\circ\!-\!\circ}{ez},$ and $d_1\pi <_{D\pi} d_2\pi$ if and only if $d_1 \in \overset{\circ\!-\!\circ}{ed_2}\}$. Let $C = \{hEg \mid E \in B; g, h \in G/\langle z \rangle$; and $he_1 g < he_2 g$ if and only if $e_1 <_E e_2\}$. Then C is an atlas on $G/\langle z \rangle$, and $(G/\langle z \rangle, C)$ is a lo.p.o. group which satisfies the cyclically ordered group conditions.*

Conversely, if (G, A) is a lo.p.o. group that satisfies the cyclically ordered group conditions, then there is a lo.p.o. group (K, B) such that (K, B) satisfies the o-group conditions, K contains an element z in its center such that $K = \cup \overline{z^n z^{n+1}}$ and if C is the atlas on $K/\langle z \rangle$ derived above, then $(K/\langle z \rangle, C)$ Eq (G, A).

Before proving a partial generalization of Rieger's theorem for lo.p.o. groups we extend the definition of a covering map to lo.p.o. sets.

Definition 5.1. If (S, A) and (R, B) are lo.p.o. sets, a mapping $\varphi : S \to R$ will be called a lo.p.o. set epimorphism, or lopo-epimorphism, if whenever $s, t \in S$; $s\varphi$, $t\varphi \in D \in B$; and $s\varphi <_D t\varphi$, there exist $c, d \in S$ such that $c\varphi = s\varphi$, $d\varphi = t\varphi$, and there is a string $\{(c_i, C_i)\}_{i=1}^n$ from $c\varphi$ to $d\varphi$ such that $C_i\varphi \subseteq D$ for $1 \leqslant i \leqslant n$.

Definition 5.2. If (S, A) and (R, B) are lo.p.o. sets then a lopo-epimorphism φ: $S \to R$ will be called a covering map if for each $s \in S$ there exists $C \in A$ and $D \in B$ such that $s \in C$, $C\varphi \subseteq D$ and $\varphi_{|C} : C \to D$ is one and order preserving.

Our last theorem is a partial generalization of Rieger's theorem. By a topological group, we mean a group G with a topology on G (not necessarily Hausdorff) such that the map $(x, y) \to xy^{-1}$ from $G \times G$ into G is continuous. We prove that if G is a topological p.o. group and N a discrete normal supgroup, then G/N is a lo.p.o. group.

Theorem 5.4. *Suppose that G is a topological group and a p.o. group, N is normal in G, $\pi: G \to G/N$ denotes the natural map and G has an open subset M such that $e \in M$, $M = M^{-1}$, and $N \cap M = \{e\}$. Then $N = \{(C\pi, \leqslant_{C\pi}) \mid e \in C \subseteq M;$ C is open; and $c_1\pi \leqslant_{C\pi} c_2\pi$ if and only if $c_1, c_2 \in C$ and $c_1 \leqslant c_2\}$ is a family of nuclei for an atlas A on G/N, and $(G/N, A)$ is a lo.p.o. group. Furthermore, $B = \{(g_1 M g_2, \leqslant) \mid g_1, g_2 \in G\}$ is an atlas on G and π is a covering map from (G, B) to $(G/N, A)$.*

Proof. We first show that N satisfies the axioms for a family of nuclei of a lo.p.o. group, i.e. N satisfies the conditions of Theorem 4.3. For axiom (i), suppose that $C\pi$, $D\pi \in N$. Then $e \in C \cap D$, $C \cap D \subseteq M$, and $C \cap D$ is open. Since π is one to one on M, $(C \cap D)\pi = C\pi \cap D\pi$ and $\leqslant_{C\pi|(C\cap D)\pi} = \leqslant_{D\pi|(C\cap D)\pi}$. Thus, $(C \cap D)\pi \in N$ and axiom (i) is satisfied. To see that the inverse of a chart in N is in N, we note that (i) $C\pi \in N$ implies $C \subseteq M$ and thus $C^{-1} \subseteq M$; (ii) $e \in C$ implies $e \in C^{-1}$; (iii) C is open implies C^{-1} is open; and (iv) the orders of $C\pi$ and $C^{-1}\pi$ agree with those of C and C^{-1}.

For axiom (iii), observe that $C\pi \in N$ implies there exist open sets E, F containing e with $EF \subseteq C$ so if $H = E \cap F^{-1}$ then $H\pi \in N$. For axiom (iv), note that $C\pi \in N$ and $g\pi \in G/N$ imply that there exists an open set D in G such that $e \in D \subseteq g^{-1}Cg$ so if $H = D \cap M$, then $H\pi \in N$. Finally, if $c \in C$ and $C\pi \in N$, then, since $(x, y) \to xy^{-1}$ is continuous, there exists an open set D containing e with $cD \subseteq C$; $(cD)\pi = c\pi D\pi \subseteq C\pi$; and $\leqslant_{D\pi}$ agrees with with $\leqslant_{C\pi}$.

We have shown that N is a family of nuclei for a lo.p.o. group. The proof that B is an atlas on G is routine. It remains to be seen that π is a covering map. So suppose that $c\pi <_{C\pi} d\pi$ for some $C\pi$ in N. Then $c\pi = c_0\pi$ and $d\pi = d_0\pi$ for some c_0, d_0 in C with $c_0 < d_0$ in G, and since this property is preserved by translation, π is a lopo-epimorphism. Finally, if $g \in G$, then $g \in Mg \in B$, and $(Mg)\pi = M\pi g\pi \in A$ so $\pi|Mg$ is order preserving. Since $\pi|M$ is one to one, $\pi|Mg$ is one to one and π is, in fact, a covering map. □

References

[B] *Blair (Roll), J.D.*: A generalization of the concept of cyclically ordered groups. Ph.D. Thesis, Bowling Green State University, 1976.

[C] *Cohn, P.M.*: Lie Groups, The University Press, Cambridge, 1967.

[F] *Fuchs, L.*: Partially Ordered Algebraic Systems, Pergamon Press, New York, 1983.

[H] *Holland, C.*: The lattice-ordered group of automorphisms of an ordered set, Mich. Math. J. *10* (1963), 399–408.

[Hu] *Husain, T.*: Introduction to Topological Groups, W.B. Saunders, Philadelphia, 1966.

[L] *Lawson, J.D.*: Ordered manifolds, invariant cone fields, and semigroups. Forum Math..

[M] *Megiddo, N.*: Partial and complete cyclic orders, Bull. Amer. Math. Soc *82* (1976), 274–276.

[N1] *Novák, V.*: Cyclically ordered sets., Czech. Math. Jour. *32(107)* (1982), 460–473.

[N2] *Novák, V.*: Cuts in cyclically ordered sets., Czech. Math. Jour. *34(109)* (1984), 322–333.

[R] *Rieger, L.S.*: On the ordered and cyclically-ordered groups I–III, Věstník Král. České Spol. Nauk *6* (1946), 1–31 *1* (1947), 1–33 *1* (1948), 1–26.

[S1] *Swierczkowski, S.*: On cyclically ordered groups, Fund. Math. *47* (1959), 161–166.

[S2] *Swierczkowski, S.*: On cyclically ordered intervals of integers, Fund. Math. *47* (1959), 167–172.

[S3] *Swierczkowski, S.*: On successive settings of arcs on the circumference of a circle, Fund. Math. *46* (1958), 187–189.

Author's address: The University of Findlay, The Division of Mathematics and Computer Science, 1000 North Main Street, Findlay, Ohio 45840-3695, U.S.A.