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A NOTE ON EMBEDDINGS MANIFOLDS INTO TOPOLOGICAL GROUPS PRESERVING DIMENSIONS

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1. INTRODUCTION

In this paper, we assume that all spaces are Tychonoff. It is well-known that topological groups are Tychonoff (e.g., see [7, p. 29]). In [2], Bel'nov proved that every spaces X can be embedded into a homogeneous space H_X such that ind $H_X = X$, Ind $H_X =$ Ind X and $\dim H_X = \dim X$ in the case when the corresponding dimension of X is finite. Also, Bel'nov asked whether every spaces X can be embedded into a topological group G with dim $G \leqslant \dim X$ (see [9]). Shakhmatov proved that if $n \neq 0, 1, 3, 7$, then the n-dimensional sphere S^n can not be embedded into an n-dimensional topological group, and he showed that in the case $\dim X = 0$, the answer to this question is positive [9]. In [6], Kimura proved that if a topological group G contains the bouquet $S^1 \vee S^1$ of two circles, then $\dim G \geqslant 2$, which implies that in the case $\dim X = 1$, the answer is negative. In [5], the author proved that if G contains the one point union $S^n \vee I$ of the n-dimensional sphere S^n and an arc I, then $\dim G \geqslant n + 1$ ($n = 1, 2, \ldots$), which implies that in the case $\dim X \geqslant 1$, the question is negative.

Also, in [9, p.182] Shakhmatov asked whether S^7 can be embedded into a topological group G with $\dim G = 7$. Note that $S^n (n = 0, 1, 3)$ is a topological group, S^7 is an H-space but not a topological group, and $S^n (n \neq 0, 1, 3, 7)$ is not an H-space (see [1]). To prove his above result, Shakhmatov essentially used the Adams' theorem that S^n ($n \neq 0, 1, 3, 7$) is not an H-space. Naturally, the following problem will be raised: *What kinds of manifols can be embedded into topological groups preserving dimensions?*

In this paper, we prove that if a topological group G contains the one point union $\mathbf{D}^n \vee I$ of an n-ball $\mathbf{D}^n$ and an arc I, then $\dim G \geqslant n+1$. The case $n = 1$ is a negative answer to a question of Kimura [6, (4.5) Question]. Next, we prove the following theorem: Let M be an n-dimensional compact manifold without boundary. Then M

can be embedded into an n-dimensional topological group if and only if M is itself a topological group. Hence S^7 can not be embedded into a topological group G with $\dim G = 7$ or $\operatorname{ind} G = 7$ or $\operatorname{Ind} G = 7$.

The author wishes to thank Professor A. Koyama for calling his attention to the above problem concerning embeddings manifolds into topological groups.

2. Embeddings into topological groups and dimensions.

Let R be the real line and Let $\mathbf{D}^n$ be the n-ball $\{(x_1, x_2, \ldots, x_n) \in R^n \mid \sum_{i=1}^{n} x_i^2 \leqslant 1\}$. Let I be the unit interval $[0,1]$ in R. Also, let S^n be the n-dimensional sphere $\{(x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} \mid \sum_{i-1}^{n+1} x_i^2 = 1\}$ and let $p = (1, 0, \ldots 0) \in S^n$. By identifying the point $* = (0, 0, \ldots, 0) \in \mathbf{D}^n$ and $0 \in I$, we obtain the one point union $(\mathbf{D}^n \vee I, *)$ of $(\mathbf{D}^n, *)$ and $(I, 0)$.

Then we have the following theorem.

Theorem 2.1. *Let G be a topological group. If G contains $\mathbf{D}^n \vee I$ $(n \geqslant 1)$, then* $\dim G \geqslant n + 1$.

To prove (2.1), we need the following well-known result (e.g., see [3, (3.2.10) Theorem]).

Theorem 2.2. *A normal space X satisfies the inequality $\dim X \leqslant n$ $(n \geqslant 0)$ if and only for every closed subset A of X and each mapping $f\colon A \to S^n$ there is an (continuous) extension $F\colon X \to S^n$ of f over X.*

Proof of (2.1). Suppose, on the contrary, that there is a topological group G with contains $\mathbf{D}^n \vee I$ and $\dim G \leqslant n$. Let $h\colon \mathbf{D}^n \vee I \to G$ be an embedding. Since G is homogeneous, we may assume that $h(*) = e$ is the unit element of the group G. We may assume that $\mathbf{D}^n$ and I are naturally the subsets of $\mathbf{D}^n \vee I$. Let $\varphi\colon \mathbf{D}^n \times I \to G$ be the homotopy defined by

$$\varphi(x, t) = h(x) \cdot h(t)$$

for $x \in \mathbf{D}^n$ and $t \in I$, where the symbol $\cdot$ denotes the group composition of G. Choose a neighborhood U of $\varphi(\partial\mathbf{D}^n \times \{0\})$ in G and a neighborhood V of $\varphi(*, 0)$ $(= h(*))$ in G such that $U \cap V = \emptyset$, where $\partial\mathbf{D}^n$ denotes the manifold boundary. Then, take a sufficiently small positive number t such that $\varphi(\partial\mathbf{D}^n \times [0, t]) \subset U$ and $\varphi(\{*\} \times [0, t]) \subset V$. Since $\varphi(\mathbf{D}^n \times \{t\})$ is an n-ball and $\varphi(*, t)$ is not contained in $\varphi(\mathbf{D}^n \times \{0\})$, we can choose a small n-ball B in $\varphi(\mathbf{D}^n \times \{t\}) \cap V$ such that $\varphi(*, t) \in B$ and $B \cap \varphi(\mathbf{D}^n \times \{0\}) = \emptyset$. Note that $B \cap (\varphi(\partial\mathbf{D}^n \times [0, t]) \cup \varphi(\mathbf{D}^n \times \{0\})) = \emptyset$. Define a map $f\colon \varphi(\mathbf{D}^n \times \{0.t\}) \cup \varphi(\partial\mathbf{D}^n \times [0,t]) \to S^n$ as follows: If x is not contained in $B, f(x) = p$, and $f|B\colon (B, \partial B) \to (B/\partial B, *') \equiv (S^n, p)$ is the natural quotient map which is obtained from B by shriking the boundary ∂B to a point $*'$. Note that $\varphi(\mathbf{D}^n \times I)$ is compact metrizable. Since $\dim G \leqslant n$, $\dim \varphi(\mathbf{D}^n \times [0,t]) \leqslant n$.

By (2.2), we have an extension $F\colon \varphi(\mathbf{D}^n \times [0,t]) \to S^n$ of f. Put $H' = F\varphi\colon \mathbf{D}^n \times I \to S^n$. Note that $H'(\partial\mathbf{D}^n \times [0,t]) = *$. Hence we obtain a homotopy $H\colon S^n \times [0,t] \to S^n$ induced by H' such that H_0 is a constant map and H_t is homotopic to the identity map of S^n, where $H_s(x) = H(x,s)$ for $0 \leqslant s \leqslant t$ and $x \in S^n$. Since S^n is not contractible, this is a contradiction. □

Remark 2.3. By (2.1), the one point union $\mathbf{D}^n \vee I (n \geqslant 1)$ can not be embedded into an n-dimensional topological group and $\dim(\mathbf{D}^n \vee I) = n$. Hence the one point union $\mathbf{D}^n \vee I (n \geqslant 1)$ is the simplest example which gives a negative answer to the question of Bel'nov. The case $n = 1$ is a negative answer to the question of Kimura [6, (4.5) Question]. Also, in the proof of (2.1) we get a contradiction to assume that $\dim \varphi(\mathbf{D}^n \times I) \leqslant n$. Since $\varphi(\mathbf{D}^n \times I)$ is compact and metrizable, we can conclude that $\mathbf{D}^n \vee I$ can not be embedded into a topological group G such that $\operatorname{ind} G \leqslant n$ or $\operatorname{Ind} G \leqslant n$ or $\dim G \leqslant n$.

The following lemma is trivial.

Lemma 2.4. *Let G be a topological group. If P is the path component containing the unit element of G, then P is a subgroup of G.*

The following is the main theorem of this paper.

Theorem 2.5. *Let M be a n-dimensional compact manifold without boundary ($n \geqslant 1$). Then M can be embedded into an n-dimensional topological group G if and only if M is itself a topological group.*

Proof. Suppose that M is not a topological group. We may assume that M is path connected and M contains the unit element e of G. Suppose, on the contrary, that G is an n-dimensional topological group G containing M. Let P is the path component of G which contains the unit element e of G. Then P is also a topological group (see (2.4)) and $P \supset M$. Since M is not a topological group, $P - M \neq \emptyset$. Take a point $x_0 \in P - M$. Since P is path connected, there is an arc A in P from x_0 to a point y_0 of M such that $A \cap M = \{y_0\}$. Since M is an n-dimensional manifold without boundary, there is a subset K of P which is homeomorphic to $\mathbf{D}^n \vee I$. Let $\varphi\colon \mathbf{D}^n \times I \to P$ be the homotopy as in the proof of (2.1). Then we see that $\varphi(\mathbf{D}^n \times I)$ is compact and matrizable with $\dim \varphi(\mathbf{D}^n \times I) \geqslant n + 1$. Hence $\dim G \geqslant n + 1$. This is a contradiction. The converse assertion is obvious. □

Remark 2.6. It is well known that an n-dimensional sphere S^n is a topological group if and only if $n = 0, 1, 3$. Hence $S^n (n \neq 0, 1, 3)$ can not be embedded into an n-dimensional topological group. The case $n = 7$ is a negative answer to a question of Shakhmatov [9, p. 182] . Also, by the proof of (2.5), we can conclude that if M is an n-dimensional compact manifold without boundary which is contained in an n-dimensional topological group G, then M is a path component of G.

In the theory of topological groups (see [7]), the structure of locally compact topological groups has been studied by may mathematicians. Especially, the following is well known as a positive answer to Hilbert's fifth problem: *A locally compact topological group which is finite-dimensional and locally (path) connected is a Lie group, in particular, a manifold without boundary* (see [7, (4.10.1) Theorem]). In the theory of locally compact topological groups, the property of being locally compact is essential (see [7] and [4]).

Now, we will give the proof of the following without the assumption of locally compactness.

Corollary 2.7. *If G is an n-dimensional topological group which contains an n-ball and G is locally path connected, then G is a Lie group.*

Proof. By (2.1), we know that G does not contain $\mathbf{D}^n \vee I$. Since G is homogeneous and locally path connected, we can see that G is an n-dimensional manifold, which implies that G is locally compact. Hence G is a Lie group. □

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