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MATRIX TRANSFORMATIONS ON NUCLEAR KÖTHE SPACES

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One of the most fundamental properties of the power series space of analytic or entire functions is that, they are simple, i.e. bounded subsets of each of them is contained in the normal hull of a single point of the space. Infact, every nuclear Köthe space is simple and in the case of a Frechét space this property is synonymous with nuclearity ([6], p. 330). Simple character plays a key role in the study of theory of bases in Frechét spaces ([4]).

Thanks to Jacob ([1]) who laid the foundation for the study of matrix transformations on simple perfect sequence spaces. The present article is a contribution in that direction. Objective of this paper is to characterize matrix transformations on nuclear Köthe-spaces and to reveal their structural behaviour. This aspects of the study of matrix transformations paves the way for a new direction which relates the structure of transformations to the associated nuclearities of the underlying spaces with bases (cf. Theorem 3.1). The last result illuminates a new direction of study, namely, the study of matrix transformation on sequence spaces, where the traditional normal topology is replaced by the fairly generalized $\sigma\mu$-topology.

§1. BACKGROUND AND TERMINOLOGY

We expect the reader to be familiar with the rudiments of locally convex spaces and for that reason it suffices to refer to some standard text, e.g. ([2], [5]). As our interest in this paper is related to sequence spaces, for the sake of completeness we recall only a very few relevant terms and results. For various other unexplained terms and facts we assume that one would turn to ([9], [11]).

We write $\lambda^{\times}$, as usual for the Köthe dual of a sequence space λ. Unless stated otherwise, it will be taken into account that each λ is equipped with the natural topology, $\eta(\lambda, \lambda^\times)$, called the *normal topology*, which is generated by the seminorms $\{p_y : y \in \lambda^\times\}$ on λ, where for x in λ, $p_y(x) = \sum_{i \geqslant 1} |x_i y_i|$.

A sequence space λ is called *simple* (cf. [5]) if every bounded set in λ is dominated by an element of λ.

A set of sequences of non-negative real numbers P will be called a *Köthe set* if it satisfies the following conditions.

(i) If a and b are two members of P there is a c in P with $a \leqslant c$ and $b \leqslant c$.

(ii) For every integer $n \in \mathbb{N}$ there is an $a \in P$ with $a_n > 0$.

The space of all sequences $x = (x_n)$, such that $p_a(x) = \sum_{n \geqslant 0} |x_n| a_n < \infty$ for each $a \in P$, is called the *Köthe space* generated by P and denoted by $\Lambda(P)$.

A Köthe set P will be called a *power set* of *infinite* type if it satisfies the additional conditions;

(iii) For each $a \in P$, $0 < a_n \leqslant a_{n+1}$ for each $n \in \mathbb{N}$.

(iv) For $a \in P$ and $b \in P$ there is a $c \in P$ with $a_n b_n \leqslant c_n$ for all $n \in \mathbb{N}$.

If P is a power set of infinite type the Köthe space $\Lambda(P)$ is called a *G_∞-space* or a *smooth sequence space* of *infinite type*.

If $\alpha = (\alpha_n)$ is a non-decreasing sequence of non-negative real numbers, the power series space of infinite type $\Lambda(\alpha)$ is the G_∞-space generated by the power set $\{R^{\alpha_n} : R \geqslant 1\}$. For $\alpha_n = \log(n+1)$ we obtain the space of all rapidly decreasing sequences s, while $\alpha_n = n$ gives us the space of entire functions Γ.

A Schauder base $\{x_n, f_n\}$ in an l.c. TVS (X, T) is said to be a *fully-λ-base*, if for each $p \in D_\times$, the mapping $\psi_p : X \to \lambda$ defined by

$$\psi_p(x) = \{p(x_n) f_n(x)\}, \quad x \in X$$

is continuous. For $\lambda = l^1$, we get the traditional absolute base.

Finally, by the notation (λ, μ) we mean the class of all matrix transformations $A = [a_{ij}] : \lambda \to \mu$; that is if $x \in \lambda$, then $Ax = y$ with $y_i = \sum_{j \geqslant 1} a_{ij} x_j$. We use the symbol $A^\perp = [a_{ji}]$ for the transpose of A. For other unexplained facts about the matrix transformation to be used here we follow [1].

As the analysis involved in this article veers round the nuclearity of the underlying Köthe spaces it will be sufficient to call on the following widely used Grothendieck-Pietsch criterion (c.f. [11], p. 195), namely,

Theorem A. *A Köthe space $\Lambda(P)$ [resp. λ] is nuclear iff to each $a \in P$ [resp. $y \in \lambda^\times$] there corresponds a $b \in P$ [resp. $z \in \lambda^\times$] such that*

$$\{a_n / b_n\} \in l^1 \quad [\text{resp. } \{y_n / z_n\} \in l^1].$$

In the case of a G_∞-space this can be strengthened and improved further, (cf. [11], p. 207) as indicated by the following

Corollary B. *A G_∞-space $\Lambda(P)$ is nuclear iff there exists an $a \in P$ with $\{1/a_n\} \in l^1$ iff for each $k \geqslant 1$ and $a \in P$, there exists a $c \in P$ and $M > 0$ such that*

$$(n+1)^{2^k} a_n \leqslant M c_n, \quad \forall n \geqslant 1.$$

§2. Characterization of Transformations

At the outset we have a result which modifies a result of Jacob (cf. [1], p. 185) and sharpens, strengthens and improves the main result contained in ([3], p. 327); ([1], p. 185).

Theorem 2.1. *Let $\Lambda(P)$ be a nuclear G_∞-space and the power series space $\Lambda(\alpha)$ be nuclear. Then for an infinite matrix $A = [a_{ij}]$, the following statements are equivalent:*

(i) $A \in (\Lambda(\alpha), \Lambda(P))$.

(ii) $A^\perp \in ((\Lambda(P))^\times, (\Lambda(\alpha))^\times)$.

(iii) *For each $x \in P$, there corresponds $R \geqslant 1$ such that*

$$|a_{ij} x_i| \leqslant R^{\alpha_j}, \quad \forall i, j \geqslant 1.$$

(iv) *For each $k, l \geqslant 1$ and $x \in P$, there exists $R \geqslant 1$ and $c > 0$ such that*

$$(i+1)^{2^k} (j+1)^{2^l} |a_{ij} x_i| \leqslant c R^{\alpha_j}, \quad \forall i, j \geqslant 1.$$

P r o o f. (i) $\Leftrightarrow$ (ii). This follows from ([3], p. 326) or ([8], p. 164).

(ii) $\Leftrightarrow$ (iii). This results from ([1], p. 184).

(iii) $\Leftrightarrow$ (iv). Since $\Lambda(P)$ is nuclear, for each $k \geqslant 1$ and $x \in P$ the famous Grothendieck-Pietsch criterion leads us to find an element $y \in P$ and $M > 0$ such that

$$(i+1)^{2^k} x_i \leqslant M y_i, \quad \forall i \geqslant 1.$$

Then for this y, in view of (iii) there corresponds a $t \geqslant 1$ for which the following inequality, namely,

$$(i+1)^{2^k} (j+1)^{2^l} |a_{ij} x_i| \leqslant M (j+1)^{2^l} |a_{ij} y_i| \leqslant M (j+1)^{2^l} t^{\alpha_j}$$

holds. As $\Lambda(\alpha)$ is nuclear once more using Corollary B we obtain $R \geqslant 1$ and $c > 0$ satisfying

$$(j+1)^{2^l} t^{\alpha_j} \leqslant cR^{\alpha_j}, \quad \forall j \geqslant 1.$$

Thus we have the inequality

$$(i+1)^{2^k}(j+1)^{2^l}|a_{ij}x_i| \leqslant McR^{\alpha_j}, \quad \forall i,j \geqslant 1,$$

which gives (iv).

(iv) $\Rightarrow$ (i). The condition (iv) guarrantees the existence of $\big(A(x)\big)_i$ for each $x \in \Lambda(\alpha)$ and $i \geqslant 1$. Let now $x \in \Lambda(\alpha)$ and $u \in P$. Then Theorem A directs us to the existence of a $v \in P$ for which we can have

$$\{u_n/v_n\} \in l^1.$$

From (iv), we find an element $R \geqslant 1$ and $c > 0$ such that

$$(i+1)^{2^k}(j+1)^{2^l}|a_{ij}v_i| \leqslant cR^{\alpha_j}, \quad \forall i,j \geqslant 1.$$

Hence

$$\sum_{i\geqslant 1}\left|\big(A(x)\big)_i u_i\right| \leqslant \sum_{i\geqslant 1}\sum_{j\geqslant 1}|a_{ij}x_j|u_i \leqslant c\sum_{i\geqslant 1}\frac{1}{(i+1)^{2^k}}\frac{u_i}{v_i}\sum_{j\geqslant 1}\frac{1}{(j+1)^{2^l}}|x_j|R^{\alpha_j} < \infty.$$

Thus, $A(x) \in \Lambda(P)$ and this completes the proof. □

To prove the next result, we recall the following relation which is to be found in ([11], p. 210).

Lemma C. *If $\Lambda(\alpha)$ is nuclear, then*

$$\Lambda(\alpha) = \{x = (x_n) : |x_n|^{1/\alpha_n} \to 0\}.$$

Note. For $\alpha = (\alpha_n) = n$, we have the space of entire sequences $\delta \equiv \Gamma$, which is a nuclear space ([11], p. 210). Hence

$$\Gamma = \{x = (x_n) : |x_n|^{1/n} \to 0\},$$

whose cross-dual is given by

$$\Gamma^\times = \{x = (x_n) : \sup_n |x_n|^{1/n} < \infty\}.$$

The following result not only includes the main result of ([3], p. 327) and a result of ([8], p. 165) and ([1], p. 186) but also tells us about the behaviour of the transformation.

Theorem 2.2. *Let P be a countable power set of infinite type. If $\Lambda(P)$ is nuclear, then the following statements are equivalent:*

(i) $A \in (\Lambda(P), \Gamma)$.

(ii) $A^{\perp} \in (\Gamma^{\times}, (\Lambda(P))^{\times})$.

(iii) *For every ε, $0 < \varepsilon < \infty$, $\exists y \in (\Lambda(P))^{\times}$ such that*

$$|a_{ij}|\varepsilon^{i} \leqslant y_j, \quad \forall i, j \geqslant 1.$$

Proof. (i) $\Leftrightarrow$ (ii). This follows along similar lines of Theorem 2.1, (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii). Suppose (ii) holds and $x \in \Gamma^{\times}$. Then the main result, namely, Theorem 3.1 (cf. [3], p. 326) allows us to find an element $y \in (\Lambda(P))^{\times}$ which satisfies the inequality

$$|a_{ij}x_i| \leqslant y_j, \quad \forall i, j \geqslant 1.$$

But for any ε with $0 < \varepsilon < \infty$, $\{\varepsilon^{i}\} \in \Gamma^{\times}$ and hence (iii) follows.

(iii) $\Rightarrow$ (i). Let $\varepsilon > 0$ be choosen arbitrarily. In view of (iii) there exists $y \in (\Lambda(P))^{\times}$ such that

$$|a_{ij}| \leqslant \varepsilon^{i} y_j.$$

Let now $x \in \Lambda(P)$. Then from the following inequality, namely

$$\Big(\sum_{j\geqslant 1} |a_{ij}x_j|\Big)^{1/i} \leqslant \Big(\sum_{j\geqslant 1} \varepsilon^{i}|y_j x_j|\Big)^{1/i} = \varepsilon M^{1/i}$$

where $M = \sum\limits_{j=1} |y_j x_j| < \infty$, it follows that $A(x) \in \Gamma$. This completes the proof. □

Exactly proceeding along same lines we can have the following result involving the nuclear space s of rapidly decreasing sequences and the simple character of l^{∞}.

Proposition 2.3. *For an infinite matrix $A = [a_{ij}]$, the following are equivalent:*

(i) $A \in (l^{1}, s)$.

(ii) $A^{\perp} \in (s^{\times}, l^{\infty})$.

(iii) *For each ε, $0 < \varepsilon < \infty$, there exists an $y \in l^{\infty}$ such that*

$$|a_{ij}|\varepsilon^{\log(i+1)} \leqslant y_j, \quad \forall i, j \geqslant 1.$$

Proof. The proof follows mutatis mutandis on lines similar to that of Theorem 2.2. □

N o t e. The obove result provides a far reaching generalization of results ([1], p. 185), ([3], p. 327).

In view of Theorem 2.1, we can still have a powerful structure regarding the matrix transformations whose underlying spaces are s and Γ. This is illustrated in

Corollary 2.4. *The following statements for an infinite matrix $A = [a_{ij}]$ are equivalent*

(i) $A \in (s, \Gamma)$.

(ii) $A^{\perp} \in (\Gamma^{\times}, s^{\times})$.

(iii) *For each $x \in \Gamma^{\times}$ there corresponds $R \geqslant 1$ such that*

$$|a_{ij} x_i| \leqslant R^{\log(j+1)}, \quad \forall i, j \geqslant 1.$$

(iv) *For every ε, $0 < \varepsilon < \infty$, there exists $R \geqslant 1$ such that*

$$|a_{ij}| \varepsilon^i \leqslant R^{\log(j+1)}, \quad \forall i, j \geqslant 1.$$

(v) *For each $k \geqslant 1$ and for every ε, $0 < \varepsilon < \infty$, we can find a $t \geqslant 1$ and $M > 0$ such that*

$$(j+1)^{2^k} |a_{ij}| \varepsilon^i \leqslant M t^{\log(j+1)}, \quad \forall i, j \geqslant 1.$$

P r o o f. Since $\{\varepsilon^i\} \in \Gamma^{\times}$ for each ε, $0 < \varepsilon < \infty$, the result follows from Theorem 2.1. □

At this stage, we would like to recall the following two results ([9], p. 18, p. 20) which strengthen and improve the famous Grothendieck-Pietsch criterion in the case of a Frechét K-spaces.

Proposition D. *Let $\Lambda(P)$ [resp. λ] be a Frechét K-space. Then $\Lambda(P)$ [resp. λ] is nuclear iff $\exists t \in l^1$ such that $tP = P$ [resp. $t\lambda^{\times} = \lambda^{\times}$].*

Proposition E. *Let λ [resp. $\Lambda(P)$] be a Frechét K-space. Then λ [resp. $\Lambda(P)$] is nuclear iff there exists a permutation $\pi \in \mathscr{P}(\mathbb{N})$ such that $\lambda^{\times} = \pi(i)\lambda^{\times}$ [resp. $\pi(i)P = P$].*

The effect of these two results leads us to the following

Theorem 2.5. *Let P and Q be two countable nuclear Köthe sets such that $\Lambda(Q)$ is a Frechét K-space and $A = [a_{ij}]$ an infinite matrix. Then the following statements* *are equivalent:*

(i) $A \in (\Lambda(P), \Lambda(Q))$.

(ii) $A^{\perp} \in ((\Lambda(Q))^{\times}, (\Lambda(P))^{\times})$.

(iii) *There exists* $t \in l^1$, *such that given any* $x \in Q$ *we can find* $y \in P$ *such that*

$$|a_{ij} t_i x_i| \leqslant y_j, \quad \forall i, j \geqslant 1.$$

(iv) *There exists a permutation* π *such that to each* $x \in Q$ *there corresponds a* $y \in P$ *such that*

$$|a_{ij} x_i| \leqslant \pi(i) y_j, \quad \forall i, j \geqslant 1.$$

Remark. The question of characterizing the matrix transformation on Frechét K-spaces, constitute the subject matter of this result.

Restricting the nuclear exponent sequence α, further, we obtain

Theorem 2.6. *Let* $\Lambda(\alpha)$ *be a nuclear power series space of infinite type. Suppose*

$$\underline{\lim}(\alpha_n - \alpha_{n-1}) = 2h > 0,$$

then the following statements regarding an infinite matrix $A = [a_{ij}]$ *are equivalent:*

(i) $A \in (l^1, \Lambda(\alpha))$.

(ii) $A^{\perp} \in ((\Lambda(\alpha))^{\times}, l^{\infty})$.

(iii) *For every* c, $0 < c < \infty$, *there correspond* $y \in l^{\infty}$ *such that*

$$|a_{ij}| c^{\alpha_i} \leqslant y_j, \quad \forall i, j \geqslant 1.$$

(iv) $|a_{ij}|^{1/\alpha_i} \to 0$ *as* $i \to \infty$ *uniformly on* $j \geqslant 1$.

Proof. Here we prove only (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i).

(ii) $\Rightarrow$ (iii). By a result ([11], p. 198), we conclude that, for any c, $0 < c < \infty$,

$$c^{\alpha_i} \in (\Lambda(\alpha))^{\times} = \{x = (x_i) \colon |x_i| \leqslant \varrho R^{\alpha_i} \text{ for some } \varrho > 0 \text{ and } R \geqslant 1\}$$

and thus follows (iii) in view of Theorem 3.1 ([3], p. 326).

(iii) $\Rightarrow$ (iv). Let $\varepsilon > 0$ be choosen arbitrarily. Taking $c = 1/\varepsilon$ in (iii) we obtain $y \in l^{\infty}$ such that

$$\begin{aligned} |a_{ij}| &\leqslant \varepsilon^{\alpha_i} y_j, \quad \forall i, j \geqslant 1 \\ &\leqslant \|y\|_{\infty} \varepsilon^{\alpha_i}. \end{aligned}$$

Hence $|a_{ij}|^{1/\alpha_i} \to 0$ as $i \to \infty$, uniformly in $j \geqslant 1$.

(iv) $\Rightarrow$ (i). Take any $x \in l^1$ and $R \geqslant 1$. Nuclearity of $\Lambda(\alpha)$ yields a number $S \geqslant 1$ such that

$$\{R^{\alpha_i}/S^{\alpha_i}\} \in l^1.$$

Choose $\varepsilon > 0$ such that $\varepsilon S < 1$, the by (iv), we can find i_0 such that

$$|a_{ij}| < \varepsilon^{\alpha_i}, \quad \forall i \geqslant i_0, \quad \forall j \geqslant 1.$$

Thus we have the following inequality,

$$\begin{aligned}\sum_{i\geqslant 1}\left|(A(x))_i\right|R^{\alpha_i} &\leqslant \sum_{i\geqslant 1} R^{\alpha_i}/S^{\alpha_i}\cdot S^{\alpha_i}\sum_{j\geqslant 1}|a_{ij}x_j| \\ &\leqslant \sum_{i\geqslant 1} R^{\alpha_i}/S^{\alpha_i}\sum_{i\geqslant 1}(\varepsilon S)^{\alpha_i}\sum|x_j| < \infty,\end{aligned}$$

which gives (i). □

We still have one more result before relating the study of transformation to Frechét-spaces with absolute bases,

Proposition 2.7. *For an infinite matrix $A = [a_{ij}]$ the following are equivalent:*

(i) $A \in (\omega, \Gamma)$.

(ii) $A^{\perp} \in (\Gamma^{\times}, \varphi)$.

(iii) *For every ε, $0 < \varepsilon < \infty$, there exists $y \in \varphi$ such that*

$$|a_{ij}|\varepsilon^i \leqslant y_j, \quad \forall i, j \geqslant 1.$$

(iv) *For any arbitrarily choosen $\varepsilon > 0$,*

$$\{a_{ij}/\varepsilon^i\}_{j\geqslant 1} \in \varphi, \quad \forall i, j \geqslant 1.$$

Proof. It is sufficient to show (iv) $\Rightarrow$ (i), in view of preceding discussions. So take any $x \in \omega$ and $R \geqslant 1$. Then the nuclearity of Γ allows us to find $S \geqslant 1$ such that

$$\{R^i/S^i\} \in l^1.$$

If we take $\varepsilon = 1/S$ in (iv), then we have the following inequality, namely,

$$\sum_{i\geqslant 1}\left|(A(x))_i\right|R^i < \sum_{i\geqslant 1} R^i/S^i \sum_{j\geqslant 1}|a_{ij}x_j|S^i = \sum_{i\geqslant 1} R^i/S^i \sum_{i\geqslant 1}\frac{|a_{ij}x_j|}{\varepsilon^i} < \infty.$$

Thus follows the assertion. □

§3. Transformation on Frechét spaces with bases

The significance of the absolute character of a Schauder base has been fully realized in view of its impact on the nuclear structure of Frechét spaces. This direction of study was subjected to intensive investigation and thus came the generalized base, namely, λ-base which has got far reaching repurcussions.

Thanks to Pietsch ([17], p. 172) who demonstrated that each complete locally convex space with an absolute base can be identified with a Köthe space. In fact, every Frechét-nuclear space X with a Schauder base $\{x_n, f_n\}$ can be topologically identified with a Köthe space $\Lambda(P)$,

$$P = \{p_j(x_n)\}_{j \geqslant 1},$$

where $\{p_j\}_{j \geqslant 1}$ is the generating family of seminorms (cf. [7], p. 173). Absolute bases have been generalized to a fairly large extent (cf. [4]). In this direction we turn to the following key result which is to be found in [4], p. 84.

Theorem F. *Let E be a sequentially complete l.c. TVS with a fully-λ-base $\{x_n, f_n\}$. If $\lambda^\times$ has an element β such that $\inf \beta_n = k > 0$ and $(\lambda, \eta(\lambda, \lambda^\times))$ is nuclear then E can be identified with a nuclear Köthe-space $\Lambda(P)$,*

$$P = \{p(x_n)y_n : y \in \lambda^\times,\ p \in \mathscr{D}_E\}.$$

The impact of this result paves the way for the following

Theorem 3.1. *Let μ be a Frechét-space with a fully-λ-base $\{x_n, f_n\}$ and η be a normal sequence space such that $\eta^\times$ is simple. If λ is nuclear and there exists $\beta \in \lambda^\times$ such that $\inf \beta_n > 0$ then for an infinite matrix $A = [a_{ij}]$ the following are equivalent:*

(i) $A \in (\eta, \mu)$.

(ii) $A^\perp \in (\mu^\times, \eta^\times)$.

(iii) *For each $k \geqslant 1$ and $y \in \lambda^\times$ there corresponds $z \in \eta^\times$ such that*

$$|a_{ij} p_k(x_i) y_i| \geqslant z_j, \quad \forall i, j \geqslant 1.$$

Remark. This theorem asserts that there is a definite relation between the structure of matrix transformations and the effect of generalized absolute bases of the underlying spaces.

§4. Transformation on Sequence Spaces with $\sigma\mu$-topologies

Corresponding to a normal sequence space $(\mu, \eta(\mu, \mu^{\times}))$ we write λ^{μ} for the generalized dual obtained by

$$\lambda^{\mu} = \{x \in \omega \colon xa \in \mu, \ \forall a \in \lambda\}.$$

For $\mu = l^1$, note that the μ-dual is the Köthe dual and the normal topology on λ is the *$\sigma\mu$-topology* which is generated by the family $\{p_{y,z} \colon y \in \lambda^{\mu}, \ z \in \mu^{\times}\}$ of seminorms, where, for x in λ,

$$p_{y,z}(x) = \sum_{n \geqslant 1} |x_n y_n z_n|.$$

By $\lambda^{\mu\mu} = (\lambda^{\mu})^{\mu}$ we mean the collection of all x in ω such that $xy \in \mu$, for all y in λ^{μ}. The relevant information regarding μ-duals can be had from ([9],[10]).

We would like to invoke the powerfull Grothendieck-Pietsch type criterion for the nuclearity of a sequence space with $\sigma\mu$-topology from ([10], p. 152).

Theorem G. *A sequence space λ with $\sigma\mu$-topology is nuclear iff $\lambda^{\mu}\mu^{\times} = l^1\lambda^{\mu}\mu^{\times}$.*

Remark. For nuclear sequence spaces $(\mu, \eta(\mu, \mu^{\times}))$, the space $(\lambda, \sigma\mu)$ is always nuclear, no matter what sequence space is choosen for λ. Theorem G when applied to $\mu = l^1$ gives the Grothendieck-Pietsch criterion for nuclearity of $(\lambda, \eta(\lambda, \lambda^{\times}))$.

We present the last result of the article which gives an alltogether different direction to carryout the study of matrix transformations involving sequence spaces with $\sigma\mu$-topologies.

Theorem 4.1. *Let μ be a perfect sequence space and λ and S be two sequence spaces equipped with $\sigma\mu$-topology such that $S^{\mu\mu} = S$ and S is nuclear. If for each $u \in S^{\mu}$ and $a \in \mu^{\times}$ there exist $z \in \lambda^{\mu}$ and $b \in \mu^{\times}$ such that*

$$(+) \qquad |a_{ij} u_i a_i| \leqslant |z_j b_j|, \quad \forall i, j \geqslant 1,$$

then $A \in (\lambda, S)$.

Proof. The condition (+) guarrantees the existence of $(A(x))_i$ for each $x \in \lambda$. In view of Theorem G, nuclearity of S gives $t \in S^{\mu}$ and $c \in \mu^{\times}$ such that

$$\{u_i a_i | t_i c_i\} \in l^1.$$

Then by (+) we obtain $z \in \lambda^\mu$ and $b \in \mu^\times$ such that

$$|a_{ij} t_i c_i| \leqslant |z_j b_j|, \quad \forall i, j \geqslant 1.$$

Thus, for any $u \in S^\mu$ and $a \in \mu^\times$, we have the following inequality,

$$\begin{aligned} \sum_{i \geqslant 1} \left|\left(A(x)\right)_i\right| |u_i a_i| &\leqslant \sum_{i \geqslant 1} \frac{|u_i a_i|}{|t_i c_i|} \sum_{j \geqslant 1} |a_{ij} x_j t_i c_i| \\ &\leqslant \sum_{i \geqslant 1} \left|\frac{u_i a_i}{t_i c_i}\right| \sum_{j \geqslant 1} |x_j z_j b_j| < \infty, \end{aligned}$$

which tells us that

$$\left\{ \left(A(x)\right)_i u_i \right\} \in \mu^{\times\times} = \mu, \quad \forall u \in S^\mu.$$

Hence $A(x) \in S^{\mu\mu} = S$, which completes the proof. □

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