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RELATIONS AND TOPOLOGIES

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Recently, ternary relations have been intensively studied by V. Novák and M. Novotný in a series of papers. In [6] these authors dealt with the problem of representation of ternary relations on a set G by binary relations on G^3. In [7] M. Novotný represented the former ones by grupoids on the power set of G. In this note we investigate the possibility of representing ternary relations on a given set by topologies in Čech's sense on this set. First of all, however, with any relation of type α (α an ordinal) we associate a topology and study these associated topologies.

1. PRELIMINARIES

Following the convention introduced by J. von Neumann, we identify ordinals with the set of their predecessors, and cardinals with their initial ordinals (see e.g. [1]).

Let α be an ordinal and G a set. By a *relation of type* α on G we understand any subset $R \subseteq G^\alpha$ (where G^α denotes the set of all mappings of α into G). The set G is then called the *underlying set* of R. In other words, a relation of type α on G is a set of sequences of type α consisting of elements of G. These relations are introduced and studied in [9]. The ordered pair (G, R) where G is a set and R is a relation of type α on G is said to be a *relational system of type* α. Given two relational systems (G, R), (H, S) of type α, a *homomorphism* of (G, R) into (H, S) is any mapping $f\colon G \to H$ such that $(x_i \mid i < \alpha) \in R \Rightarrow \big(f(x_i) \mid i < \alpha\big) \in S$.

We shall use some fundamental concepts of the category theory—they can be found e.g. in [5]. For an ordinal α, by Rel_α we denote the category of relational systems of type α with homomorphisms as morphisms.

By a *topology* on a set G we mean a topology in Čech's sense [3], i.e. a mapping $u\colon \exp G \to \exp G$ satisfying $u\emptyset = \emptyset$, $X \subseteq G \Rightarrow X \subseteq uX$, and $X \subseteq Y \subseteq G \Rightarrow uX \subseteq uY$. The ordered pair (G, u) is then called a *topological space* and the elements of G are called *points*. If (G, u) is a topological space and $x \in G$ a point, then a set $X \subseteq G$ is said to be a *neighborhood* of x if $x \notin u(G-X)$. A *complete system of neighborhoods* of x is any system $\mathcal{T}(x)$ of neighborhoods of x such that for each neighborhood X of x there exists $Y \in \mathcal{T}(x)$ with $Y \subseteq X$. The topology u is uniquely determined by complete systems of neighborhoods $\mathcal{T}(x)$ of all $x \in G$ because for any subset $Z \subseteq G$ and any point $x \in G$ we have $x \in uZ \Leftrightarrow Z \cap X \neq \emptyset$ for every $X \in \mathcal{T}(x)$.

According to [3], for a topology u on a set G we consider the following two axioms:

$$X \subseteq G \Rightarrow uuX = uX \qquad F\text{-}axiom,$$

$$X, Y \subseteq G \Rightarrow u(X \cup Y) = uX \cup uY \qquad A\text{-}axiom.$$

Next, for any cardinal $n > 1$ we introduce another axiom:

$$X \subseteq G \Rightarrow uX = \bigcup\{uA \mid A \subseteq X, \operatorname{card} A < n\} \qquad S_n\text{-}axiom.$$

A topology fulfilling a λ-axiom ($\lambda \in \{F, A, S_n\}$) is said to be a *$\lambda$-topology*, a topology fulfilling both a λ-axiom and a μ-axiom is said to be a *$\lambda\mu$-topology*. The topologies of Bourbaki [2], most frequently understood under topologies in literature, are exactly the FA-topologies. In [4], S_2-topologies are called *quasi-discrete closure operations* and it is shown that they coincide with reflexive binary relations. Of course, any S_2-topology is an A-topology. Next, any S_n-topology is an S_m-topology whenever $m > n$. Since any topology u on a set G is obviously an S_n-topology for each cardinal n with $n > \operatorname{card} G$, there exists a least cardinal n for which u is an S_n-topology. In [3] this cardinal is mentioned as an important invariant of the topology u. Clearly, if $n \leqslant \omega_0$, then any AS_n-topology is an S_2-topology. Finally, let us note that we need the axiom of choice whenever we consider the S_n-axiom with $n > \omega_1$.

Lemma 1. *Let $n > 1$ be a cardinal and let u be an S_n-topology on a set G. For any $x \in G$ put $\mathcal{T}(x) = \{X \subseteq G \mid$ for each subset $A \subseteq G$ fulfilling $\operatorname{card} A < n$ and $x \in uA$ there exists a non-empty subset $X_A \subseteq A$ such that $X = \bigcup\{X_A \mid A \subseteq G, \operatorname{card} A < n, x \in uA\}\}$. Then $\mathcal{T}(x)$ is a complete system of neighborhoods of x in (G, u).*

Proof. As u is an S_n-topology, a subset $X \subseteq G$ is a neighborhood of x in (G, u) iff $x \in uA \Rightarrow A \cap X \neq \emptyset$ for each subset $A \subseteq G$ with $\operatorname{card} A < n$. Therefore $\mathcal{T}(x)$ is a system of neighborhoods of x in (G, u). Let $Y \subseteq G$ be an arbitrary neighborhood of x in (G, u). For any subset $A \subseteq G$ with $\operatorname{card} A < n$ and $x \in uA$ put $X_A = A \cap Y$ ($\neq \emptyset$) and $Z = \bigcup\{X_A \mid A \subseteq G, \operatorname{card} A < n, x \in uA\}$. Then clearly $Z \subseteq Y$ and $Z \in \mathcal{T}(x)$. The lemma is proved. $\square$

For any two topologies u, v on a set G we put $u \leqslant v$ iff $uX \subseteq vX$ for each $X \subseteq G$. If (G, u), (H, v) are topological spaces, then a *continuous mapping* of (G, u)

into (H, v) is any mapping $f: G \to H$ fulfilling $f(uX) \subseteq vf(X)$ whenever $X \subseteq G$. By Top we denote the category of topological spaces with continuous mappings as morphisms.

2. Topologies associated with relations of type α

Throughout this section, α denotes an ordinal with $\alpha > 1$, and $|\alpha|$ denotes the least cardinal fulfilling $|\alpha| \geqslant \alpha$.

Let R be a relation of type α on a set G. Then for any subset $X \subseteq G$ we put

$$u_R X = X \cup \{x \in G \mid \text{ there exist } (x_i \mid i < \alpha) \in R \text{ and an ordinal } i_0,\\ 0 < i_0 < \alpha, \text{ such that } x = x_{i_0} \text{ and } x_i \in X \text{ for all } i < i_0\}.$$

Clearly, u_R is a topology on G. For relations R of type 2 the topologies u_R coincide with those dealt with in [8]. In the next section we shall study topologies u_R for relations of the particular type 3.

For any object (G, R) of Rel_α we put $F_\alpha(G, R) = (G, u_R)$ and for any morphism f in Rel_α we put $F_\alpha f = f$.

Theorem 1. *F_α is a faithful (covariant) functor from Rel_α into Top.*

Proof. Let (G, R), (H, S) be relational systems of type α and let $f: G \to H$ be a homomorphism of (G, R) into (H, S). Let $X \subseteq G$ be a subset and let $y \in f(u_R X)$. Then there exists $x \in u_R X$ with $y = f(x)$. If $x \in X$, then $y \in f(X) \subseteq u_S f(X)$. Suppose $x \notin X$. Then there exist $(x_i \mid i < \alpha) \in R$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Consequently, there exist $\big(f(x_i) \mid i < \alpha\big) \in S$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $y = f(x_{i_0})$ and $f(x_i) \in f(X)$ for all $i < i_0$. Hence $y \in u_S f(X)$ and the inclusion $f(u_R X) \subseteq u_S f(X)$ is valid. Therefore f is a continuous mapping of the space (G, u_R) into (H, u_S). Thus F_α is a covariant functor from Rel_α into Top and it is evident that F_α is faithful. □

Proposition 1. *Let R be a relation of type α and let $n = |\alpha|$. Then u_R is an S_n-topology.*

Proof. Let G be the underlying set of R and let $X \subseteq G$ be a subset. Let $x \in u_R X$ be a point. If $x \in X$, then $x \in u_R\{x\} \subseteq \bigcup\{u_R A \mid A \subseteq X, \operatorname{card} A < n\}$. Suppose $x \notin X$. Then there exist $(x_i \mid i < \alpha) \in R$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Hence $\{x_i \mid i < i_0\} \subseteq X$ and $\operatorname{card}\{x_i \mid i < i_0\} \leqslant \operatorname{card} i_0$. But $\operatorname{card} i_0 < \alpha = n$ whenever α is a cardinal and $\operatorname{card} i_0 \leqslant \operatorname{card} \alpha < n$ whenever α is not a cardinal. Since $x \in u_R\{x_i \mid i < i_0\}$, we have $x \in \bigcup\{u_R A \mid A \subseteq X, \operatorname{card} A < n\}$. Therefore the inclusion $u_R X \subseteq \bigcup\{u_R A \mid A \subseteq X, \operatorname{card} A < n\}$ is valid and the proof is complete. □

By virtue of Proposition 1 one can use Lemma 1 for defining u_R by determining complete systems of neighborhoods of all points of G.

For any topology u on a set G we denote by R_u the relation of type α on G defined as follows:

$$(x_i \mid i < \alpha) \in R_u \Leftrightarrow x_{i_0} \in u\{x_i \mid i < i_0\} \text{ for any } i_0 \text{ with } 0 < i_0 < \alpha.$$

Obviously, for each relation S of type α we have $S \subseteq R_{u_S}$.

Proposition 2. *For any topology v we have $u_{R_v} \leqslant v$.*

Proof. Let v be a topology on a set G, let $X \subseteq G$ be a subset and $x \in u_{R_v}X$ a point. If $x \in X$, then $x \in vX$. Suppose $x \notin X$. Then there exist $(x_i \mid i < \alpha) \in R_v$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Consequently, $x \in v\{x_i \mid i < i_0\} \subseteq vX$. Thus $u_{R_v}X \subseteq vX$ and the assertion is proved. □

Theorem 2. *Let v be an S_n-topology on a set G where $n = |\alpha|$. Then the following conditions are equivalent:*

(i) $v = u_{R_v}$,

(ii) *there exists a relation R of type α on G such that $v = u_R$,*

(iii) *if $\alpha > 2$ and $X \subseteq G$ is a subset with $1 < \operatorname{card} X = m < n$, then for any point $x \in vX - \bigcup\{vA \mid A \subseteq X, \operatorname{card} A < m\}$ there exists a subset $Y \subseteq X$ with $\operatorname{card} Y = m$ such that both $x \in vY$ and there exists a bijection $p\colon m \to Y$ with the property $p(j) \in v\{p(i) \mid i < j\}$ whenever $0 < j < m$.*

Proof. For $\alpha = 2$ the assertion states that $v = u_{R_v}$, which is well known—see [4]. Suppose $\alpha > 2$.

The implication (i) $\Rightarrow$ (ii) is trivial.

Let the condition (ii) be true. Let $X \subseteq G$ be a subset with $1 < \operatorname{card} X = m < n$ and let $x \in vX - \bigcup\{vA \mid A \subseteq X, \operatorname{card} A < m\}$. Then obviously $x \notin X$. Thus $x \in u_R X - X$ and $x \notin u_R A$ for any subset $A \subseteq X$ with $\operatorname{card} A < m$. Consequently, there exist $(x_i \mid i < \alpha) \in R$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $x = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Put $Y = \{x_i \mid i < i_0\}$. Then $x \in vY$ and since $Y \subseteq X$, we have $\operatorname{card} Y = m$ and $x_j \in v\{x_i \mid i < j\}$ whenever $0 < j < i_0$. Put $p(0) = x_0$ and for each ordinal j with $0 < j < m$, having defined $p(i)$ for all $i < j$, put $p(j) = x_k$ where $k < i_0$ is the least ordinal such that $x_k \neq p(i)$ for all $i < j$. Then clearly $p : m \to Y$ is a bijection with the property $p(j) \in v\{p(i) \mid i < j\}$ whenever $0 < j < m$. The implication (ii) $\Rightarrow$ (iii) is proved.

Let (iii) be true. Let $X \subseteq G$ be a set with $1 < \operatorname{card} X < n$ and let $y \in vX$. If there exists a point $x \in X$ such that $y \in v\{x\}$, then putting $x_0 = x$ and $x_i = y$

for all ordinals i with $0 < i < \alpha$ we get $(x_i \mid i < \alpha) \in R_v$. Hence $y \in u_{R_v}\{x\} \subseteq u_{R_v}X$. Suppose $y \notin v\{x\}$ for every $x \in X$. Let $Y \subseteq X$ be a subset with the minimal cardinality such that $y \in vY$. Denote $\operatorname{card} Y = m$. Since $y \notin \bigcup\{vA \mid A \subseteq Y, \operatorname{card} A < m\}$, there exists a subset $Z \subseteq Y$ with $\operatorname{card} Z = m$ such that both $y \in vZ$ and there exists a bijection $p\colon m \to Z$ with the property $p(j) = v\{p(i) \mid i < j\}$ whenever $0 < j < m$. For each $i < m$ put $x_i = p(i)$ and for each i with $m \leqslant i < \alpha$ put $x_i = y$. Then $(x_i \mid i < \alpha) \in R_v$ and consequently $y \in u_{R_v}\{p(i) \mid i < m\} = u_{R_v}Z \subseteq u_{R_v}X$. We have proved the inclusion $vX \subseteq u_{R_v}X$. Thus, as v and u_{R_v} are S_n-topologies, we have $v \leqslant u_{R_v}$. Since the inverse inclusion is valid according to Proposition 2, the equality $v = u_{R_v}$ is true. Therefore (iii) $\Rightarrow$ (i) and the proof is complete. □

Theorem 2 immediately results in

Corollary 1. *Let S, T be relations of type α on a given set. Then $u_S \neq u_T$ iff $R_{u_S} \neq R_{u_T}$.*

The following proposition is obvious:

Proposition 3. *Let R be a relation of type α. Then u_R is an S_2-topology iff for every $(x_i \mid i < \alpha) \in R$ the following condition is true:*

for any ordinal i_1, $1 < i_1 < \alpha$, with the property that $x_i \neq x_{i_1}$ for all $i < i_1$ there exist $(y_j \mid j < \alpha) \in R$ and ordinals i_0, j_0, $i_0 < i_1$, $0 < j_0 < \alpha$, such that $y_{j_0} = x_{i_1}$ and $y_j = x_{i_0}$ for all $j < j_0$.

Proposition 4. *Let R be a relation of type α and let $\alpha \leqslant \omega_0$. If u_R is an F-topology, then u_R is an S_2-topology.*

Proof. Let G be the underlying set of R and let $X \subseteq G$ be a subset with $\operatorname{card} X < \alpha$. We are to prove that $u_R X \subseteq \bigcup_{x \in X} u_R\{x\}$. To this end, let $y \in u_R X$ be a point. If $y \in X$, then $y \in u_R\{y\} \subseteq \bigcup_{x \in X} u_R\{x\}$. Let $y \notin X$. Then there exist $(x_i \mid i < \alpha) \in R$ and an ordinal i_0, $0 < i_0 < \alpha$, such that $y = x_{i_0}$ and $x_i \in X$ for all $i < i_0$. Consequently, $y \in u_R\{x_0, \dots, x_{i_0-1}\}$ and $\{x_0, \dots, x_j\} \subseteq u_R\{x_0, \dots, x_{j-1}\}$ whenever $1 \leqslant j \leqslant i_0 - 1$. Hence, we have $y \in u_R\{x_0, \dots, x_{i_0-1}\} \subseteq u_R u_R\{x_0, \dots, x_{i_0-2}\} = u_R\{x_0, \dots, x_{i_0-2}\} \subseteq u_R u_R\{x_0, \dots, x_{i_0-3}\} = u_R\{x_0, \dots, x_{i_0-3}\} \subseteq \dots = u_R\{x_0\}$. Thus $y \in \bigcup_{x \in X} u_R\{x\}$ and the inclusion $u_R X \subseteq \bigcup_{x \in X} u_R\{x\}$ is proved. □

3. Topologies associated with ternary relations

In this section we further investigate topologies associated with relations of type α for the special case $\alpha = 3$, i.e. for ternary relations. Clearly, for any ternary relation R on a set G the S_3-topology u_R on G is given by

$$X \subseteq G \Rightarrow u_R X = X \cup \{z \in G \mid \exists x \in X, y \in G\colon (x, z, y) \in R \text{ or } \exists x,\\ y \in X\colon (x, y, z) \in R\}.$$

In [6] and [7] some properties of ternary relations are introduced and subsequently used such as symmetry, asymmetry, antisymmetry, transitivity and cyclicity. However, among them only the cyclicity is useful for the study of topologies associated with ternary relations. For our purpose we introduce several other properties of ternary relations:

A ternary relation R on a set G will be called

irreflexive if $(x, y, z) \in R \Rightarrow y \neq x \neq z$,

irreversible if $(x, y, z) \in R \Rightarrow (y, x, p) \notin R$ for any $p \in G$,

feebly regular if $(x, y, p) \in R$, $(y, z, q) \in R \Rightarrow (x, y, z) \in R$,

regular if R is feebly regular and $(x, y, p) \in R$, $(x, z, q) \in R \Rightarrow (x, y, z) \in R$,

feebly translative if $(x, y, z) \in R$, $(y, p, q) \in R \Rightarrow \exists r \in G\colon (x, p, r) \in R$,

translative if R is feebly translative and $(x, y, z) \in R \Rightarrow \exists r \in G\colon (x, z, r) \in R$ or $(y, z, r) \in R$,

cyclic if $(x, y, z) \in R \Rightarrow (y, z, x) \in R$.

Theorem 3. *Let (G, R), (H, S) be ternary relational systems and $f\colon G \to H$ a continuous injective mapping of the space (G, u_R) into (H, u_S). If R is irreflexive and S is both irreflexive and regular, then f is a homomorphism of (G, R) into (H, S).*

Proof. Let $(x, y, z) \in R$. Then $y \in u_R\{x\}$ and $z \in u_R\{x, y\}$. Hence $f(y) \in u_S\{f(x)\}$ and $f(z) \in u_S\{f(x), f(y)\}$. As R is irreflexive and f is injective, we have $f(y) \neq f(x) \neq f(z)$. Thus, by virtue of the irreflexivity of S, it follows from $f(y) \in u_S\{f(x)\}$ that there exists $p \in H$ with $\big(f(x), f(y), p\big) \in S$. If $f(y) = f(z)$, then the regularity of S implies $\big(f(x), f(y), f(z)\big) \in S$. Suppose $f(y) \neq f(z)$. As S is irreflexive, it follows from $f(z) \in u_S\{f(x), f(y)\}$ that $\big(f(x), f(y), f(z)\big) \in S$ or $\big(f(y), f(x), f(z)\big) \in S$ or there exists $q \in H$ such that $\big(f(x), f(z), q\big) \in S$ or there exists $r \in H$ such that $\big(f(y), f(z), r\big) \in S$. If $\big(f(y), f(x), f(z)\big) \in S$, then the regularity of S yields $\big(f(x), f(y), f(z)\big) \in S$, which is a contradiction with the irreflexivity of S. In both cases $\big(f(x), f(z), q\big) \in S$ and $\big(f(y), f(z), r\big) \in S$ the regularity of S results in $\big(f(x), f(y), f(z)\big) \in S$. Thus, $\big(f(x), f(y), f(z)\big) \in S$ always holds and the statement is proved. □

Corollary 2. *Let R, S be ternary relations on a given set. If R is irreflexive and S is both irreflexive and regular, then the equivalence $R \subseteq S \Leftrightarrow u_R \leqslant u_S$ is valid.*

Proof. Let G be a set and let α be an ordinal. Then clearly $\subseteq$ and $\leqslant$ are orderings of the fibres of G in Rel_α and Top, respectively, i.e. for any two relations R, S of type α (topologies u, v) on G we have $R \subseteq S$ ($u \leqslant v$) iff the identity mapping of G is a homomorphism (continuous mapping) of (G, R) into (G, S) (of (G, u) into (G, v)). By this fact, the corollary follows from Theorems 1 and 3. □

Corollary 3. *Let R, S be irreflexive and regular ternary relations on a given set. Then $R \neq S$ implies $u_R \neq u_S$.*

Proof. Let G be the underlying set of both R and S and let id_G denote the identity mapping of G. Assume $u_R = u_S$. Then id_G is a continuous mapping of (G, u_R) into (G, u_S) as well as of (G, u_s) into (G, u_R). Thus, by Theorem 3, id_G is a homomorphism of (G, R) into (G, S) as well as of (G, S) into (G, R). Therefore $R \subseteq S$ and $S \subseteq R$, i.e. $R = S$. This proves the statement. □

We denote by $\widetilde{\mathrm{Rel}_3}$ the full subcategory of Rel_3 whose objects are precisely the ternary relational systems (G, R) for which R is both irreflexive and regular. As a consequence of Theorem 1 and Corollary 3 we get

Corollary 4. *The functor F_3 (restricted onto $\widetilde{\mathrm{Rel}_3}$) is an embedding of the category $\widetilde{\mathrm{Rel}_3}$ into* Top.

Next, for any set G we denote by $\mathcal{R}(G)$ and $\mathcal{U}(G)$ the fibres of G in $\widetilde{\mathrm{Rel}_3}$ and Top, respectively, i.e. $\mathcal{R}(G)$ is thee set of all irreflexive and regular ternary relations on G ordered by the set inclusion and $\mathcal{U}(G)$ is the set of all topologies on G ordered by $\leqslant$. Then Corollaries 2 and 3 result in

Corollary 5. *For any set G the correspondence $R \mapsto u_R$ defines an embedding of the ordered set $\mathcal{R}(G)$ into $\mathcal{U}(G)$.*

Remark. By Theorem 2, for any S_3-topology u on a set G there exists a ternary relation R on G such that $u = u_R$ if and only if the following condition is fulfilled:

for any two points $x, y \in G$ the implication $u\{x, y\} \neq u\{x\} \cup u\{y\} \Rightarrow x \in u\{y\}$ or $y \in u\{x\}$ is valid.

Thus, if we denote by Top_3 the full subcategory of Top whose objects pre precisely the topological spaces (G, u) for which u is an S_3-topology fulfilling the condition mentioned above, then Corollary 4 is still valid when Top is replaced by Top_3. Similarly, in Corollary 5 we can consider the fibre $\mathcal{U}(G)$ of G in Top_3 instead of in Top.

E x a m p l e 1. Let ϱ be an asymmetric binary relation on a set G. Let us define $R_\varrho \subseteq G^3$ as follows:

$$(x, y, z) \in R_\varrho \Leftrightarrow x\varrho y \text{ and either } x\varrho z \text{ or } y\varrho z.$$

Then R_ϱ is an irreflexive and regular (and irreversible) ternary relation on G. In particular, if $<$ is a strict order (i.e. an asymmetric and transitive binary relation) on G, then $(x, y, z) \in R_<$ iff $x < y$ and $x < z$, i.e. iff x is a lower bound of the set $\{y, z\}$.

Obviously, if R is a cyclic ternary relation, then u_R is an S_2-topology. Unfortunately, if a non-empty cyclic ternary relation is regular, then it is not irreflexive (because any regular ternary relation R fulfills $(x, y, z) \in R \Leftrightarrow (x, y, y) \in R$). Therefore Corollary 3 can not be applied to cyclic ternary relations. Nevertheless, we have

Proposition 5. *Let R, S be irreflexive, feebly regular and cyclic ternary relations on a given set. Then $R \neq S$ implies $u_R \neq u_S$.*

P r o o f. Let G be the underlying set of both R and S. Assume $R \neq S$. Without loss of generality we can suppose that $R \not\subseteq S$. Then there exists $(x, y, z) \in R$ such that $(x, y, z) \notin S$. Suppose $u_R = u_S$. Since $y \in u_R\{x\}$ and $z \in u_R\{y\}$, we have $y \in u_S\{x\}$ and $z \in u_S\{y\}$. The irreflexivity and cyclicity of R imply $x \neq y \neq z$. Therefore there exist $p, q \in G$ with $(x, y, p) \in S$ and $(y, z, q) \in S$. Thus, by the feeble regularity of S, we have $(x, y, z) \in S$. But this is a contradiction. Hence $u_R \neq u_S$. □

E x a m p l e 2. Let $G = \{x, y, z, p, q\}$ and let $R \subseteq G^3$ be defined as follows: $R = \{(x, y, z), (x, y, p), (y, p, q), (y, z, q)\}$. Denote by R^c the cyclic hull of R, i.e. the least (w.r.t. the set inclusion) cyclic ternary relation on G fulfilling $R \subseteq R^c$. Then R^c is an irreflexive, feebly regular and cyclic ternary relation on G.

Theorem 4. *Let R be a ternary relation. If R is translative, then u_R is an F-topology.*

P r o o f. Let G be the underlying set of R and let R be translative. As u_R is evidently and S_2-topology, u_R is an F-topology iff $u_Ru_R\{x\} \subseteq u_R\{x\}$ holds for any $x \in G$. Let $x \in G$, $y \in u_Ru_R\{x\}$ and suppose $y \notin u_R\{x\}$. Then there exists $p \in u_R\{x\}$ with $y \in u_R\{p\}$. Because $x \neq p \neq y$, we have $(x, x, p) \in R$ or there exists $q \in G$ such that $(x, p, q) \in R$, and either $(p, p, y) \in R$ holds or there exists $r \in G$ such that $(p, y, r) \in R$. By virtue of the translativity of R there always exists $s \in G$ with $(x, y, s) \in R$. Consequently, $y \in u_R\{x\}$, which is a contradiction. Therefore $u_Ru_R\{x\} \subseteq u_R\{x\}$ and the proof is complete. □

E x a m p l e 3. Let ϱ be a preorder on a set G. Let us define a ternary relation R on G as follows:

$$(x, y, z) \in R \Leftrightarrow x\varrho y, y\varrho z.$$

Then R is translative.

Obviously, a cyclic ternary relation is translative iff it is feebly translative. Thus, Theorem 4 results in

Corollary 6. *Let R be a cyclic ternary relation. If R is feebly translative, then u_R is an F-topology.*

E x a m p l e 4. Let $G = \{x, y, z, p\}$ and let $R = \{(x, y, z), (x, z, p), (x, p, y), (y, p, z), (x, x, x,), (y, y, y), (z, z, z), (p, p, p)\}$. Then R^c (see Example 2) is a feebly translative cyclic ternary relation on G.

Proposition 6. *Let R be an irreflexive and irreversible ternary relation. If u_R is an F-topology, then R is translative.*

P r o o f. Let G be the underlying set of R and let u_R be an F-topology. Let $(x, y, z) \in R$, $(y, p, q) \in R$. Then $p \in u_R\{y\}$ and $y \in u_R\{x\}$, hence $p \in u_R u_R\{x\} = u_R\{x\}$. Since R is irreversible, we have $p \neq x$. Thus, by the irreflexivity of R, there exists $r \in G$ with $(x, p, r) \in R$. By Proposition 3, u_R is an S_2-topology. Therefore it follows from $(x, y, z) \in R$ that $z \in u_R\{x\}$ or $z \in u_R\{y\}$. If $y = z$, then $(x, z, z) \in R$. Suppose $y \neq z$. Because $x \neq z$ and R is irreflexive, there exists $r \in G$ with $(x, z, z) \in R$ or $(y, z, z) \in R$. Hence R is translative. □

References

[1] *Bar-Hillel, Y., Fraenkel, A. A., and Lévy, A.*: Foundations of Set Theory, North Holand, Amsterdam, 1973.

[2] *Bourbaki, N.*: Topologie générale, Eléments de Mathématique, I. part., livre III, Paris, 1940.

[3] *Čech, E.*: Topological spaces, Topological papers of Eduard Čech, Academia, Prague, 1968.

[4] *Čech, E.*: Topological Spaces (Revised by Z. Frolík and M. Katětov), Academia, Prague, 1966.

[5] *Mac Lane, S.*: Categories for the Working Mathematician, Springer-Verlag, New York-Heidelberg-Berlin, 1971.

[6] *Novák, V., and Novotný, M.*: Transitive ternary relations and quasiorderings, Arch. Math. Brno *25* (1989), 5–12.

[7] *Novotný, M.*: Ternary structures and grupoids, Czech. Math. J. *41* (1991), 90–98.

[8] *Šlapal, J.*: On closure operations induced by binary relations, Rev. Roum. Math. Pures et Appl. *33* (1988), 623–630.

[9] *Šlapal, J.*: Relations of type α, Zeitschr. für Math. Logik und Grundl. der Math. *34* (1988), 563–573.

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