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TOPOLOGICAL RESULTS ON SEQUENCES $\{n_k x\}_{k=1}^{\infty}$ AND THEIR APPLICATIONS IN THE THEORY OF TRIGONOMETRIC SERIES

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0. Introduction

All results of this paper are based on Theorem 1.1 which describes from the topological point of view the behaviour of fractional parts of numbers $n_k x$ $(k = 1, 2, \ldots)$, where $x \in \mathbf{R}$ and $\{n_k\}_{k=1}^{\infty}$ is a given sequence of positive integers. In the second part of the paper we give a new proof of the categorical analogue of the well-known theorem of Cantor and Lebesgue.

Let us recall the usual notation. In what follows under $\mathbf{R} = (-\infty, \infty)$ we understand the metric space of all real numbers endowed with the Euclidean metric. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers then $\{a_n\}_n'$ denotes the set of all *limit* points of the sequence $\{a_n\}_{n=1}^{\infty}$. For $M \subset \mathbf{R}$ and $t \in \mathbf{R}$ the symbol $M + t$ denotes the set $\{x + t : x \in M\}$. The symbol Q stands for the set of all rational numbers. For $t \in \mathbf{R}$ the symbol $[t]$ denotes the integral part of t and $(t) = t - [t]$ the fractional part of t. Further, λ is the Lebesgue measure in $\mathbf{R}$ and $\|x\|$ the distance of the real number x from the nearest integer, i.e. $\|x\| = \min\{(t), 1 - (t)\}$.

1. Sequences $\{(n_k x)\}_{k=1}^{\infty}$

Metric properties of sequences

$$\{(n_k x)\}_{k=1}^{\infty} \qquad (x \in \mathbf{R}); \tag{1}$$

$n_1 < n_2 < \ldots < n_k < \ldots$ being a fixed sequence of natural numbers, have been investigated in several papers (see e.g. [2], [4]). In the following theorem we describe the behaviour of these sequences from the topological point of view.

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Theorem 1.1. *Let $n_1 < n_2 < \ldots < n_k < \ldots$ be a sequence of positive integers. Denote by $W(n_1, n_2, \ldots)$ the set of all $x \in \mathbf{R}$ for which $\{(n_k x)\}'_k = [0,1]$. Then $W(n_1, n_2 \ldots)$ is a residual set in $\mathbf{R}$ which can be expressed as the union of a G_δ-set and a countable set.*

Proof. Denote by D_0 the set of all numbers of the form $\frac{v}{n_j}$, where v is an integer, $0 \leqslant v < n_j$ $(j = 1, 2, \ldots)$. Put

$$D = \bigcup_{h=-\infty}^{\infty} (D_0 + h)$$

(h runs over all integers). Obviously D is a countable set. Put $S = \mathbf{R} \setminus D$. In the sequel S will be regarded as a metric subspace of $\mathbf{R}$. Further, S is evidently a G_δ-set in $\mathbf{R}$.

For $r \in [0,1]$ we put

$$A(r) = \{x \in S \colon r \in \{(n_k x)\}'_k\}.$$

We shall show that $A(r)$ is a dense G_δ-set in S.

By the definition of $A(r)$ we have

$$A(r) = \bigcap_{k=1}^{\infty} \bigcap_{p=1}^{\infty} \bigcup_{j=p}^{\infty} \left\{x \in S \colon |(n_j x) - r| < \frac{1}{k}\right\}. \tag{2}$$

For a fixed j the function $g_j : S \to \mathbf{R}, g_j(x) = (n_j x)$ is continuous on S and so according to (2) the set $A(r)$ is a G_δ-set in S.

It is proved in [4] (see also [2]) that the set of all $x \in \mathbf{R}$ for which $\{(n_k x)\}_{k=1}^{\infty}$ is dense in [0,1] has full measure.

This implies the density of $W(n_1, n_2, \ldots)$ in S. (Note that this fact can be obtained also from Theorem 4.1 of [6] p. 32 according to which the sequence $\{n_k x\}_{k=1}^{\infty}$ is uniformly distributed mod 1 for almost all $x \in \mathbf{R}$.) Hence $A(r)$ is a dense G_δ-set in S and therefore it is a residual set in S (cf. [7], p. 49). But then also the set $A = \bigcap_{r \in Q \cap [0,1]} A(r)$ is residual in S. It can be easily checked that $A = S \cap W(n_1, n_2 \ldots)$ and so A is a residual G_δ-set in S. Since S is a G_δ-set in $\mathbf{R}$, the set A is a G_δ-set in $\mathbf{R}$. In the equality $\mathbf{R} = A \cup (S \setminus A) \cup D$ each of the sets $S \setminus A$, D is a set of the first Baire category. Therefore A is residual in $\mathbf{R}$. The assertion now follows at once from the equality

$$W(n_1, n_2, \ldots) = A \cup [W(n_1, n_2 \ldots) \cap D].$$

□

2. Categorical analogue of the Cantor-Lebesgue theorem

Recall that a set $E \subseteq [0,1]$ is said to be an R-set if there exists a trigonometric series

$$\sum_{n=1}^{\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx) \tag{3}$$

which pointwise converges on E but $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$ does not hold (see [1], p. 721–722). We now introduce the following new notion:

A set $E \subset [0,1]$ is called a Y-set if there exists a trigonometric series (3) such that for each $x \in E$ we have

$$\lim_{n\to\infty}(a_n \cos 2\pi nx + b_n \sin 2\pi nx) = 0$$

but

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

does not hold.

In [3] a set $E \subset [0,1]$ has been called a weak D-set with respect to a sequence $n_1 < n_2 < \dots$ of natural numbers provided $\lim_{k\to\infty} \|n_k x\| = 0$ for each $x \in E$. Weak D-sets have been studied in the theory of trigonometric series as "sets admitting a sequence with limit zero" (cf. [1], p. 732; [3]).

It is easy to see that every R-set and also every weak D-set are Y-sets. According to [5] the converse is not true. The following theorems of Cantor and Lebesgue belong to classical results in the theory of trigonometric series.

Theorem A. *If $E \subset \mathbb{R}$, $\lambda(E) > 0$, and*

$$\lim_{n\to\infty}(a_n \cos nx + b_n \sin nx) = 0$$

for each $x \in E$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

(cf. [10], p. 279).

Theorem A′. *If $E \subset \mathbb{R}$, $\lambda(E) > 0$ and a trigonometric series $\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ (pointwise) converges on E, then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

(cf. [1], *p. 174;* [10], *p. 279).*

Obviously Theorem A′ is an easy consequence of Theorem A. We introduce it for reasons which will become evident when we shall investigate the possibility of conversion of these theorems (see Remark 2.1 in the next part of the paper).

In connection with Theorems A, A′ a question arises whether their categorical analogues are valid, i.e. whether such theorems hold which can be obtained from theorems A, A′ if we replace the assumption $\lambda(E) > 0$ by the assumption that E is a set of the second Baire category. In [1], (p. 736) it is proved that such an analogue is true for Theorem A′.

Theorem A can be formulated as "every Y-set has measure zero" while Theorem A′ states that every R-set has measure zero.

Let us remark that Theorems A, A′ have been generalized in several directions (see e.g. [8]). From the results of [9] the categorical analogoue of Theorem A can be deduced (see Theorem 2.1 below). A proof of Theorem 2.1 was published already in the year 1909 in [11]. Here we give a simple proof of this theorem based on our Theorem 1.1.

Theorem 2.1. *Let $A \subset \mathbb{R}$ be a set of the second Baire category and let*

$$\lim_{n\to\infty} (a_n \cos nx + b_n \sin nx) = 0$$

for each $x \in A$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.

From Theorem 2.1 we immediately get

Theorem 2.1′. *Let A be a set of the second Baire category and let a trigonometric series $\sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ converge pointwise on A. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.*

Thus Theorem 2.1 asserts that every Y-set is a set of the first Baire category while Theorem 2.1′ states that every R-set is a set of the first Baire category.

P r o o f o f T h e o r e m 2 . 1 . Write $a_n \cos nx + b_n \sin nx$ in the form

$$a_n \cos nx + b_n \sin nx = \varrho_n \cos(nx - \alpha_n),$$

where

$$\varrho_n = \sqrt{a_n^2 + b_n^2}, \quad a_n = \varrho_n \cos \alpha_n, \quad b_n = \varrho_n \sin \alpha_n, \quad \alpha_n \in [0, 2\pi] \qquad (n = 1, 2, \ldots).$$

The assertion of Theorem 2.1 is equivalent to the assertion that $\varrho_n \to 0$ $(n \to \infty)$.

We prove that $\varrho_n \to 0$ $(n \to \infty)$. We shall proceed indirectly. Assume that $\lim_{n\to\infty} \varrho_n = 0$ does not hold. Then there exists a number $\eta > 0$ and a sequence $n_1 < n_2 < \dots$ of natural numbers such that

$$\varrho_{n_k} \geqslant \eta > 0 \quad (k = 1, 2, \dots). \tag{4}$$

We can already assume that $\{\alpha_{n_k}\}_{k=1}^{\infty}$ converges and

$$\lim_{k\to\infty} \alpha_{n_k} = \alpha \in [0, 2\pi]. \tag{5}$$

The assumption of Theorem 2.1 is equivalent to the equality

$$\lim_{n\to\infty} \varrho_n \cos(nx - \alpha_n) = 0$$

for $x \in A$. From this by virtue of (4) we get (for $x \in A$)

$$\lim_{k\to\infty} \cos(n_k x - \alpha_{n_k}) = 0. \tag{6}$$

Put for brevity

$$V_k = V_k(x) = \cos(n_k x - \alpha_{n_k}) \qquad (k = 1, 2, \dots).$$

Then we have

$$V_k = \cos n_k x \cos \alpha_{n_k} + \sin n_k x \sin \alpha_{n_k} \qquad (k = 1, 2, \dots). \tag{7}$$

Put $t = \frac{x}{2\pi}$. Then $t \in [0, 1]$ if $x \in [0, 2\pi]$. We have $n_k x = 2\pi n_k t, n_k t = [n_k t] + (n_k t)$ and so we get $\cos n_k x = \cos 2\pi(n_k t), \sin n_k x = \sin 2\pi(n_k t)$.

Further, put

$$X_k = X_k(t) = \cos 2\pi(n_k t) \qquad (k = 1, 2, \dots).$$

Then according to (7) we obtain

$$\sin \alpha_{n_k} \cdot \sqrt{1 - X_k^2} = V_k - X_k \cos \alpha_{n_k}.$$

Squaring and calculating a little we get

$$X_k^2 - 2X_k V_k \cos \alpha_{n_k} + (V_k^2 - \sin^2 \alpha_{n_k}) = 0. \tag{8}$$

This is a quadratic equation for X_k, and so

$$X_k = \frac{2V_k \cos \alpha_{n_k} \pm \sqrt{D_k}}{2} \tag{9}$$

where

$$(10) \qquad D_k = 4V_k^2 \cos^2 \alpha_{n_k} - 4V_k^2 + 4\sin^2 \alpha_{n_k} = 4(1 - V_k^2)\sin^2 \alpha_{n_k}.$$

According to (6) for all sufficiently large k (e.g. for $k > k_0$) the inequality $V_k^2 < 1$ holds. Hence for $x \in A$ and $k > k_0$ we have $D_k \geqslant 0$ and so we can calculate X_k from (9). If $k \to \infty$ then by virtue of (5)–(7) and (10) we obtain

$$(11) \qquad \lim_{k\to\infty} X_k = \lim_{k\to\infty} \cos(n_k x - \alpha_{n_k}) = \pm|\sin\alpha|.$$

We have proved that if $x \in A \cap [0, 2\pi]$ then (11) holds, i.e.

$$(11') \qquad \lim_{k\to\infty} \cos 2\pi(n_k t) = \pm|\sin\alpha|$$

where $t = \frac{x}{2\pi} \in [0, 1]$.

Since the functions $g_k(x) = (n_k x)$ $(k = 1, 2, \ldots)$ are periodic (with the period 1), Theorem 1.1 implies that the set M of all $t \in [0, 1]$ with

$$\{(n_k t)\}_k' = [0, 1]$$

is residual in $[0, 1]$. By virtue of continuity of the function cosine the limit points of the sequence $\{\cos 2\pi(n_k t)\}_{k=1}^{\infty}$ fill up the whole interval $[-1, 1]$ if t belongs to M. Therefore if $x \in A \cap [0, 2\pi]$, then $t = \frac{x}{2\pi}$ belongs to $[0, 1] \setminus M$ (see (11′)) and $[0, 1] \setminus M$ is a set of the first category. Further, the function $t = \frac{x}{2\pi}$ $(x \in [0, 2\pi])$ is a homeomorphism of $[0, 2\pi]$ onto $[0, 1]$. Hence according to the previous considerations the set of all $x \in A \cap [0, 2\pi]$ is a set of the first category. By periodicity of the functions cosine, sine we see that the set A is a set of the first category in $\mathbf{R}$. However, this contradicts the assumption of the theorem. □

Using the method of the proof of Theorem 2.1 we can give a new proof of Theorem A.

P r o o f o f T h e o r e m A. We proceed indirectly. Similarly as in the proof of Theorem 2.1 we can derive the relations (4)–(11′). Then we apply the quoted result from [6] (Theorem 4.1, p. 32). Owing to this result the set of all $y \in [0, 2\pi]$ for which $\{(n_k y)\}_{k=1}^{\infty}$ is dense in $[0, 1]$ is a set of full measure. Hence by continuity of cosine we see that the set of all $t \in [0, 1]$ for which (11′) holds is a null set, and so the set $[0, 2\pi] \cap E$ is also a null set. Owing to periodicity of cosine we deduce from this that $\lambda(E) = 0$. This contradicts the assumption of the theorem. □

R e m a r k 2.1. Theorems A and 2.1 can be conversed. This is not true for Theorems A′ and 2.1′. This is shown by the following example given by H. Steinhaus (cf. [10], p. 474–477):

The trigoniometric series

$$\sum_{k=3}^{\infty} \frac{\cos k(x - \ln\ln k)}{\ln k} = \sum_{k=3}^{\infty} (a_k \cos kx + b_k \sin kx),$$

$$a_k = \frac{\cos(k \ln\ln k)}{\ln k}, \quad b_k = \frac{\sin(k \ln\ln k)}{\ln k}$$

$(k = 3, 4, \ldots)$ satisfies the condition $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$, but as is proved in [10], it converges nowhere.

3. Weak D-sets

We can deduce from Theorem 2.1 that every weak D-set has Lebesgue measure zero. Actually, it is known that a weak D-set is an H_σ-set (see [1], p. 732) and therefore it has Lebesgue measure zero. We present here a direct proof of this fact based on our Theorem 1.1.

Theorem 3.1. *Let $n_1 < n_2 < \ldots$ be a sequence of natural numbers. Then $\lambda(E(n_1, n_2 \ldots)) = 0$, where $E(n_1, n_2, \ldots)$ is a weak D-set with respect to the sequence $n_1, n_2, \ldots$.*

Proof. Choose a fixed $\alpha \in (0, 1)$ and put

$$M(\alpha) \left\{ x \in \mathbb{R} : \alpha \notin \{(n_k x)\}'_k \right\}.$$

As we have already seen, for almost all $x \in \mathbb{R}$ we have $\{(n_k x)\}'_k = [0, 1]$ (cf. [4] or [6], p. 32). Hence it suffices to show that

$$E(n_1, n_2, \ldots) \subset M(\alpha). \tag{12}$$

Let $x \in E(n_1, n_2, \ldots)$ and suppose that x does not belong to $M(\alpha)$. Then there exists a sequence $k_1 < k_2 < \ldots$ of natural numbers such that

$$\lim_{j\to\infty} (n_{k_j} x) = \alpha.$$

Since $\alpha \in (0, 1)$ there is an $\eta > 0$ such that for all sufficiently large j (e.g. for $j > j_0$) we have

$$(n_{k_j} x) \in (\eta, 1 - \eta). \tag{13}$$

This yields

$$(13') \qquad 1 - (n_{k_j} x) \in (\eta, 1 - \eta).$$

From (13), (13′) we get (for $j > j_0$).

$$\|n_{k_j} x\| = \min\left\{(n_{k_j} x), 1 - (n_{k_j} x)\right\} \in (\eta, 1 - \eta)$$

and this contradicts the fact that x belongs to $E(n_1, n_2, \ldots)$. □

Theorem 3.2. *Let $n_1 < n_2 < \ldots$ be a sequence of natural numbers. Then the set $E^*(n_1, n_2, \ldots)$ of all $x \in \mathbb{R}$ with*

$$\lim_{k \to \infty} \|n_k x\| = 0$$

is an $F_{\sigma\delta}$-set of the first Baire category in $\mathbb{R}$.

Proof. By the definition of the set $E^*(n_1, n_2, \ldots)$ we have

$$(14) \qquad E^*(n_1, n_2, \ldots) = \bigcap_{k=1}^{\infty} \bigcup_{s=1}^{\infty} \bigcap_{j=s}^{\infty} \left\{x \in \mathbb{R} \colon \|n_j x\| \leqslant \frac{1}{k}\right\}.$$

For a fixed j the function $h_j \colon \mathbb{R} \to \mathbb{R}$, $h_j(x) = \|n_j x\|$ ($x \in \mathbb{R}$) is continuous on $\mathbb{R}$. Owing to this fact and to (14) it is obvious that $E^*(n_1, n_2, \ldots)$ is an $F_{\sigma\delta}$-set. Further, $E^*(n_1, n_2, \ldots)$ is a subset of the complement of the set $W(n_1, n_2, \ldots)$ occuring in Theorem 1.1. So by Theorem 1.1 the set $E^*(n_1, n_2, \ldots)$ is a set of the first Baire category in $\mathbb{R}$. □

Corollary. *Every weak D-set is a set of the first category.*

Remark 3.1. By a suitable choic of the numbers n_k ($k = 1, 2, \ldots$) the set $E^*(n_1, n_2, \ldots)$ can be made dense in $\mathbb{R}$. For example, choose $n_k = k!$ ($k = 1, 2, \ldots$). Then all rational numbers belong to $E^*(n_1, n_2, \ldots)$.

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