CHARACTERIZING THE MAXIMUM GENUS OF A CONNECTED GRAPH

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In this paper a generalization of Tutte's theorem on perfect matchings and a generalization of Rado's theorem on independent transversals will be used for characterizing the maximum genus of a connected graph.

0. By a graph we mean here a graph in the sense of [4], i.e. a pseudograph in the sense of [2]. A graph G is determined by its vertex set $V(G)$, its edge set $E(G)$, and its incidence relation between edges and vertices. A graph in the sense of [2] will be called here a simple graph, similarly as in [4] or [18]. Note that a simple graph G is determined by $V(G)$ and $E(G)$ only.

A trivial graph (i.e. a graph with only one vertex and no edge) will be considered to be 2-edge-connected. Any maximal 2-edge-connected subgraph of a graph G will be referred to as leaf of G.

Let G be a graph. We denote by $c(G)$ the number of components of G. We define $p(G) = |V(G)|$, $q(G) = |E(G)|$, and $\beta(G) = q(G) - p(G) + c(G)$; thus, if G is connected, then $\beta(G) = q(G) - p(G) + 1$. Moreover, we denote by $b(G)$ or $b^{\lambda}(G)$ the number of components F_1 of G such that $\beta(F_1)$ is odd, or the number of leaves F_2 of G such that $\beta(F_2)$ is odd, respectively.

Let G be a connected graph. We denote by $\mathscr{A}_G$ the set of all $A \subseteq E(G)$ such that $G - A$ is connected. We denote by $\mathscr{T}(G)$ the set of all spanning trees of G. If $T \in \mathscr{T}(G)$, then we denote by $\mathscr{A}_G(T)$ the set of all $A \subseteq E(G) - E(T)$. Clearly,

$$\mathscr{A}_G = \bigcup_{T \in \mathscr{T}(G)} \mathscr{A}_G(T).$$

For every graph G we denote by $\Gamma(G)$ the set of all integers i such that there exists a 2-cell embedding of G into the closed orientable surface of genus i (for the above mentioned concepts of topological graph theory the reader is referred to [17] or to Chapter 5 of [2]). As follows from the properties of 2-cell embeddings, $\Gamma(G)$ is finite for every graph G. Moreover, $\Gamma(G) \neq \emptyset$ if and only if G is connected. Duke [5] proved that if G is a connected graph, $i, k \in \Gamma(G)$ and j is an integer such that $i < j < k$, then $j \in \Gamma(G)$. (As was proved in [14], this result does not hold for signed graphs.) For every connected graph G, the maximum genus $\gamma_M(G)$ of G is defined as the maximum integer in $\Gamma(G)$. As was shown in [11], $\gamma_M(G) \leqslant [\beta(G)/2]$ for every connected graph G. Since the beginning of the seventies many papers concerning the maximum genus have been written. (The maximum nonorientable genus has been also studied. Ringel [13] proved that the maximum nonorientable genus of a connected graph G is equal to $\beta(G)$.)

The maximum genus of a connected graph was determined by Homenko, Ostroverkhy and Kusmenko [8] and independently by Xuong [19]. We will present the result obtained in [19]. The result obtained in [8] looks rather dissimilarly but in substance it is the same.

If G is a connected graph and $T \in \mathscr{T}(G)$, then we denote by $x_G(T)$ the number of components F of $G - E(T)$ such that $|E(F)|$ is odd.

Theorem A ([19]). *Let G be a connected graph. Then*

$$\gamma_M(G) = \tfrac{1}{2}\big(\beta(G) - \min_{T \in \mathscr{T}(G)} x_G(T)\big).$$

For the case when $\gamma_M(G) = [\frac{1}{2}\beta(G)]$, the formula was proved independently by Jungerman [9].

If G is a connected graph and $A \subseteq E(G)$, then we denote

$$y_G(A) = c(G - A) + b(G - A) - 1 - |A|.$$

Proposition A. *If G is a connected graph, then*

$$\begin{aligned} \max_{A_0 \subseteq E(G)} \big(b^\lambda(G - A_0) - |A_0|\big) &= \max_{A \subseteq E(G)} y_G(A) \\ &= \max_{A_1 \in \mathscr{A}_G} \big(b^\lambda(G - A_1) - |A_1|\big). \end{aligned}$$

Proof (outlined). Let $A \subseteq E(G)$; there exists $A' \subseteq A$ such that $G - A'$ is connected and $|A - A'| = c(G - A) - 1$; we can see that $b^\lambda(G - A') \geqslant b(G - A)$. Let $A_1 \in \mathscr{A}_G$; there exists $A'' \subseteq E(G)$ such that $A_1 \subseteq A''$ and the set of components of $G - A''$ is the same as the set of leaves of $G - A_1$; hence $|A'' - A_1| = c(G - A'') - 1$. Finally, let $A_0 \subseteq E(G)$; there exists $A^* \subseteq A_0$ such that $A^* \in \mathscr{A}_G$ and $b^\lambda(G - A^*) = b^\lambda(G - A_0)$. The result of the proposition easily follows. □

Homenko and Glukhov [7] and independently Nebeský [10] have found that for any connected graph G,

$$\min_{T\in\mathcal{T}(G)} x_G(T)$$

can be expressed as the maximum of a function. Homenko and Glukhov [7] proved that if G is a connected graph, then

$$\min_{T\in\mathcal{T}(G)} x_G(T) = \max_{A\subseteq E(G)} \big(b^{\lambda}(G-A) - |A|\big).$$

The present author proved the following theorem:

Theorem B ([10]). *If G is a connected graph, then*

$$\min_{T\in\mathcal{T}(G)} x_G(T) = \max_{A\subseteq E(G)} y_G(A).$$

Note that Širáň and Škoviera [15] generalized Theorems A and B to signed graphs. In Section 2 of the present paper an extension of Theorem B will be given.

1. Let G be a connected graph different from a tree, and let $T \in \mathcal{T}(G)$. It is clear that if e_1 and e_2 are distinct edges in $E(G) - E(T)$, then the subgraph $T + e_1 + e_2$ of G has at least one and at most two nontrivial (i.e. cyclic) leaves. We denote by $G\#T$ the simple graph with

$$V(G\#T) = E(G) - E(T)$$

and with the property that

$ef \in E(G\#T)$ if and only if the subgraph $T + e + f$ of G has only one nontrivial leaf

for any distinct $e, f \in E(G) - E(T)$.

Lemma 1. *Let G be a nontrivial 2-edge-connected graph, and let $T \in \mathcal{T}(G)$. Then $G\#T$ is connected.*

P r o o f. We assume, to the contrary, that $G\#T$ is not connected. Then there exist E_1, $E_2 \subseteq \mathcal{A}_G(T)$ such that $E_1 \neq \emptyset \neq E_2$, $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = E(G) - E(T)$, and that $T + e_1 + e_2$ has two nontrivial leaves for any $e_1 \in E_1$ and $e_2 \in E_2$. We denote by $\mathcal{E}$ the set of all $E \in \mathcal{A}_G(T)$ with the properties that $E \cap E_1 \neq \emptyset \neq E \cap E_2$ and the subgraph $T + E$ of G has only one nontrivial leaf. Clearly, $\mathcal{E} \neq \emptyset$.

Consider $E_0 \in \mathscr{E}$ such that no proper subset of E_0 belongs to $\mathscr{E}$. We can see that $|E_0| \geqslant 3$. Without loss of generality we will assume that $|E_0 \cap E_2| \geqslant 2$. Consider an arbitrary $e_0 \in E_0 \cap E_2$. Obviously, $E_0 - \{e_0\} \notin \mathscr{E}$. According to the definition, $T + (E_0 - \{e_0\})$ has at least two nontrivial leaves. Clearly, there exists a leaf F_1 of $T + (E_0 - \{e_0\})$ such that $E(F_1) \cap E_1 \neq \emptyset$. Denote $E^* = E(F_1) - E(T)$. Since $T + (E_0 - \{e_0\})$ has at least two nontrivial leaves, we conclude that E^* is a proper subset of $E_0 - \{e_0\}$, and therefore $E^* \cup \{e_0\}$ is a proper subset of E_0. Hence $E^* \cup \{e_0\} \notin \mathscr{E}$.

On the other hand, F_1 is a nontrivial leaf of $T+(E_0-\{e_0\})$ and $T+(E_0-\{e_0\})+e_0$ has only one nontrivial leaf. It is easy to see that $T + \big(E(F_1) - E(T)\big) + e_0$ has only one nontrivial leaf. Thus we get $E^* \cup \{e_0\} \in \mathscr{E}$, which is a contradiction. The lemma is proved. □

Corollary. *Let G be a connected graph different from a tree, and let $T \in \mathscr{T}(G)$. Then there exists a bijection φ of the set of all nontrivial leaves of G onto the set of all components of $G\#T$ such that*

$$V\big(\varphi(F)\big) = E(F) - E(T)$$

for each nontrivial leaf F of G.

Proof is obvious.

Let G be a graph. If M is a matching in G and $u \in V(G)$ is such that u is incident with no edge in M, then we say that u is an unsaturated vertex of M. A matching M in G is referred to as a maximum matching in G if $|M_0| \leqslant |M|$ for every matching M_0 in G.

If H is a graph, then we denote by $c_0(H)$ the number of components F of H such that $p(F)$ is odd. We shall need the following theorem:

Theorem C (Berge [3]). *Let G be a graph. Then the number of unsaturated vertices of a maximum matching in G is equal to*

$$\max_{U \subset V(G)} \big(c_0(G-U) - |U|\big).$$

Note that Theorem C is a generalization of Tutte's theorem on perfect matchings [16].

If G is a connected graph different from a tree and $T \in \mathscr{T}(G)$, then we shall denote by $z_G(T)$ the number of unsaturated vertices of a maximum matching in $G\#T$.

Lemma 2. *Let G be a connected graph different from a tree, and let $T \in \mathscr{T}(G)$. Then*

$$z_G(T) = \max_{A \in \mathscr{A}_G(T)} \left(b^\lambda(G - A) - |A|\right).$$

Proof. According to Theorem C,

$$z_G(T) = \max_{A \subset E(G) - E(T)} \Big(c_0\big((G\#T) - A\big) - |A|\Big).$$

Consider an arbitrary $A \subset E(G) - E(T)$. The corollary implies that

$$c_0\big((G - A)\#T\big) = b^\lambda(G - A).$$

It is easy to see that

$$(G\#T) - A = (G - A)\#T.$$

Obviously, $b^\lambda\big(G - \big(E(G) - E(T)\big)\big) = 0$. Hence, the statement of the lemma follows. □

In the next section we will prove that if G is a connected graph different from a tree, then there exists $T \in \mathscr{T}(G)$ such that

$$\min_{T_0 \in \mathscr{T}(G)} x_G(T_0) = x_G(T) = z_G(T) = \max_{T_1 \in \mathscr{T}(G)} z_G(T_1).$$

2. The following proposition can be easily proved:

Proposition B. *If G is a connected graph, then*

$$y_G(A) \equiv \beta(G) \pmod 2$$

for every $A \subseteq E(G)$.

For the proof see [10].

If G is a connected graph, then we denote by $\mathscr{M}(G)$ the set of all $A \subseteq E(G)$ such that

$$y_G(A) = \max_{A' \subseteq E(G)} y_G(A')$$

and $y_G(A'') < y_G(A)$ for every $A'' \subseteq E(G)$ such that A is a proper subset of A''.

A complete proof of the next Lemma can be found in [10].

Lemma A. *Let G be a connected graph, let $A \in \mathscr{M}(G)$, and let F be a component of $G - A$. If $\beta(F)$ is even, then $q(F) = 0$. If $\beta(F)$ is odd, then $F - e$ is connected and*

$$\max_{A_F \subseteq E(F-e)} y_{F-e}(A_F) = 0$$

for each $e \in E(F)$.

Proof (outlined). The case when $\beta(F)$ is even is clear. Let $\beta(F)$ be odd. Consider an arbitrary $e \in E(F)$. Since $A \in \mathscr{M}(G)$, we get that $F - e$ is connected. Let $A_F \subseteq E(F - e)$. Then

$$y_G(A) > y_G(A \cup \{e\} \cup A_F) = y_G(A) + y_{F-e}(A_F) - 2,$$

and thus $y_{F-e}(A_F) < 2$. Proposition B implies that $y_{F-e}(A_F) \leqslant 0$. Since $y_{F-e}(\emptyset) = 0$, the proof is complete. □

We shall need a theorem from the intersection of matroid theory and transversal theory; see Wilson [18], for example. Corollary 33B in [18] can be reformulated as follows:

Theorem D. *Consider a matroid on a finite nonempty set A with rank function r. Let $D_1, \ldots, D_k$ ($k \geqslant 1$) be nonempty subsets of A. Denote $\mathscr{D} = (D_1, \ldots, D_k)$. Then the maximum size of an independent transversal of $\mathscr{D}$ is equal to*

$$k - \max_{I \subseteq \{1,\ldots,k\}} \Big(|I| - r\Big(\bigcup_{i \in I} D_i\Big)\Big).$$

Clearly, Theorem D is a generalization of Rado's theorem on independent transversals [12].

We are now prepared to prove the main result of the present paper.

Theorem 1. *Let G be a connected graph different from a tree. Then there exists $T \in \mathscr{T}(G)$ such that*

$$\min_{T_0 \in \mathscr{T}(G)} x_G(T_0) = x_G(T) = \max_{A \in E(G)} y_G(A) = z_G(T) = \max_{T_1 \in \mathscr{T}(G)} z_G(T_1).$$

Proof. For every connected graph H we denote

$$x_H = \min_{T \in \mathscr{T}(G)} x_H(T) \qquad \text{and} \qquad y_H = \max_{A \subseteq E(G)} y_H(A).$$

We shall prove that

(I) there exists $T \in \mathscr{T}(G)$ such that $x_G(T) \leqslant y_G \leqslant z_G(T)$ and $y_G \leqslant x_G$ and that

(II) $\max_{T_1 \in \mathscr{T}(G)} z_G(T_1) = y_G$.

(I) We proceed by induction on $q(G)$. Since G is different from tree, we get that $q(G) \geqslant 1$. The case when $q(G) = 1$ is obvious. Let $q(G) \geqslant 2$. Consider an arbitrary $A \in \mathscr{M}(G)$. Let $\mathscr{B}$ denote the set of all components F of $G - A$ such that $\beta(F)$ is odd. We put $k = b(G - A)$. Since G is not a tree and $A \in \mathscr{M}(G)$, we can see that $k \geqslant 1$. There exist mutually distinct components B_1, ..., B_k of $G - A$ such that $\mathscr{B} = \{B_1, \ldots, B_k\}$. For every $i \in \{1, \ldots, k\}$ we denote by N_i the set of all $e \in A$ such that e is incident with a vertex of B_i. For $I \subseteq \{1, \ldots, k\}$ we denote

$$N_I = \bigcup_{i \in I} N_i.$$

Let r denote the mapping of exp A into the set of integers defined as follows:

$$r(A_0) = |A_0| - c(G - A_0) + 1 \qquad \text{for every } A_0 \subseteq A.$$

If $I \subseteq \{1, \ldots, k\}$, then

$$|I| - r(N_I) = |I| - |N_I| + c(G - N_I) - 1 \leqslant y_G(N_I) \leqslant y_G.$$

It is easy to see that

$$k - r(N_{\{1,\ldots,k\}}) = y_G(A).$$

Thus,

$$\max_{I \subseteq \{1,\ldots,k\}} (|I| - r(N_I)) = y_G.$$

It is not difficult to see that r is the rank function of a matroid on A. According to Theorem D, the maximum size of an independent partial transversal of $(N_1, \ldots, N_k)$ is equal to $k - y_G$. Thus, without loss of generality we will assume that there exists an independent transversal of $(N_1, \ldots, N_{k-y_G})$. This means that there exist mutually distinct $a_1, \ldots, a_{k-y_G} \in A$ such that

$$a_i \in N_i, \qquad \text{for each } i \in \{1, \ldots, k - y_G\},$$

and $G - a_1 - \ldots - a_{k-y_G}$ is connected. Denote

$$A^* = A - \{a_1, \ldots, a_{k-y_G}\}.$$

We can see that $|A^*| = c(G - A) - 1$.

Let $i \in \{1, \ldots, k\}$. We choose an edge e_i of B_i such that if $i \leqslant k - y_G$, then there exists a vertex incident with both a_i and e_i. According to Lemma A, $B_i - e_i$ is connected and $y_{B_i - e_i} = 0$. If $B_i - e_i$ is not a tree, then it follows from the induction hypothesis that there exists $T_i \in \mathscr{T}(B_i - e_i)$ such that $x_{B_i - e_i}(T_i) = 0$. If $B_i - e_i$ is a tree, we put $T_i = B_i - e_i$.

We denote by T the subgraph of G induced by the set of edges

$$A^* \cup E(T_1) \cup \ldots \cup E(T_k).$$

Clearly, T is a spanning tree of G. It is easy to see that $x_G(T) \leqslant y_G$.

According to Lemma 2,

$$z_G(T) = \max_{A_0 \in \mathscr{A}_G(T)} \big(b^\lambda(G - A_0) - |A_0|\big).$$

Since $|A^*| = c(G - A) - 1$, we can see that

$$b^\lambda(G - \{a_1, \ldots, a_{k-y_G}\}) - |\{a_1, \ldots, a_{k-y_G}\}| = y_G.$$

Hence $y_G \leqslant z_G(T)$.

Consider $T' \in \mathscr{T}(G)$ such that $x_G(T') = x_G$. Let

$\mathscr{B}_{\mathrm{con}} = \{B \in \mathscr{B};$ the subgraph of T' induced by $V(B)$ is connected$\}$.

It is not difficult to see that $q(F)$ is odd for at least $|B_{\mathrm{con}}| - |A - E(T)|$ components F of $G - E(T')$. Thus

$$x_G(T') \geqslant |B_{\mathrm{con}}| - |A - E(T')| = |B_{\mathrm{con}}| - |A| + |A \cap E(T')|.$$

Moreover, we can see that

$$|A \cap E(T')| \geqslant c(T' - A) - 1 \quad \text{and} \quad c(T' - A) \geqslant c(G - A) + |B - B_{\mathrm{con}}|.$$

We get that $x_G(T') \geqslant y_G(A)$, and thus $x_G \geqslant y_G$.

(II) If we combine Lemma 2 with Proposition A, we obtain

$$\begin{aligned}\max_{T_1 \in \mathscr{T}(G)} z_G(T_1) &= \max_{T_1 \in \mathscr{T}(G)} \max_{A_{T_1} \in \mathscr{A}_G(T_1)} \big(b^\lambda(G - A_{T_1}) - |A_{T_1}|\big) \\ &= \max_{A \in \mathscr{A}_G} \big(b^\lambda(G - A') - |A'|\big) = y_G.\end{aligned}$$

The proof of the theorem is complete. □

Theorem 1 is an extension of Theorem B. In the proof of Theorem 1 some ideas from [10] were utilized. On the other hand, the proof of Theorem 1 shows that Theorem B can be proved by using Theorem D; then the role of Theorem D is similar to the role of Hall's theorem on distinct representatives [6] in Anderson's proof [1] of Tutte's theorem on perfect matchings.

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