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STRICT COMPLETIONS OF $\mathscr{L}_0^*$-GROUPS

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Since an $\mathscr{L}_0^*$-group can have several nonequivalent completions (see [12], [2], [3]), it is natural to develop a classification of $\mathscr{L}_0^*$-group completions which would enable us to select completions having required properties. To this end we introduce the notion of a strict $\mathscr{L}_0^*$-completion. Answering a question asked by J. Novák in [13] about the number of closure dense $\mathscr{L}_0^*$-group completions of the $\mathscr{L}_0^*$-group $\mathbf{Q}$ of rational numbers, we prove that $\mathbf{Q}$ has exactly $\exp(\exp(\omega))$ strict completions. For every $\mathscr{L}_0^*$-group $\mathbf{G}$, its Novák completion $\nu\mathbf{G}$ is the finest strict completion of $\mathbf{G}$. We show that the $\mathscr{L}_0^*$-group $\mathbf{R}$ of real numbers is the coarsest strict completion of $\mathbf{Q}$, and give a sufficient condition for an $\mathscr{L}_0^*$-group to have the coarsest strict completion.

We consider only abelian groups but the notion of a strict completion can be applied to the nonabelian case and also to other $\mathscr{L}_0^*$-algebras, e.g. to $\mathscr{L}_0^*$-rings.

By an $\mathscr{L}_0^*$-group $\mathbf{G} = (G, \mathsf{L}, +)$ we understand a group $(G, +)$ equipped with a sequential convergence $\mathsf{L} \subset G^N \times G$ satisfying all four axioms of convergence (concerning constants, subsequences, uniqueness of limits and the Urysohn axiom) and compatible with the group structure (if sequences $\langle x_n\rangle$ and $\langle y_n\rangle$ converge to x and y, respectively, then their difference $\langle x_n - y_n\rangle$ converges to $x - y$). Denote by NIN the set of all mappings of N into N and by MON its subset of all strictly monotone mappings; for $f, g \in \mathrm{NIN}$, $f > g$ means that $f(n) > g(n)$ for all $n \in N$. If $S = \langle S(n)\rangle$ is a sequence and $s \in \mathrm{MON}$, then $S \circ s$ denotes the subsequence of S the n-th term of which is $S(s(n))$. We say that S is Cauchy if for each $s, t \in \mathrm{MON}$ the sequence $S \circ s - S \circ t = \langle S(s(n)) - S(t(n))\rangle$ converges to 0. Sometimes we start with an $\mathscr{L}_0$-group $(G, \mathsf{L}, +)$, i.e. L need not satisfy the Urysohn axiom, and then pass to the $\mathscr{L}_0^*$-group $(G, \mathsf{L}^*, +)$, where L^* is the Urysohn modification of L (cf. [2], [5]). For every abelian $\mathscr{L}_0^*$-group $\mathbf{G}$ its Novák completion $\nu\mathbf{G}$ is the categorical one ([2]), and there are nonabelian $\mathscr{L}_0^*$-groups having no (two-sided) completion ([4]). Further information on sequential convergence spaces and groups can be found, e.g., in [3], [8], [9] and the references therein.

The notion a precompletion has been introduced by the authors in [6] in connection with $\mathscr{L}_0^*$-group convergences relatively coarse with respect to a subgroup. In the present paper it will be used for constructing $\mathscr{L}_0^*$-group completions.

Definition 0.1. Let $\mathbf{G} = (G, \mathsf{L}, +)$ and $\mathbf{H} = (H, \mathsf{L}_H, +)$ be $\mathscr{L}_0^*$-groups such that:

(i) G is a topologically dense (iterated closure) subgroup of H and $\mathsf{L}_H \restriction G = \{(S, x) \in \mathsf{L}_H\,;\ S \in G^N,\ x \in G\} = \mathsf{L}$;
(ii) each Cauchy sequence in $\mathbf{G}$ convergences in $\mathbf{H}$.
Then $\mathbf{H}$ is said to be a *precompletion* of $\mathbf{G}$. If G is closure dense (first closure) in H, then we speak of a *dense precompletion*.

Remark 0.2. Let $\mathbf{G} = (G, \mathsf{L}, +)$ be an $\mathscr{L}_0^*$-group, $\nu\mathbf{G} = (G_1, \mathsf{L}_1^*, +)$ its Novák completion and let $\mathbf{H} = (H, \mathsf{L}_H, +)$ be a dense precompletion of $\mathbf{G}$. As pointed out in [6], G_1 and H are isomorphic with G pointwise fixed and the isomorphism as mapping of $\nu\mathbf{G}$ onto $\mathbf{H}$ is continuous. Thus H can be identified with G_1 and L_H can be considered as an $\mathscr{L}_0^*$-group convergence for G_1 coarser that L_1^*.

1. Strict extensions

Besides the fact that an $\mathscr{L}_0^*$-group can have several nonequivalent $\mathscr{L}_0^*$-group completions, the convergence in the growth (ideal points) of a completion can be quite independent of the convergence in the original group. The next example shows that this is true even for $\mathbf{Q}$.

Example 1.1. Consider the $\mathscr{L}_0^*$-group $\mathbf{Q} = (Q, \mathsf{L}, +)$ of rational numbers and its Novák completion $\nu\mathbf{Q} = (R, \mathsf{L}_1^*, +)$. Recall that L_1^* ia an $\mathscr{L}_0^*$-group convergence strictly finer that the usual metric convergence L_m for R. If a sequence L_1^*-converges to 0, then it contains a subsequence of the form $\langle r - r_n \rangle$, where $\langle r_n \rangle$ is an L-Cauchy sequence of rational numbers L_m-converging to r (cf. [12]).

Let B be a Hamel basis of R over Q such that $1 < b < 2$ for all $b \in B$ and let $S = \langle b_n \rangle$ be a one-to-one sequence in B. We show that there is an $\mathscr{L}_0^*$-group convergence L_c for R such that $\mathsf{L}_1^* \subset \mathsf{L}_c$, $(R, \mathsf{L}_c, +)$ is a completion of $\mathbf{Q}$ and $(\langle b_n \rangle, 0) \in \mathsf{L}_c$.

First, we show that the sequence $\langle b_n \rangle$ is L_1^*-free at 0 (cf. [1]), i.e., we construct an $\mathscr{L}_0^*$-group convergence for R containing L_1^* in which $\langle b_n \rangle$ converges to 0. In fact, we construct the smallest one of all such convergences. Let us denote it by L_S. Indeed (cf. Theorem 3.3 in [5]), L_S can be constructed via the group $\mathsf{L}_S^{\leftarrow}(0)$ of all sequences L_S-converging to 0. It is a specific subgroup of R^N containing $\langle b_n \rangle$ and all sequences L_1^*-converging to 0. The only nontrivial part of the construction is to show that L_S has unique limits. Accordingly, it suffices to prove that whenever

$k \in N$, $z_i \in Z \setminus \{0\}$, $s_i \in \mathrm{MON}$, $i = 1, \ldots, k$, $r \in R$, $\langle r_n \rangle \in Q^N$, $(\langle r_n \rangle, r) \in \mathsf{L}_m$, then no sequence T of the form $T = \left\langle r - r_n + \sum\limits_{i=1}^{k} z_i(S \circ s_i)(n) \right\rangle$ is a constant nonzero sequence. Contrariwise, suppose that for some T there is an $a \in R$, $a \neq 0$, such that $-r_n + \sum\limits_{i=1}^{k} z_i(S \circ s_i)(n) = -r + a$ for all $n \in N$. Without loss of generality we can assume that for all $n \in N$ we have $(S \circ s_i)(n) \neq (S \circ s_j)(n)$ whenever $i \neq j$ and the sets $\{(S \circ s_1)(m), \ldots, (S \circ s_k)(m)\}$ and $\{(S \circ s_1)(n), \ldots, (S \circ s_k)(n)\}$ are disjoint whenever $m \neq n$ (otherwise we can replace T by some of its subsequences having the desired properties). Now, the sequence $\langle r_n \rangle$ (of rational numbers) contains either a constant subsequence or a one-to-one subsequence. Since $\{b_n; n \in N\}$ are Q-linearly independent, this is clearly a contradiction.

Secondly, it follows from the construction of L_S that no unbounded sequence of real numbers L_S-converges. Since each bounded sequence of rational numbers contains a subsequence L_1^*-converging to some $r \in R$, we have $\mathsf{L}_S \restriction Q = \mathsf{L}$.

Thirdly, as shown in [6], there is an $\mathscr{L}_0^*$-group convergence L_c for R such that $\mathsf{L}_S \subset \mathsf{L}_c$ and L_c is coarse with respect to L (i.e. if L' is an $\mathscr{L}_0^*$-group convergence for R such that $\mathsf{L}_c \subset \mathsf{L}'$ and $\mathsf{L}' \restriction Q = \mathsf{L}$, then $\mathsf{L}_c = \mathsf{L}'$).

Finally, in [6] it has been proved that the divisibility of $\mathbf{R}$ implies that L_c is complete. Thus $(R, \mathsf{L}_c, +)$ is a completion of $\mathbf{Q}$.

In the realm of filter convergence structures (see e.g. [7]) there is a natural way to control the convergence in the growth of an extension by means of the convergence in the ground space. Namely, let X be a dense subspace of Y. For $A \subset Y$ denote by $\hat{A}$ the adherence of A in Y. Then Y is said to be a strict extension of X if the following implication holds true: if a filter $\mathscr{F}$ converges in Y to a point y, then there is a filter $\mathscr{G}$ converging in Y to y such that $X \in \mathscr{G}$ and $\mathscr{F}$ is finer that the filter $\hat{\mathscr{G}}$ generated by $\{\hat{G}; G \in \mathscr{G}\}$.

There are several possibilities how to adopt strictness to $\mathscr{L}$-structures. The following one seems to suit our purpose, namely, to control the convergence in the growth from "below".

Definition 1.2. Let $\mathbf{Y} = (Y, \mathsf{L})$ be an $\mathscr{L}_0^*$-space and let X be a topologically dense subset of Y. We say that $\mathbf{Y}$ is a *strict extension* of $\mathbf{X} = (X, \mathsf{L} \restriction X)$ if the following holds:

(s) Let $(\langle y_n \rangle, y) \in \mathsf{L}$, $y_n \in Y \setminus X$, $n \in N$. Then there are a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ and sequences S_k in X, $k \in N$, such that $(\langle S_k(n) \rangle, y'_k) \in \mathsf{L}$, $k \in N$, and for each $g \in \mathrm{NIN}$ we have $(\langle S_n(g(n)) \rangle, y) \in \mathsf{L}$.

If $\mathbf{X}$ and $\mathbf{Y}$ are $\mathscr{L}_0^*$-groups ($\mathscr{L}_0^*$-rings, etc), and $\mathbf{Y}$ is a precompletion, resp. completion, of $\mathbf{X}$ such that (s) holds, then $\mathbf{Y}$ is said to be a *strict precompletion*, resp. *completion*, of $\mathbf{X}$.

Remark 1.3. Observe that if $\mathbf{Y}$ is a strict extension of $\mathbf{X}$, then X is closure dense in Y. In particular, every strict precompletion is a dense precompletion. If X is not closure dense in Y, then the notion of strictness has to be modified. Natural examples of completions in which the original group is not closure dense in the completion are spaces of functions equipped with the pointwise convergence, e.g., continuous functions in the space of all Borel measurable functions. It might be interesting to investigate strictness of such extensions.

Let $\mathbf{Q} = (Q, \mathsf{L}, +)$ and $\mathbf{R} = (R, \mathsf{L}_m, +)$ be the $\mathscr{L}_0^*$-groups of rational numbers and real numbers, respectively. Let $\nu\mathbf{Q} = (R, \mathsf{L}_1^*, +)$ be the Novák completion of $\mathbf{Q}$. The proof of the next proposition is straightforward and therefore it is omitted.

Proposition 1.4. (i) *Let L_g be an $\mathscr{L}_0^*$-group convergence for R such that $\mathsf{L}_1^* \subset \mathsf{L}_g \subset \mathsf{L}_m$. Then $(R, \mathsf{L}_g, +)$ is a strict precompletion of $\mathbf{Q}$.*

(ii) *Up to a homeomorphic isomorphism pointwise fixed on Q, every strict precompletion of Q is of the form $(R, \mathsf{L}_g, +)$, where L_g is an $\mathscr{L}_0^*$-group convergence such that $\mathsf{L}_1^* \subset \mathsf{L}_g \subset \mathsf{L}_m$.*

(iii) *Every Fréchet precompletion of $\mathbf{Q}$ is strict.*

Remark 1.5. In [6] it is proved that $\mathbf{Q}$ has a unique (up to a homeomorphic isomorphism pointwise fixed on Q) Fréchet precompletion, viz. $\mathbf{R}$.

2. Strict completions of rationals

This section is devoted to the following question. How many nonequivalent strict completions does $\mathbf{Q}$ possess? Besides $\nu\mathbf{Q}$ and $\mathbf{R}$, the categorical $\mathscr{L}_0^*$-ring completion $\varrho\mathbf{Q} = (R, \mathsf{L}_2^*, +, .)$ of $\mathbf{Q}$ constructed in [10] is a strict $\mathscr{L}_0^*$-group completion of $\mathbf{Q}$ (this follows by Proposition 1.4, since $\mathsf{L}_1^* \subset \mathsf{L}_2^* \subset \mathsf{L}_m$). Let $\{1\} \cup B$ be a Hamel basis of the linear space R over Q. For each $A \subset B$ we construct a strict $\mathscr{L}_0^*$-group completion $\varrho_A\mathbf{Q} = (R, \mathsf{L}_A^*, +)$ of $\mathbf{Q}$ such that $\varrho_\emptyset\mathbf{Q} = \nu\mathbf{Q}$, $\varrho_B\mathbf{Q} = \varrho\mathbf{Q}$ and if $C, D \subset B$, $C \neq D$, then $\varrho_C\mathbf{Q}$ and $\varrho_D\mathbf{Q}$ are nonequivalent since the identity is not a homeomorphism. Thus the number of nonequivalent strict $\mathscr{L}_0^*$-group completions of $\mathbf{Q}$ is at least $\exp(\exp(\omega))$. Since each $\mathscr{L}_0^*$-group convergence for R is a subset of $R^N \times R$, $\mathbf{Q}$ has exactly $\exp(\exp(\omega))$ strict $\mathscr{L}_0^*$-group completions.

Let us recall (cf. [10]) the construction of $\varrho\mathbf{Q} = (R, \mathsf{L}_2^*, +)$. First, L_2 is the set of all pairs $(S, x) \in R^N \times R$ such that S is of the form $\langle S(n)\rangle = \langle S_0(n) + S_1(n)a_1 + \ldots +$

$S_k(n)a_k\rangle$, where $k \in N$, $a_i \in B$, $i = 1, \ldots, k$, S_i is a Cauchy sequence in $\mathbf{Q}$, $i = 0, 1, \ldots, k$, and $x = x_0 + x_1 a_1 + \ldots + x_k a_k$, where x_i is the L_1-limit of the Cauchy sequence S_i, $i = 0, 1, \ldots, k$. Secondly, L_2^* is the Urysohn modification of L_2.

Further, observe that L_1^* in $\nu\mathbf{Q} = (R, \mathsf{L}_1^*, +)$ is, in fact, a special case of L_2^*. Indeed, if in the definition of L_2 via $\langle S(n)\rangle = \langle S_0(n) + S_1(n)a_1 + \ldots + S_k(n)a_k\rangle$ and $x = x_0 + x_1 a_1 + \ldots + x_k a_x$, all S_i for $i \neq 0$ are constant sequences $\langle x_i\rangle$, $x_i \in Q$, then we get exactly the definition of L_1.

Construction. Let $\{1\} \cup B$ be a Hamel basis of R over Q. Fix $A \subset B$. Define $\mathsf{L}_A \subset R^N \times R$ to be the set of all pairs (S, x) such that S is of the form $\langle S(n)\rangle = \langle S_0(n) + S_1(n)a_1 + \ldots + S_k(n)a_k\rangle$, where $k \in N$, $a_i \in B$, S_0 and S_i are Cauchy sequences in $\mathbf{Q}$ and S_i is a constant sequence whenever $a_i \notin A$, $i = 1, \ldots, k$, and $x = x_0 + x_1 a_1 + \ldots + x_k a_k$, where x_i is the L_1-limit of S_i, $i = 0, 1, \ldots, k$. Let L_A^* be the Urysohn modification of L_A.

Theorem 2.1. *Let $\{1\} \cup B$ be a Hamel basis of R over Q. Let A be a subset of B. Then*

(i) *L_A^* is an $\mathscr{L}_0^*$-group convergence for R and $(R, \mathsf{L}_A^*, +)$ is a strict completion of $\mathbf{Q}$;*

(ii) *if $A = \emptyset$, then $(R, \mathsf{L}_A^*, +)$ is the Novák $\mathscr{L}_0^*$-group completion $\nu\mathbf{Q}$ of $\mathbf{Q}$;*

(iii) *if $A = B$, then $(R, \mathsf{L}_A^*, +)$ is the categorical $\mathscr{L}_0^*$-ring completion $\varrho\mathbf{Q}$ of $\mathbf{Q}$;*

(iv) *if $C, D \subset B$, $C \neq D$, then $(R, \mathsf{L}_C^*, +)$ and $(R, \mathsf{L}_D^*, +)$ are nonequivalent strict completions of $\mathbf{Q}$.*

Proof. As a straightforward consequence of the fact that $\{1\} \cup B$ is a Hamel basis of R over Q it follows that $(R, \mathsf{L}_A^*, +)$, $A \subset B$, is an $\mathscr{L}_0^*$-group completion of $\mathbf{Q}$. Since $\mathsf{L}_1^* \subset \mathsf{L}_A^* \subset \mathsf{L}_m$, by Proposition 1.4 it is a strict completion. This proves (i). Assertions (ii) and (iii) follow by comparing the definitions of L_1^*, L_2^* and L_A^*. Finally, (iv) follows from the fact that if S is a one-to-one Cauchy sequence in $\mathbf{Q}$ and $a \in C \setminus D$, then the sequence $\langle S(n)a\rangle$ L_C^*-converges and fails to L_D^*-converge. $\square$

Corollary 2.2. *There are exactly* $\exp(\exp(\omega))$ *nonequivalent strict $\mathscr{L}_0^*$-group completions of $\mathbf{Q}$.*

This result has been announced also by P. Simon.

Remark 2.3. According to Proposition 1.4, to study nonequivalent strict $\mathscr{L}_0^*$-group precompletions of $\mathbf{Q}$ it suffices to consider the set Γ of all $\mathscr{L}_0^*$-group convergences L_g for R such that $\mathsf{L}_1^* \subset \mathsf{L}_g \subset \mathsf{L}_m$. Observe that Γ partially ordered by inclusion forms a complete lattice.

Problem 2.4. Is $\varrho\mathbf{Q}$ an $\mathscr{L}_0^*$-field completion of $\mathbf{Q}$?

Problem 2.5. (i) Characterize all strict $\mathscr{L}_0^*$-group completions of $\mathbf{Q}$.
(ii) Is every strict $\mathscr{L}_0^*$-group precompletion of $\mathbf{Q}$ compete?

Observe that if there a noncomplete strict precompletion H of $\mathbf{Q}$, then the ideal points of its Novák completion $\nu\mathsf{H}$ (which is a strict completion of H) can be considered as group hyperreal numbers.

3. Regular extensions

In this section we study strict completions of (abelian) $\mathscr{L}_0^*$-group in general. In particular, we show that the Novák completion of an $\mathscr{L}_0^*$-group is strict and hence, up to an equivalence, the finest of all its strict completions. We characterize $\mathscr{L}_0^*$-groups having the coarsest (up to an equivalence) strict precompletion, and give a sufficient condition for the existence of the coarsest strict completion.

Theorem 3.1. *Let $\mathbf{G} = (G, \mathsf{L}, +)$ be an $\mathscr{L}_0^*$-group and let $\nu\mathbf{G} = (G_1, \mathsf{L}_1^*, +)$ be its Novák completion. Then $\nu\mathbf{G}$ is a strict completion of $\mathbf{G}$.*

Proof. Let $(\langle y_n \rangle, y) \in \mathsf{L}_1^*$, $y_n \in G_1 \setminus G$, $n \in N$. Then there is a subsequence $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that $(\langle y'_n \rangle, y) \in \mathsf{L}_1$ and hence (cf. [12]) there is an L-Cauchy sequence $\langle r_n \rangle$ in G L_1^*-converging to a point $r \in G_1$ such that for all $n \in N$ we have $y'_n = y + r - r_n$. Let $\langle q_n \rangle$ be an L-Cauchy sequence in G L_1^*-converging to y. For $n, k \in N$ put $S_k(n) = q_{n+k} + r_{n+k} - r_k$. Since for each $k \in N$ the sequence $\langle S_k(n) \rangle$ L_1^*-converges to y'_k and for each $g \in \mathrm{NIN}$ the sequence $\langle S_n(g(n)) \rangle$ L_1^*-converges to y, $\nu\mathbf{G}$ is a strict completion of $\mathbf{G}$. $\square$

Corollary 3.3. *Let $\mathbf{G}$ be an abelian $\mathscr{L}_0^*$-group and let $\nu\mathbf{G}$ be its Novák completion. Up to an equivalence, $\nu\mathbf{G}$ is the finest of all strict completions of $\mathbf{G}$.*

An analysis of the construction of $\nu\mathbf{G}$ reveals that $\nu\mathbf{G}$ satisfies a strictness condition stronger that (s).

Lemma 3.3. *Let $\mathbf{G} = (G, \mathsf{L}, +)$ be an $\mathscr{L}_0^*$-group and let $\nu\mathbf{G} = (G_1, \mathsf{L}_1^*, +)$ be its Novák completion. Assume that $\langle y_n \rangle$ is a sequence in $G_1 \setminus G$ L_1^*-converging to $y \in G_1$. Then*

(i) there is a finite nonvoid set $\{a_1, \dots, a_k\} \subset G_1$ such that each y_n, $n \in N$, belongs to some coset $G + a_i$, $i = 1, \dots, k$;

(ii) let $i \in \{1, \dots, k\}$. Assume that for some $s \in \mathrm{MON}$ we have $y_{s(n)} \in G + a_i$ for all $n \in N$. Then there are a point $r \in G_1$ and a sequence $\langle r_n \rangle$ in G such that $(\langle r_n \rangle, r) \in \mathsf{L}_1^*$, $a_i = y + r$, and $y_{s(n)} = y + r = r_n$ for all $n \in N$.

Proof. Both assertions (i) and (ii) follow easily from the definition of L_1^* (cf. [12]). □

Proposition 3.4. *Let $\mathbf{G} = (G, \mathsf{L}, +)$ be an $\mathscr{L}_0^*$-group and let $\nu\mathbf{G} = (G_1, \mathsf{L}_1^*, +)$ be its Novák completion. Let $(\langle y_n \rangle, y) \in \mathsf{L}_1^*$, $y_n \in G_1 \setminus G$, $n \in N$. Then there are sequences S_k in G such that $(\langle S_k(n) \rangle, y_k) \in \mathsf{L}_1$, $k \in N$, and for each $g \in \mathrm{NIN}$ we have $(\langle S_n(g(n)) \rangle, y) \in \mathsf{L}_1^*$.*

Proof. According to (i) in Lemma 3.3, there are two possibilities. 1. All y_n, $n \in N$, belong to the same coset $G + a$, $a \in G_1$. In this case we copy the proof of Theorem 3.1 with $y'_n = y_n$. 2. There are $k \in N$, $k > 1$, and $a_i \in G_1$, $i = 1, \dots, k$, such that each y_n belongs to one of the cosets $G + a_i$, $i = 1, \dots, k$. In this case the proof of Theorem 3.1 has to be slightly modified. The details are omitted. □

On the one hand, strictness imposes a restriction on the convergence in the growth of an extension and, on the other hand, various notions of regularity force the convergence to be as coarse as possible.

Definition 3.5. Let $\mathbf{Y} = (Y, \mathsf{L})$ be a strict extension of $\mathbf{X} = (X, \mathsf{L} \restriction X)$ such that the following condition is satisfied:

(r) Let $\langle y_n \rangle$ be a sequence in $Y \setminus X$ and let $y \in Y$. Assume that there is a sequence $\langle S_n \rangle$ of sequences S_n of points of X such that, for each $k \in N$, S_k L-converges to y_k and, for each $g \in \mathrm{NIN}$, the sequence $S_g = \langle S_n(g(n)) \rangle$ L-converges to y. Then $\langle y_n \rangle$ L-converges to y.

Then $\mathbf{Y}$ is said to be a *regular extension* of $\mathbf{X}$. If $\mathbf{Y}$ and $\mathbf{X}$ are $\mathscr{L}_0^*$-groups, $\mathbf{Y}$ is a strict precompletion or completion, of $\mathbf{X}$ such that (r) holds, then $\mathbf{Y}$ is said to be a *regular precompletion*, or *completion*, of $\mathbf{X}$, respectively.

Proposition 3.6. *Let $\mathbf{G} = (G, \mathsf{L}, +)$ be an $\mathscr{L}_0^*$-group and let $\mathbf{H} = (H, \mathsf{L}_H, +)$ be its regular precompletion.*

(i) If $\mathbf{F} = (F, \mathsf{L}_F, +)$ is a strict precompletion of $\mathbf{G}$, then there is a continuous isomorphism of F onto H with G pointwise fixed;

(ii) Up to a homeomorphic isomorphism leaving G pointwise fixed, $\mathbf{H}$ is uniquely determined as the coarsest strict precompletion of $\mathbf{G}$.

Proof. The assertions follow directly from Definition 3.5 and Remark 0.2. □

Our final goal is to find conditions on an $\mathscr{L}_0^*$-group $\mathbf{G} = (G, \mathsf{L}, +)$ guaranteeing the existence of its regular precompletion and regular completion, respectively. By Remark 0.2, to construct a strict precompletion of $\mathbf{G}$ it suffices to construct a suitable $\mathscr{L}_0^*$-group convergence for G_1 coarser than L_1^*. Observe that for topologically dense extensions the notion of an inductive regularity can be introduced in a natural way.

Definition 3.7. Let $\mathbf{G}$ be an $\mathscr{L}_0^*$-group. Let $\mathscr{S} = \langle S_n \rangle$ be a sequence of sequences S_n in $\mathbf{G}$ such that

(i) for each $n \in N$, S_n is a Cauchy sequence;

(ii) for each $g \in \mathrm{NIN}$, $S_g = \langle S_n(g(n)) \rangle$ is a Cauchy sequence and if $g, h \in \mathrm{NIN}$, then the Cauchy sequences S_g and S_h belong to the same equivalence class $v(\mathscr{S})$ of Cauchy sequences in $\mathbf{G}$.

Then $\mathscr{S}$ is said to be a *Cauchy double sequence* and $v(\mathscr{S})$ is said to be the *vertex* of $\mathscr{S}$.

Remark 3.8. Let $\mathbf{G}$ be an $\mathscr{L}_0^*$-group and let $\langle S_n \rangle$ be a sequences S_n in $\mathbf{G}$. Then condition (ii) in Definition 3.7 is satisfied iff for each $g, h \in \mathrm{NIN}$ and for each $v, w \in \mathrm{MON}$ the sequence $\langle S_{v(n)}(g(v(n))) - S_{w(n)}(h(w(n))) \rangle$ converges to 0.

Let $\mathbf{G} = (G, \mathsf{L}, +)$ be an $\mathscr{L}_0^*$-group, let $\nu\mathbf{G} = (G_1, \mathsf{L}_1^*, +)$ be its Novák completion, and let $(G_1, \mathsf{L}_g, +)$, $\mathsf{L}_1^* \subset \mathsf{L}_g$, be a regular precompletion of $\mathbf{G}$. It is easy to see that $\mathbf{G}$ satisfies the following condition:

(rp) If $\mathscr{S}$ and $\mathscr{T}$ are Cauchy double sequences in $\mathbf{G}$ such that, for each $n \in N$, $\mathscr{S}(n)$ and $\mathscr{T}(n)$ are equivalent divergent Cauchy sequences, then $v(\mathscr{S}) = v(\mathscr{T})$.

Using the free group technique, it is not difficult to construct an $\mathscr{L}_0^*$-group not satisfying condition (rp).

Let $\mathbf{G} = (G, \mathsf{L}, +)$ be an $\mathscr{L}_0^*$-group and let $\nu\mathbf{G} = (G_1, \mathsf{L}_1^*, +)$ be its Novák completion. Define $\mathsf{L}_r \subset G_1^N \times G_1$ as follows:

$(S, y) \in \mathsf{L}_r$ if either $S(n) \in G$ for all $n \in N$ and

$(S, y) \in \mathsf{L}_1^*$, or $S(n) \in G_1 \setminus G$ for all $n \in N$ and there is a Cauchy double sequence $\mathscr{S}$ in $\mathbf{G}$ such that for each $k \in N$ the sequence $\mathscr{S}(k)$ L_1^*-converges to $S(k)$ and $v(\mathscr{S}) = y$.

Proposition 3.9. (i) L_r *is an* $\mathscr{L}$*-group convergence for* G_1 *such that* $\mathsf{L}_1^* \subset \mathsf{L}_r$.

(ii) *Let* $\mathbf{G}$ *satisfy condition* (rp). *Then* L_r *is an* $\mathscr{L}_0$*-group convergence for* G_1 *and* $(G, \mathsf{L}_r^*, +)$ *is a regular precompletion of* $\mathbf{G}$.

P r o o f. Both assertions follow easily from the definition of L_r. □

Corollary 3.10. *An* $\mathscr{L}_0^*$*-group* $\mathbf{G}$ *has a regular precompletion iff it satisfies condition* (rp).

Proposition 3.11. *Let* $\mathbf{G}$ *be an* $\mathscr{L}_0^*$*-group. If* $\mathbf{G}$ *is Fréchet, then it has a regular precompletion.*

P r o o f. Let $\mathscr{S}$ and $\mathscr{T}$ be Cauchy double sequences in $\mathbf{G}$ such that, for each $n \in N$, $\mathscr{S}(n)$ and $\mathscr{T}(n)$ are equivalent divergent Cauchy sequences. Since for each $k \in N$ the sequence $U_k = \mathscr{S}(k) - \mathscr{T}(k)$ converges to 0 and $\mathbf{G}$ is a Fréchet group, for each $f \in \mathrm{MON}$ there are $g \in NIN$, $s \in \mathrm{MON}$ such that $g > f$ and the sequence $\langle U_{s(n)}(g(s(n)))\rangle$ converges to 0 (cf. [11]). But $\langle \mathscr{S}(n)(g(n))\rangle$ and $\langle \mathscr{T}(n)(g(n))\rangle$ are Cauchy sequences and $\langle U_{s(n)}(g(s(n)))\rangle = \langle \mathscr{S}(s(n))(g(s(n)))\rangle - \langle \mathscr{T}(s(n))(g(s(n)))\rangle$ implies $v(\mathscr{S}) = v(\mathscr{T})$. Thus $\mathbf{G}$ satisfies condition (rp) and, by Proposition 3.9, has a regular precompletion. □

Let $\mathbf{G}$ be an $\mathscr{L}_0^*$-group. Consider the following conditions:

(cd1) For each $n \in N$ let S_n and T_n be equivalent Cauchy sequences. If $\mathscr{S} = \langle S_n\rangle$ is a Cauchy double sequence, then $\mathscr{T} = \langle T_n\rangle$ is a Cauchy double sequence and $v(\mathscr{S}) = v(\mathscr{T})$.

(cd2) For each $n \in N$ let S_n be a divergent Cauchy sequence. Then either there is a mapping $s \in MON$ such that $\langle S_{s(n)}\rangle$ is as Cauchy double sequence, or there are mappings $s, t \in \mathrm{MON}$ such that for each $w \in \mathrm{MON}$ the sequence $\langle S_{s(w(n))} - S_{t(w(n))}\rangle$ of Cauchy sequences $S_{s(w(n))} - S_{t(w(n))}$ fails to be a Cauchy double sequence.

It is easy to see that condition (cd1) implies condition (rp). Further, (cd1) implies that if a sequence $\langle x_n\rangle$ converges to x and for each $k \in N$ a sequence T_k converges to x_k, then $\langle T_k\rangle$ is Cauchy double sequence and x is its vertex.

Theorem 3.12. *Let* $\mathbf{G} = (G, \mathbf{L}, +)$ *be an* $\mathscr{L}_0^*$*-group. If* $\mathbf{G}$ *satisfies conditions* (cd1) *and* (cd2), *then* $(G_1, \mathbf{L}_r^*, +)$ *is a regular completion of* $\mathbf{G}$.

P r o o f. Since (cd1) implies (rp), it follows from Proposition 3.9 that $(G_1, \mathbf{L}_r^*, +)$ is a regular precompletion of $\mathbf{G}$. We shall prove that it is complete.

Let $\langle y_n\rangle$ be a one-to-one $\mathbf{L}_r^*$-Cauchy sequence such that $y_n \in G_1 \setminus G$ for all $n \in N$. Clearly, it suffices to show that some of its subsequences $\mathbf{L}_r$-converges. For each $n \in N$ let S_n be a Cauchy sequence in $\mathbf{G}$ $\mathbf{L}_r$-converging to y_n. According to (cd2), there are two possibilities.

Case 1. There is a mapping $s \in \mathrm{MON}$ such that $\langle S_{s(n)}\rangle$ is a Cauchy double sequence in $\mathbf{G}$. Then $\langle y_{s(n)}\rangle$ $\mathbf{L}_r$-converges to the vertex of $\langle S_{s(n)}\rangle$.

Case 2. There are mappings $s, t \in \mathrm{MON}$ such that $\langle S_{s(w(n))} - S_{t(w(n))}\rangle$ fails to be a Cauchy double sequence for all $w \in \mathrm{MON}$. Since $\langle y_{s(n)} - y_{t(n)}\rangle$ $\mathbf{L}_r^*$-converges to 0, there exists a mapping $w \in \mathrm{MON}$ such that $\langle y_{s(w(n))} - y_{t(w(n))}\rangle$ $\mathbf{L}_r$-converges to 0. Hence, it follows from condition (cd1) that Case 2 cannot occur. This completes the proof. □

There are some interesting questions concerning the partially ordered set of all strict completions of an $\mathscr{L}_0^*$-group. We mention two of them:

Problem 3. (i) Characterize $\mathscr{L}_0^*$-groups having a regular completion.
(ii) Characterize Fréchet $\mathscr{L}_0^*$-groups having a Fréchet regular completion.

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