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CLOSED BOUNDED SETS IN INDUCTIVE LIMITS OF \mathcal{K} -SPACES

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A web W in a vector space F is a countable family of balanced subsets of F , arranged in "layers". The first layer of the web consists of a sequence $(A_p : p = 1, 2, \dots)$ whose union absorbs each point of F . For each set A_p of the first layer there is a sequence $(A_{pq} : q = 1, 2, \dots)$ of sets, called the sequence determined by A_p , such that

$$A_{pq} + A_{pq} \subset A_p \text{ for each } q,$$

$$\bigcup \{A_{pq} : q = 1, 2, \dots\} \text{ absorb each point of } A_p.$$

Further layers are made up in a corresponding way so that each set of the k -th layer is indexed by a finite row of k integers and at each step the above mentioned two conditions are satisfied. Suppose that one chooses a set A_p from the first layer, then a set A_{pq} of the sequence determined by A_p and so on. The resulting sequence $\mathcal{S} = (A_p, A_{pq}, A_{pqr}, \dots)$ is called a strand. Whenever we are dealing with only one strand we can simplify the notation by writing $W_1 = A_p$, $W_2 = A_{pq}$ etc.; thus $\mathcal{S} = (W_k)$ is a strand where for each k , W_k is a set in the k -th layer. We will work only with locally convex spaces and also assume that each member for a web is absolutely convex.

Let $S = (W_n)$ be a strand. Consider $x_n \in W_n$ and the series $\sum_{n=1}^{\infty} x_n$. The space F is webbed if the series $\sum_{n=1}^{\infty} x_n$ is convergent for any choice of x_n in W_n . The standard references for webs in locally convex spaces are [1], [2] and [3].

It is shown in [5] that $E = \text{indlim } E_n$, where all spaces E_n are Fréchet, is regular iff it is locally complete. In [6] T. Gilsdorf substitutes the Fréchet spaces E_n by webbed \mathcal{K} -spaces and proves that E is regular iff it is locally Baire. In this paper the same substitution is used to generalize results by Qui Jing Hui, see [7], on closed bounded sets in inductive limits.

Theorem 1. Let (E, τ) be a webbed locally convex space. Let $B \subset E$ be closed. If B is a Baire disk then B is bounded.

Proof. Denote by η the topology in E_B induced by the system of neighborhoods of zero $\{(1/nB) \cap V : V \in \tau, V \text{ closed, and } n \in \mathbf{N}\}$.

First. Let's prove that (E_B, η) is a webbed space. Let W be a web in E consisting of the sets $A_{pq} \dots r$. We will construct a web in E_B in the following way: the first layer will be $\{A_p \cap B : p = 1, 2, \dots\}$, the second layer $\{A_{pq} \cap (\frac{1}{2}B) : q = 1, 2, \dots\}$ and so on. Take a string (W'_n) in E_B , consider any $x_n \in W'_n$ and the series $\sum_{n=1}^{\infty} x_n$. Note that for each W'_n there exist a W_n such that $W'_n \subset W_n \subset E$ where

$$\begin{aligned} W'_1 &= A_p \cap B \subset A_p = W_1 \subset E \\ W'_2 &= A_{pq} \cap B \subset A_{pq} = W_2 \subset E \end{aligned}$$

and so on.

Since E is webbed for any choice of x_n in W_m the series $\sum_{m=1}^{\infty} x_m$ is convergent in E therefore also for any x_m in W'_m . That means that the sequences $y_m = \sum_{n=1}^m x_n$ converges in E so [Theorem 3.2.4 p.59,8] it converges in (E_B, η) . Then E_B, η is webbed.

Now the map $id: (E_B, \eta) \rightarrow (E_B, p_B)$ is continuous, (E_B, η) is webbed, and (E_B, p_B) is a Baire space. Hence it is also open by [Theorem 3.2, p.59,2]. That means $id(B \cap V)$ is a neighborhood of 0 in (E_B, p_B) . So there exists $\lambda > 0$ such that $\lambda B \subset B \cap V \subset V$, i.e. B is bounded in E . \square

Lemma. Let (E, τ) be a locally convex space. Let $B \subset E$ be closed. If (E, τ) is a \mathcal{K} -space then (E_B, η) is a \mathcal{K} -space.

Proof. Let $x_n \rightarrow 0$ in (E_B, η) . Then $x_n \rightarrow 0$ in (E, τ) because $id: (E_B, \eta) \rightarrow (E, \tau)$ is continuous. Hence there is a subsequence (x_{n_k}) of (x_n) and $x \in E$ such that $\sum_{k=1}^{\infty} x_{n_k} = x$. Let $y_m = \sum_{k=1}^m x_{n_k}$. Then (y_m) is a sequence of elements in E_B , and it converges in (E, τ) hence it also converges in $(E_B, \tau/E_B)$, so [Theorem 3.2.4, p.59,8] y_m converges to x in (E_B, η) . Since B is closed in (E, τ) and (y_m) is a bounded sequence in (E_B, τ) we have for some λ that $x \in \lambda B \subset E_B$. This means (E_B, η) satisfies property \mathcal{K} . \square

Let $E_1 \subset E_2 \subset \dots$ be a sequence of locally convex spaces with all identity maps: $(E_n, \tau_n) \rightarrow (E_{n+1}, \tau_{n+1})$ continuous and $E = \text{indlim } E_n$.

Theorem 2. *Let each space (E_n, τ_n) be a webbed \mathcal{K} -space. Let a set B be an absolutely convex subset of E_n . If B is bounded and closed in some (E_m, τ_m) , where $m > n$, then B is bounded and closed in (E_n, τ_n) .*

Proof. Since B is closed in (E_m, τ_m) and $id: (E_n, \tau_n) \rightarrow (E_m, \tau_m)$ is continuous, B is also closed in (E_n, τ_n) . Now B is bounded and closed in (E_m, τ_m) , where (E_m, τ_m) is webbed and has property \mathcal{K} , so following the proof of Lemma, (E_B, p_B) also has property \mathcal{K} . By [4] any metrizable \mathcal{K} -space is Baire. We conclude that (E_B, p_B) is Baire and B is a Baire disk. By Theorem 1, B is bounded in (E_n, τ_n) . \square

Theorem 3. *Let each space (E_n, τ_n) be a webbed \mathcal{K} -space. Let a set B be absolutely convex, bounded, and closed in $(E, \tau) = \text{indlim}(E_n, \tau_n)$. If (E, τ) is locally Baire and $B \subset E_n$ for some n in \mathbf{N} then B is bounded and closed in (E_n, τ_n) .*

Proof. It follows from Theorem 2 and Theorem 3 in [6]. \square

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