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ON COMPLEX RADON MEASURES I

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1. INTRODUCTION

If Ψ is a right continuous complex function of finite variation in $\mathbf{R}^n$, then Ψ induces a complex Lebesgue-Stieltjes measure m_Ψ on $\mathbf{R}^n$, whose domain is a δ-ring M_Ψ containing all the compact subsets of $\mathbf{R}^n$. If $\mathscr{R} = M_\Psi \cap \mathscr{B}(\mathbf{R}^n)$, it is well-known (e.g. vide [6], [8]) that M_Ψ (resp., m_Ψ) is the Lebesgue completion of $\mathscr{R}$ (resp., of $m_\Psi|\mathscr{R}$) with respect to $v(m_\Psi|\mathscr{R},\mathscr{R})$ (resp., to $\mathscr{R}$). Conversely, if μ is a complex measure on a δ-ring $\mathscr{D}$ containing the compact subsets of $\mathbf{R}^n$ and $\mathscr{D}$ is the Lebesgue completion of the δ-ring $\mathscr{R} = \mathscr{D} \cap \mathscr{B}(\mathbf{R}^n)$ with respect to $|\mu||\mathscr{R}$ and $\{E \in \mathscr{B}(\mathbf{R}^n)\colon |\mu|^*(E) < \infty\} = \mathscr{R}$, then there is a right continuous function Ψ of finite variation on $\mathbf{R}^n$ such that $M_\Psi = \mathscr{D}$ and $\mu = m_\Psi$. This result is essentially the same as Theorem 54.2 of [6], where McShane considers μ to be real.

The object of the present paper and the succeeding one is to generalize the above mentioned results to complex Radon measures on a locally compact Hausdorff space X. Since $\mathscr{K}_0 \neq \mathscr{K}$ in general, we are led to the study of various types of regular extensions of positive and complex measures defined on $\mathscr{D}(\mathscr{K}_0)$.

With each $\theta \in \mathscr{K}(X)^*$ we associate canonically a unique complex measure μ_θ defined on a δ-ring M_θ containing $\mathscr{D}(\mathscr{K})$ and in the present paper we study the regularity properties of μ_θ. Among other results, we show that $M_\theta = M_{|\theta|}$ and $\mu_{|\theta|} = v(\mu_\theta, M_\theta)$. In the succeeding paper [9] we obtain the generalization of the said theorem of [6] and give some characterizations of bounded and unbounded complex Radon measures.

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In this section we fix the notation and terminology. Also we give some definitions and results from the literature.

X denotes a locally compact Hausdorff space and $C_c(X)$ (resp., $C_c^r(X)$) is the vector space of all continuous complex (resp., real) valued functions with compact support in X. $\mathscr{K}_0$ (resp., $\mathscr{K}$) is the family of all compact $G_\delta s$ (resp., compact subsets) of X, while $\mathscr{U}$ is the class of all open subsets of X. For a class $\mathscr{C}$ of subsets of X, $\mathscr{D}(\mathscr{C})$ (resp., $\mathscr{S}(\mathscr{C})$) is the δ-ring (resp., σ-ring) generated by $\mathscr{C}$. We denote $\mathscr{S}(\mathscr{U})$ by $\mathscr{B}(X)$, $\mathscr{S}(\mathscr{K})$ by $\mathscr{B}_c(X)$ and $\mathscr{S}(\mathscr{K}_0)$ by $\mathscr{B}_0(X)$. The members of $\mathscr{B}(X)$ are called the Borel subsets of X; those of $\mathscr{B}_0(X)$ the Baire subsets of X and finally, those of $\mathscr{B}_c(X)$ are called σ-Borel (since $E \in \mathscr{B}(X)$ belongs to $\mathscr{B}_c(X)$ if and only if E is σ-bounded).

The locally convex spaces $\mathscr{K}(X)$ and $\mathscr{K}(X, \mathbf{R})$ are as in Bourbaki [1] and $\mathscr{K}(X)^*$ (resp., $\mathscr{K}(X,\mathbf{R})^*$) is the topological dual of $\mathscr{K}(X)$ (resp., of $\mathscr{K}(X,\mathbf{R})$). For a functional $\theta \in \mathscr{K}(X)^*$, we refer to [1] for the concepts of $\operatorname{Re}\theta$, $\operatorname{Im}\theta$, $|\theta|$ and for those of θ^+ and θ^- when θ is real in the sense of [1].

In the following proposition is given the concept of Lebesgue completion of a complex measure on a δ-ring. As its proof is of routine nature (vide [3]) we omit it.

Proposition 2.1. *Let ν be a complex measure on a δ-ring $\mathscr{R}$. Let $\mathscr{R}^* = \{E \cup N : E \in \mathscr{R},\ N \subset M \in \mathscr{R}$ with $|\nu|(M) = 0\}$, where $|\nu| = v(\nu, \mathscr{R})$. For such $E \cup N \in \mathscr{R}^*$, let $\tilde{\nu}(E \cup N) = \nu(E)$. Then $\mathscr{R}^*$ is a δ-ring, $\mathscr{R}^* \supset \mathscr{R}$, $\tilde{\nu}$ is well defined, $\tilde{\nu}|\mathscr{R} = \nu$ and $\tilde{\nu}$ is a complex measure on $\mathscr{R}^*$. We say that $\tilde{\nu}$ (resp., $\mathscr{R}^*$) is the Lebesgue completion of ν (resp., of $\mathscr{R}$) relative to $\mathscr{R}$ (resp., to ν).*

By measure (on a ring) we mean a positive measure.

From Chapter IV of Bourbaki [1], we have the following theorem. The reader may also refer to [5], [7], [8] and [10].

Theorem 2.2. *Let θ be a positive linear functional on $C_c(X)$. Let $M^+(X) = \{f : X \to [0,\infty],\ f$ lower semi-continuous$\}$ and $\mathscr{F}(X) = \{f : X \to [0,\infty]\}$. Let*

$$\mu_\theta^*(E) = \inf_{\chi_E \leqslant g \in M^+(X)} \sup \{\theta(\psi) : \psi \leqslant g, \psi \in C_c^+(X)\}.$$

for $E \subset X$. Then:

(i) *μ_θ^* is an outer measure on $\mathscr{P}(X)$.*

(ii) *Let $M_{\mu_\theta^*} = \{E \subset X : E$ is μ_θ^*-measurable$\}$. Then $M_{\mu_\theta^*}$ is a σ-algebra and contains $\mathscr{B}(X)$. We denote $\mu_\theta^*|M_{\mu_\theta^*}$ by $\tilde{\mu}_\theta$.*

(iii) *$\mu_\theta^*(K) < \infty$, $K \in \mathscr{K}$.*

(iv) *$\mu_\theta^*(E) = \inf\{\mu_\theta^*(U) : E \subset U \in \mathscr{U}\}$, $E \subset X$.*

(v) $\mu_\theta^*(E) = \sup\{\mu_\theta^*(K) \colon K \subset E, K \in \mathscr{K}\}$, *if* $\mu_\theta^*(E)$ *is* σ*-finite and* $E \in M_{\mu_\theta^*}$ *or if* $E \in \mathscr{U}$.

(vi) *Given* $E \in M_{\mu_\theta^*}$ *with* $\mu_\theta^*(E)$ σ*-finite, there exist* A, $B \in \mathscr{B}(X)$ *such that* $A \subset E \subset B, A$ σ*-compact and* $\mu_\theta^*(B \setminus A) = 0$. *Consequently,* $\mu_\theta^*(E) = \mu_\theta^*(A)$.

(vii) *Let* $\check{\mu}_\theta = \mu_\theta^* | \mathscr{B}(X)$. *Then*

$$\theta(f) = \int_X f \, \mathrm{d}\check{\mu}_\theta, \quad f \in C_c(X).$$

Besides, if ν *is a measure on* $\mathscr{B}(X)$ *satisfying* (iii)–(v) *above and if* $\theta(f) = \int_X f \, \mathrm{d}\nu$, $f \in C_c(X)$, *then* $\nu = \check{\mu}_\theta$.

3. Regular extensions of positive measures

We introduce several notions of regularity for positive and complex measures defined on certain δ-rings or σ-rings of subsets of X and study the existence of regular extensions. These results play a key role in the next section.

Definition 3.1. A measure μ on $\mathscr{B}_0(X)$ (resp., on $\mathscr{B}_c(X)$, on $\mathscr{B}(X)$) is a Baire (resp., a σ-Borel, a Borel) measure if $\mu(K) < \infty$ for $k \in \mathscr{K}_0$ (resp., for $K \in \mathscr{K}$).

Definition 3.2. Let $\mathscr{R}$ be a ring of sets in X with $\mathscr{D}(\mathscr{K}_0) \subset \mathscr{R}$ or $\mathscr{D}(\mathscr{K}) \subset \mathscr{R}$. A measure μ defined on $\mathscr{R}$ is said to be ***$\mathscr{R}$-regular*** if

(i) $\mu(K) < \infty$, $K \in \mathscr{K} \cap \mathscr{R}$;
(ii) $\mu(E) = \inf\{\mu(U) \colon E \subset U \in \mathscr{U} \cap \mathscr{R}\}$ for $E \in \mathscr{R}$; and
(iii) $\mu(E) = \sup\{\mu(C) \colon C \subset E,\ C \in \mathscr{K} \cap \mathscr{R}\}$ for $E \in \mathscr{R}$.

A complex measure ν on $\mathscr{R}$ is said to be ***$\mathscr{R}$-regular*** if, given $E \in \mathscr{R}$ and $\varepsilon > o$, there exist $C \in \mathscr{K} \cap \mathscr{R}$ and $U \in \mathscr{U} \cap \mathscr{R}$ such that $C \subset E \subset U$ and $|\nu(G)| < \varepsilon$ for every $G \in \mathscr{R}$ with $G \subset U \setminus C$.

Definition 3.3. A Borel measure μ on $\mathscr{B}(X)$ is said to be ***Radon-regular*** if

(i) $\mu(E) = \inf\{\mu(U) \colon E \subset U \in \mathscr{U}\}$ for $E \in \mathscr{B}(X)$

and

(ii) $\mu(U) = \sup\{\mu(C) \colon C \subset U,\ C \in \mathscr{K}\}$, for $U \in \mathscr{U}$.

Proposition 3.4.

(i) *A Baire measure is* $\mathscr{B}_0(X)$*-regular.*
(ii) *A measure* μ_0 *on* $\mathscr{D}(\mathscr{K}_0)$ *with* $\mu_0(K) < \infty$ *for* $K \in \mathscr{K}_0$ *is* $\mathscr{D}(\mathscr{K}_0)$*-regular.*
(iii) *A complex measure* ν_0 *on* $\mathscr{D}(\mathscr{K}_0)$ *is* $\mathscr{D}(\mathscr{K}_0)$*-regular.*
(iv) *If* μ *is a Radon-regular measure on* $\mathscr{B}(X)$*, then (ii) of Definition 3.3 holds for* $E \in \mathscr{B}(X)$ *if* $\mu(E)$ *is* σ*-finite.*

Proof. By Proposition 13, §14of [2], $\mathscr{K} \cap \mathscr{B}_0(X) = \mathscr{K} \cap \mathscr{D}(\mathscr{K}_0) = \mathscr{K}_0$ and hence (i) holds by the first part of Theorem 52.G of [4]. Corollary 1 on p. 347 of [2] implies (ii) and (iii) while (iv) is immediate from Theorem 10.30 of [5]. □

Lemma 3.5. *Let μ be a Radon-regular measure on $\mathscr{B}(X)$. Then:*
(i) *If $\nu = \mu|\mathscr{D}(\mathscr{K})$, then ν is $\mathscr{D}(\mathscr{K})$-regular.*
(ii) *If $w = \mu|\mathscr{B}_c(X)$, then w is $\mathscr{B}_c(X)$-regular.*

Proof. Because of Proposition 3.4 (iv) it suffices to verify (ii) of Definition 3.2 for ν and w. If $E \in \mathscr{D}(\mathscr{K})$, then by Proposition 11, §14 of [2] and by the Radon-regularity of μ, the result holds for ν. If $E \in \mathscr{B}_c(X)$, then $E \subset \bigcup_1^\infty C_n$, $C_n \in \mathscr{K}$ and applying (i) to $E \cap C_n$ for each n, it can be shown that $w(E)$ satisfies (ii) of Definition 3.2. □

Lemma 3.6. (i) *Let μ_1 and μ_2 be $\mathscr{B}_c(X)$-regular measures. If*

$$\int_X f\,\mathrm{d}\mu_1 = \int_X f\,\mathrm{d}\mu_2, \quad f \in C_c(X), \tag{1}$$

then $\mu_1 = \mu_2$.
(ii) *If (1) holds for two Baire measures μ_1 and μ_2, then $\mu_1 = \mu_2$.*

Proof. (i) This is immediate by Theorem 56.E of [4].
(ii) For $K \in \mathscr{K}_0$, by Theorem 55.A of [4] there exists $f_n \downarrow \chi_K$, $f_n \in C_c^+(X)$ so that by the Lebesgue dominated convergence theorem and by the hypothesis, $\mu_1(K) = \mu_2(K)$. Consequently, $\mu_1 = \mu_2$ by Proposition 3.4 (i). □

Theorem 3.7. *Let μ_0 be a Baire measure on X. Then:*
(i) *μ_0 has a unique extension μ to $\mathscr{B}_c(X)$ such that μ is a $\mathscr{B}_c(X)$-regular measure.*
(ii) *μ_0 has a unique extension ν to $\mathscr{B}(X)$ such that ν is a Radon-regular measure.*
(iii) *If μ and ν are as in (i) and (ii), then $\mu = \nu|\mathscr{B}_c(X)$.*

Proof.
(i) This is the same as Theorem 54.D of [4].
(ii) Let

$$\theta(f) = \int_X f\,\mathrm{d}\mu_0, \quad f \in C_c(X). \tag{1}$$

Since θ is a positive linear functional on $C_c(X)$, by Theorem 2.2 there exists a unique Radon-regular Borel measure $\nu = \check{\mu}_\theta$ such that $\theta(f)$ is also given by the integral in (1) with μ_0 replaced by ν. If $w = \nu|\mathscr{B}_0(X)$, then w is a Baire measure and $\theta(f)$ is also given by the integral in (1) with μ_0 replaced by w. Thus, by Lemma 3.6

(ii) we conclude that $w = \mu_0$ and hence ν is a Radon-regular extension of μ_0. The uniqueness of ν follows from Proposition 3.4 (iv) and Theorem 2.2.

(iii) Follows from the uniqueness part of (i) and Lemma 3.5 (ii). □

Theorem 3.8. *Every $\mathscr{B}_c(X)$-regular measure μ has a unique extension $\hat{\mu}$ to $\mathscr{B}(X)$ as a Radon-regular measure.*

Proof. Apply the proof of Theorem 3.7 (ii) considering

$$\theta(f) = \int_X f \,\mathrm{d}\mu, \quad f \in C_c(X)$$

and using Lemma 3.6 (i) in the place of Lemma 3.6 (ii). □

Theorem 3.9. *Let μ be a $\mathscr{D}(\mathscr{K})$-regular measure. Then:*

(i) *The unique extension $\hat{\mu}$ of μ to $\mathscr{B}_c(X)$ as a measure is $\mathscr{B}_c(X)$-regular.*

(ii) *μ admits a unique extension w to $\mathscr{B}(X)$ as a Radon-regular measure and $w|\mathscr{B}_c(X) = \hat{\mu}$, where $\hat{\mu}$ is as in* (i).

Proof.

(i) Since μ is finite on $\mathscr{D}(\mathscr{K})$, μ admits a unique extension $\hat{\mu}$ to $\mathscr{S}(\mathscr{D}(\mathscr{K})) = \mathscr{B}_c(X)$ as a measure. Clearly, $\hat{\mu}(K) < \infty$ for $K \in \mathscr{K}$. Let $E \in \mathscr{B}_c(X)$. Then $E \subset \bigcup_1^\infty C_n$, $C_n \in \mathscr{K}$. Since $E \cap C_n \in \mathscr{D}(\mathscr{K})$ and μ is $\mathscr{D}(\mathscr{K})$-regular, given $\varepsilon > o$, there exists $U_n \in \mathscr{U} \cap \mathscr{D}(\mathscr{K})$ such that $E \cap C_n \subset U_n$ and $\mu(U_n) - \mu(E \cap C_n) < \varepsilon/2^n$. If $U = \bigcup_1^\infty U_n$, then $U \in \mathscr{U} \cap \mathscr{B}_c(X)$ and $E \subset U$. If $\hat{\mu}(E) = \infty$, then $\hat{\mu}(U) = \infty$. If $\hat{\mu}(E) < \infty$, then $\hat{\mu}(U) - \hat{\mu}(E) \leqslant \sum_1^\infty \left(\mu(U_n) - \mu(E \cap C_n)\right) < \varepsilon$. Thus $\hat{\mu}$ satisfies (ii) of Definition 3.2 with $\mathscr{R} = \mathscr{B}_c(X)$. Again, letting $E_n = \bigcup_{k=1}^n (E \cap C_k)$ we have $E_n \in \mathscr{D}(\mathscr{K})$ and $E_n \uparrow E$ so that

$$\begin{aligned}\hat{\mu}(E) = \sup_n \mu(E_n) &= \sup_n \sup \left\{\mu(C) : C \subset E_n,\ C \in \mathscr{K}\right\} \\ &\leqslant \sup \left\{\mu(C) : C \subset E,\ C \in \mathscr{K}\right\} \\ &\leqslant \hat{\mu}(E)\end{aligned}$$

and hence $\hat{\mu}$ is $\mathscr{B}_c(X)$-regular.

(ii) Taking $\hat{\mu}$ as in (i) and applying Theorem 3.8 to $\hat{\mu}$, we obtain a Radon-regular extension w of $\hat{\mu}$ and, hence of μ, to $\mathscr{B}(X)$. Besides, w is unique by the uniqueness part of Theorem 3.8. □

With each $\theta \in \mathscr{K}(X)^*$ we associate a unique complex measure μ_θ (vide Definition 4.3) defined canonically on a δ-ring M_θ which contains $\mathscr{D}(\mathscr{K})$ and call μ_θ the complex Radon measure induced by θ. The object of this section is to study the properties of μ_θ and M_θ when θ is real and when is arbitrary.

We use the notation of μ_θ^*, $M_{\mu_\theta^*}$ and $\check{\mu}_\theta$ as in Theorem 2.2. We denote $\mu_\theta^*|M_{\mu_\theta^*}$ by $\tilde{\mu}_\theta$ and call $\tilde{\mu}_\theta$ (resp., $\check{\mu}_\theta$) the Radon (resp., the Borel-Radon) measure induced by the positive linear functional θ on $C_c(X)$.

Theorem 4.1. *A measure μ on $\mathscr{B}(X)$ is Borel-Radon if and only if μ is Radon-regular.*

Proof. By Theorem 2.2 the condition is necessary. Conversely, if μ is Radon-regular, let

$$\theta(f) = \int_X f \,\mathrm{d}\mu, \quad f \in C_c(X).$$

Then θ is a positive linear functional and by the uniqueness part of Theorem 2.2 (vii) and by Proposition 3.4 (iv) we conclude that $\mu = \check{\mu}_\theta$. □

Proposition 4.2. *Let $\theta \in \mathscr{K}(X)^*$ and let $\theta_1 = \operatorname{Re}\theta$; $\theta_2 = \operatorname{Im}\theta$. Then:*

(i) *If $M_\theta = \{A \subset X : \tilde{\mu}_{\theta_j^+}(A) < \infty, \tilde{\mu}_{\theta_j} - (A) < \infty, j = 1, 2\}$, then M_θ is a δ-ring and contains $\mathscr{D}(\mathscr{K})$.*

(ii) *Let $M(\theta) = M_{\mu^*_{\theta_1^+}} \cap M_{\mu^*_{\theta_1^-}} \cap M_{\mu^*_{\theta_2^+}} \cap M_{\mu^*_{\theta_2^-}}$. Then $M(\theta)$ is a σ-algebra and contains $\mathscr{B}(X)$. The members of $M(\theta)$ are called θ-measurable sets.*

(iii) *Let*

$$\mu_\theta(E) = \left\{(\tilde{\mu}_{\theta_1^+} - \tilde{\mu}_{\theta_1^-}) + i(\tilde{\mu}_{\theta_2^+} - \tilde{\mu}_{\theta_2^-})\right\}(E), \quad E \in M_\theta.$$

Then μ_θ is a complex measure on M_θ.

(iv) *If θ is real, then μ_θ is real.*

(v) *If θ is positive, then μ_θ is positive and finite.*

Proof. This is immediate from Theorem 2.2. □

Definition 4.3. Let $\theta \in \mathscr{K}(X)^*$. The complex measure μ_θ in Proposition 4.2 (iii) is called the *complex Radon measure induced* by θ and the δ-ring M_θ is called the *domain* of μ_θ.

Notation 4.4. Let μ_θ be the complex Radon measure induced by θ, $\theta \in \mathscr{K}(X)^*$. We denote $v(\mu_\theta, M_\theta)$ by $|\mu_\theta|$. If ν is a real measure on a δ-ring $\mathscr{R}_2$ and $\mathscr{R}_1$ is another δ-ring such that $\mathscr{R}_2 \supset \mathscr{R}_1$, then

$$(\nu, \mathscr{R}_1, \mathscr{R}_2)^+(E) = \sup\left\{\nu(F) \colon F \subset E,\ F \in \mathscr{R}_1\right\}$$

and

$$(\nu, \mathscr{R}_1, \mathscr{R}_2)^-(E) = -\inf\{\nu(F) : F \subset E,\ F \in \mathscr{R}_1\}$$

for $E \in \mathscr{R}_2$. If ν is real or complex on $\mathscr{R}_2$, then

$$v(\nu, \mathscr{R}_1, \mathscr{R}_2)(E) = \sup\Big\{\sum_{i=1}^{n} |\nu(E_i)| : \{E_i\}_1^n \subset \mathscr{R}_1,\ E_i \cap E_j = \emptyset,\quad i \neq j,$$
$$\text{and } \bigcup_1^n E_i \subset E\Big\}$$

for $E \in \mathscr{R}_2$.

Theorem 4.5. *Let $\theta \in \mathscr{K}(X)^*$, θ real. Then the following assertions hold:*

(i) *μ_θ is M_θ-regular.*

(ii) *μ_θ is of finite variation on M_θ.*

(iii) *$\mu_\theta^+ \leqslant \mu_{\theta^+}$ and $\mu_\theta^- \leqslant \mu_{\theta^-}$ in M_θ, where $\mu_\theta = \mu_\theta^+ - \mu_\theta^-$ is the Jordan decomposition of μ_θ in M_θ.*

(iv) $|\mu_\theta|$, μ_θ^+ and μ_θ^- are M_θ-regular.

(v) $\mu_\theta^+|\mathscr{D}(\mathscr{K}) = (\mu_\theta|\mathscr{D}(\mathscr{K}))^+$; $\mu_\theta^-|\mathscr{D}(\mathscr{K}) = (\mu_\theta|\mathscr{D}(\mathscr{K}))^-$ and $v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K})) = |\mu_\theta||\mathscr{D}(\mathscr{K})$.

(vi) $|\mu_\theta||\mathscr{D}(\mathscr{K})$, $\mu_\theta^+|\mathscr{D}(\mathscr{K})$, $\mu_\theta^-|\mathscr{D}(\mathscr{K})$ and $\mu_\theta|\mathscr{D}(\mathscr{K})$ are $\mathscr{D}(\mathscr{K})$-regular.

(vii) (a) $(\mu_\theta, \mathscr{D}(\mathscr{K}), M_\theta)^+(E) = \mu_\theta^+(E)$, $E \in M_\theta$.

(b) $(\mu_\theta, \mathscr{D}(\mathscr{K}), M_\theta)^-(E) = \mu_\theta^-(E)$, $E \in M_\theta$.

(c) $v(\mu_\theta, \mathscr{D}(\mathscr{K}), M_\theta)(E) = |\mu_\theta|(E)$, $E \in M_\theta$.

(viii) *Given $E \in M_\theta$, there exist A, B in $\mathscr{B}(X) \cap M_\theta$ such that $A \subset E \subset B$ and $\mu_{\theta^+}(B \setminus A) = \mu_{\theta^-}(B \setminus A) = 0$. Then $\mu_{\theta^+}(E) = \mu_{\theta^+}(A)$, $\mu_{\theta^-}(E) = \mu_{\theta^-}(A)$, and μ_θ (resp., M_θ) is the Lebesgue completion of $\mu_\theta|\mathscr{R}$ (resp., of $\mathscr{R}$) with respect to $\mathscr{R}$ (resp., to $\mu_\theta|\mathscr{R}$), where $\mathscr{R} = \mathscr{B}(X) \cap M_\theta$.*

Proof. (i) Since μ_{θ^+} and μ_{θ^-} are M_θ-regular by Theorem 2.2, the assertion (i) holds.

(ii) This is immediate from the fact that M_θ is a δ-ring. (Vide Corollary 2 on p. 48 of [2]).

(iii) Since $\mu_\theta = \mu_{\theta^+} - \mu_{\theta^-}$ in M_θ and M_θ is a δ-ring, (iii) follows from the proof of Proposition 16, §3 of [2].

(iv) Since μ_{θ^+} and μ_{θ^-} are M_θ-regular by Theorem 2.2, the result follows from (iii).

(v) As in the proof of (iii) we have $(\mu_\theta|\mathscr{D}(\mathscr{K}))^+ \leqslant \mu_\theta^+|\mathscr{D}(\mathscr{K})$. Let $E \in \mathscr{D}(\mathscr{K})$ and $\varepsilon > 0$. By (iv), given $F \in M_\theta$, there exists $C_F \in \mathscr{K}$ such that $C_F \subset F$ and

$|\mu_\theta|(F \setminus C_F) < \varepsilon$. Consequently, we have

$$\begin{aligned}(\mu_\theta|\mathscr{D}(\mathscr{K}))^+(E) &= \sup\{\mu_\theta(F)\colon F \subset E,\ F \in \mathscr{D}(\mathscr{K})\}\\ &\geqslant \sup\{\mu_\theta(C_F)\colon F \subset E,\ F \in M_\theta\}\\ &\geqslant \sup\{\mu_\theta(F) - \varepsilon\colon F \subset E,\ F \in M_\theta\}\\ &= \mu_\theta^+(E) - \varepsilon.\end{aligned}$$

Thus $(\mu_\theta|\mathscr{D}(\mathscr{K}))^+ \geqslant \mu_\theta^+|\mathscr{D}(\mathscr{K})$ so that $(\mu_\theta|\mathscr{D}(\mathscr{K}))^+ = \mu_\theta^+|\mathscr{D}(\mathscr{K})$. Similarly, it follows that $(\mu_\theta|\mathscr{D}(\mathscr{K}))^- = \mu_\theta^-|\mathscr{D}(\mathscr{K})$ and consequently,

$$|\mu_\theta||\mathscr{D}(\mathscr{K}) = (\mu_\theta^+ + \mu_\theta^-)|\mathscr{D}(\mathscr{K}) = v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K})).$$

(vi) Let $E \in \mathscr{D}(\mathscr{K})$ and $\varepsilon > 0$. By (iv) there exist $U \in \mathscr{U} \cap M_\theta$ and $K \in \mathscr{K}$ such that $K \subset E \subset U$ and $|\mu_\theta|(U \setminus K) < \varepsilon$. As $\bar{E} \in \mathscr{K}$, by Proposition 11, §14 of [2] there exists $U_0 \in \mathscr{U} \cap \mathscr{D}(\mathscr{K})$ such that $\bar{E} \subset U_0$. If $V = U \cap U_0$, then $E \subset V$, $V \in \mathscr{U} \cap \mathscr{D}(\mathscr{K})$ and $|\mu_\theta|(V \setminus K) < \varepsilon$. Thus $|\mu_\theta||\mathscr{D}(\mathscr{K})$ is $\mathscr{D}(\mathscr{K})$-regular and consequently, (vi) holds.

(vii) Let $E \in M_\theta$. By (iv) and (v) we have

$$\begin{aligned}\mu_\theta^+(E) &= \sup\{\mu_\theta^+(C)\colon C \subset E, C \in \mathscr{K}\}\\ &= \sup_{\substack{C\subset E\\ C\in\mathscr{K}}} \sup\{\mu_\theta(F)\colon F \subset C, F \in \mathscr{D}(\mathscr{K})\}\\ &\leqslant \sup\{\mu_\theta(F)\colon F \subset E, F \in \mathscr{D}(\mathscr{K})\}\\ &\leqslant \sup\{\mu_\theta(F)\colon F \subset E, F \in M_\theta\}\\ &= \mu_\theta^+(E)\end{aligned}$$

and thus (a) holds. Similarly, (b) is proved.

For $E \in M_\theta$, given $\varepsilon > 0$, there exists a partition $\{E_i\}_1^n$ of E in M_θ such that

$$\sum_{i=1}^n |\mu_\theta(E_i)| > |\mu_\theta|(E) - \tfrac{1}{2}\varepsilon.$$

By (i), for each i there exists $C_i \in \mathscr{K}$ such that $C_i \subset E_i$ and $|\mu_\theta(G)| < \frac{\varepsilon}{2n}$ for $G \in M_\theta$ with $G \subset E_i \setminus C_i$. Then $\sum_1^n |\mu_\theta(C_i)| \geqslant \sum_1^n |\mu_\theta(E_i)| - \varepsilon/2 > |\mu_\theta|(E) - \varepsilon$. Thus $v(\mu_\theta, \mathscr{D}(\mathscr{K}), M_\theta)(E) \geqslant |\mu_\theta|(E)$. Since the reverse inequality is obvious, (c) holds.

(viii) Since $E \in M_\theta \subset M_{\mu_{\theta+}^*} \cap M_{\mu_{\theta-}^*}$ and $\mu_{\theta+}^*(E)$ and $\mu_{\theta-}^*(E)$ are finite, by Theorem 2.2 (vi) there exist $A, B \in \mathscr{B}(X)$ with $A \subset E \subset B$ and $\mu_{\theta+}(B \setminus A) = \mu_{\theta-}(B \setminus A) = 0$. Now, by (vii) (c), for $F \in \mathscr{R}$ we have

$$|\mu_\theta|(F) = v(\mu_\theta, \mathscr{D}(\mathscr{K}), M_\theta)(F) \leqslant v(\mu_\theta|\mathscr{R}, \mathscr{R})(F) \leqslant v(\mu_\theta, M_\theta)(F) = |\mu_\theta|(F)$$

and hence from (iii) it follows that

$$v(\mu_\theta|\mathscr{R},\mathscr{R})(B\setminus A)=|\mu_\theta|(B\setminus A)\leqslant(\mu_{\theta+}+\mu_{\theta-})(B\setminus A)=0.$$

This proves (viii). □

Theorem 4.6. *Let $\theta\in\mathscr{K}(X)^*$, θ real. Let μ_θ^+ and μ_θ^- be as in Theorem 4.5 (iii). Then:*

(i) *$\mu_\theta^+|\mathscr{D}(\mathscr{K})$ and $\mu_\theta^-|\mathscr{D}(\mathscr{K})$ admit unique extensions $\hat\mu_\theta^+$ and $\hat\mu_\theta^-$, respectively, to $\mathscr{B}(X)$ as Radon-regular measures and $\hat\mu_\theta^+(E)=\mu_\theta^+(E)$, $\hat\mu_\theta^-(E)=\mu_\theta^-(E)$ for $E\in\mathscr{B}_c(X)\cap M_\theta$.*

(ii) *If $\nu=|\mu_\theta||\mathscr{D}(\mathscr{K})$, then ν has a unique extension $\hat\nu$ to $\mathscr{B}(X)$ as a Radon-regular measure and $\hat\nu(E)=|\mu_\theta|(E)$ for $E\in B_c(X)\cap M_\theta$.*

(iii) *There exists a positive linear functional Ψ on $C_c(X)$ such that $\hat\nu=\breve\mu_\Psi$, where $\hat\nu$ is as in (ii).*

(iv) *$|\theta|\leqslant\Psi$, where Ψ is as in* (iii).

(v) *$\mu_{\theta+}=\mu_\theta^+$ and $\mu_{\theta-}=\mu_\theta^-$ in M_θ.*

(vi) *$M_\theta=M_{|\theta|}$ and $|\mu_\theta|=\mu_{|\theta|}$.*

Proof. (i) By Theorems 4.5 (vi) and 3.9 (ii), $\mu_\theta^+|\mathscr{D}(\mathscr{K})$ has a unique extension $\hat\mu_\theta^+$ to $\mathscr{B}(X)$ as a Radon-regular measure. Then by Lemma 3.5 (ii), $\hat\mu_\theta^+|\mathscr{B}_c(X)$ is $\mathscr{B}_c(X)$-regular and hence by Theorem 4.5 we have

$$\begin{aligned}\hat\mu_\theta^+(E)&=\sup\{\hat\mu_\theta^+(C)\colon C\subset E,\ C\in\mathscr{K}\}\\&=\sup\{\mu_\theta^+(C)\colon C\subset E,\ C\in\mathscr{K}\}\\&=\mu_\theta^+(E)\end{aligned}$$

for $E\in\mathscr{B}_c(X)\cap M_\theta$. Similarly, the other result holds.

(ii) By Theorems 4.5 (vi) and 3.9 (ii) such an extension $\hat\nu$ of $|\mu_\theta||\mathscr{D}(\mathscr{K})$ exists uniquely. An argument similar to that in the proof of (i) shows that $\hat\nu(E)=|\mu_\theta|(E)$ for $E\in\mathscr{B}_c(X)\cap M_\theta$.

(iii) This is immediate from Theorem 4.1.

(iv) Let $f\in C_c(X)$ with $\operatorname{supp}f=K$. Then $\mathscr{B}(K)\subset\mathscr{D}(\mathscr{K})$. As $\theta(f)=\theta^+(f)-\theta^-(f)$, by Theorem 2.2 we have

$$\theta(f)=\int_X f\,\mathrm{d}\breve\mu_{\theta+}-\int_X f\,\mathrm{d}\breve\mu_{\theta-}=\int_K f\,\mathrm{d}(\mu_\theta|\mathscr{B}(K))$$

since $f(x)=0$ for $x\in X\setminus K$, $\mathscr{B}(K)=\mathscr{B}(X)\cap K$ and μ_θ is a real measure on $\mathscr{B}(K)$. Thus,

$$\begin{aligned}|\theta(f)|&\leqslant\int_K|f|\,\mathrm{d}v(\mu_\theta|\mathscr{B}(K),\mathscr{B}(K))\\&\leqslant\int_X|f|\,\mathrm{d}\hat\nu=\int_X|f|\,\mathrm{d}\breve\mu_\Psi=\Psi(|f|)\end{aligned}$$

since $v(\mu_\theta|\mathscr{B}(K), \mathscr{B}(K)) \leqslant v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K})) = |\mu_\theta||\mathscr{D}(\mathscr{K})$ by Theorem 4.5 (v). Thus $|\theta| \leqslant \Psi$.

(v) By (iv) and by Proposition 15, §1, Chapter IV of [1]

$$\mu^*_{|\theta|} \leqslant \mu^*_\Psi. \tag{1}$$

By the same proposition of [1] we have

$$\mu_{|\theta|} = \mu_{\theta+} + \mu_{\theta-} \text{ in } \mathscr{B}(X) \cap M_\theta. \tag{2}$$

Consequently, from Theorem 4.5 (iii) and from (ii), (1) and (2) it follows that

$$|\mu_\theta|(E) = (\mu_\theta^+ + \mu_\theta^-)(E) \leqslant (\mu_{\theta+} + \mu_{\theta-})(E) = \mu_{|\theta|}(E) \leqslant \mu^*_\Psi(E) = |\mu_\theta|(E)$$

for $E \in \mathscr{B}_c(X) \cap M_\theta$. Thus

$$|\mu_\theta|(E) = \mu_{|\theta|}(E), \quad E \in \mathscr{B}_c(X) \cap M_\theta. \tag{3}$$

Since $\mu_\theta^+ \leqslant \mu_{\theta+}$ and $\mu_\theta^- \leqslant \mu_{\theta-}$ by Theorem 4.5 (iii), we conclude from (2) that

$$\mu_\theta^+(E) = \mu_{\theta+}(E) \text{ and } \mu_\theta^-(E) = \mu_{\theta-}(E) \tag{4}$$

for $E \in \mathscr{B}_c(X) \cap M_\theta$.

Now, let $E \in M_\theta$. By Theorem 4.5 (viii) there exist A, $B \in \mathscr{B}(X) \cap M_\theta$ such that $A \subset E \subset B$ with $\mu_{\theta+}(B \setminus A) = \mu_{\theta-}(B \setminus A) = 0$. Since $E \in M_\theta$, by Theorem 2.2 (vi) we can assume A to be σ-compact so that $A \in \mathscr{B}_c(X) \cap M_\theta$. Then, by Proposition 15, §1, Chapter IV of [1] we have $\mu_{|\theta|}(B \setminus A) = (\mu_{\theta+} + \mu_{\theta-})(B \setminus A) = 0$ and hence $\mu^*_{|\theta|}(B \setminus A) = 0$. Consequently, $E \setminus A \in M_{|\theta|}$. Besides, by the same proposition of [1] we have $\check{\mu}_{|\theta|}(A) = (\check{\mu}_{\theta+} + \check{\mu}_{\theta-})(A) < \infty$ and as $A \in \mathscr{B}(X)$, we conclude that $A \in M_{|\theta|}$. Therefore, $E = A \cup (E \setminus A) \in M_{|\theta|}$ and thus we have shown that

$$M_\theta \subset M_{|\theta|}. \tag{5}$$

Besides, by (4)

$$\mu_{|\theta|}(E) = \mu_{|\theta|}(A) = (\mu_{\theta+} + \mu_{\theta-})(A) = (\mu_\theta^+ + \mu_\theta^-)(A) = (\mu_\theta^+ + \mu_\theta^-)(E) = |\mu_\theta|(E)$$

so that by (5) we have

$$\mu_{|\theta|}|M_\theta = |\mu_\theta|. \tag{6}$$

Also, from (4) it follows that $\mu_{\theta+}(E) = \mu_{\theta+}(A) = \mu_\theta^+(A) = \mu_\theta^+(E)$; $\mu_{\theta-}(E) = \mu_{\theta-}(A) = \mu_\theta^-(A) = \mu_\theta^-(E)$ and hence $\mu_{\theta+} = \mu_\theta^+$ and $\mu_{\theta-} = \mu_\theta^-$ in M_θ.

(vi) In view of (5) and (6), it suffices to show that $M_{|\theta|} \subset M_\theta$. Let $E \in M_{|\theta|}$. Then by Theorem 2.2 (vi) there exist A, $B \in \mathscr{B}_c(X) \cap M_{|\theta|}$ such that $A \subset E \subset B$, $\mu_{|\theta|}(B \setminus A) = 0$ and $\mu_{|\theta|}(E) = \mu_{|\theta|}(A)$. Since $\check{\mu}_{\theta\pm} \leqslant \check{\mu}_{|\theta|}$ on $\mathscr{B}(X)$ by proposition 15, §1, Chapter IV of [1], it follows that $A \in \mathscr{B}_c(X) \cap M_\theta$. Besides, as $\mu_{|\theta|}(B \setminus A) = 0$, by the same proposition of [1], $\check{\mu}_{\theta+}(B \setminus A) = \check{\mu}_{\theta-}(B \setminus A) = 0$ so that $\mu^*_{\theta+}(E \setminus A) = \mu^*_{\theta-}(E \setminus A) = 0$. Thus $E \setminus A \in M_\theta$ and hence $E \in M_\theta$. □

Now we pass on to the study of the properties of μ_θ when θ is arbitrary in $\mathscr{K}(X)^*$.

Theorem 4.7. *Let $\theta \in \mathscr{K}(X)^*$. Then the following assertions hold:*

(i) *μ_θ is M_θ-regular.*

(ii) *μ_θ is of finite variation in M_θ.*

(iii) *$|\mu_\theta|$ is M_θ-regular.*

(iv) $v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K})) = |\mu_\theta||\mathscr{D}(\mathscr{K})$.

(v) *$|\mu_\theta||\mathscr{D}(\mathscr{K})$ is $\mathscr{D}(\mathscr{K})$-regular and hence $\mu_\theta|\mathscr{D}(\mathscr{K})$ is $\mathscr{D}(\mathscr{K})$-regular.*

(vi) $v(\mu_\theta, \mathscr{D}(\mathscr{K}), M_\theta) = |\mu_\theta|$.

(vii) *Given $E \in M_\theta$, there exist A, $B \in \mathscr{B}(X) \cap M_\theta$ such that $A \subset E \subset B$ and $|\mu_\theta|(B \setminus A) = 0$. Consequently, $\mu_\theta(E) = \mu_\theta(A)$ and μ_θ (resp., M_θ) is the Lebesgue completion of $\mu_\theta|\mathscr{R}$ (resp., of $\mathscr{R}$) with respect to $\mathscr{R}$ (resp., to $\mu_\theta|\mathscr{R}$), where $\mathscr{R} = \mathscr{B}(X) \cap M_\theta$.*

(viii) *If $\nu = |\mu_\theta||\mathscr{D}(\mathscr{K})$, then ν has a unique extension $\hat{\nu}$ to $\mathscr{B}(X)$ as a Radon-regular measure and $\hat{\nu}(E) = |\mu_\theta|(E)$ for $E \in \mathscr{B}_c(X) \cap M_\theta$.*

(ix) *There exists a positive linear functional ψ on $C_c(X)$ such that $\hat{\nu} = \check{\mu}_\psi$, where ν is as in* (viii). *Besides $|\theta| \leqslant \psi$.*

Proof. Let $\theta_1 = \operatorname{Re}\theta$ and $\theta_2 = \operatorname{Im}\theta$.

(i) As $M_\theta = M_{\theta_1} \cap M_{\theta_2}$ and $U_1 \cap U_2 \in M_\theta$ for $U_i \in \mathscr{U} \cap M_{\theta_i}$, $i = 1, 2$, (i) is immediate from Theorem 4.5 (i).

(ii) The proof is similar to that of Theorem 4.5 (ii).

(iii) This is an immediate consequence of (i) and the inequality

$$|\mu_\theta|(E) \leqslant 4 \sup\left\{|\mu_\theta(F)| : F \subset E,\ F \in M_\theta\right\}.$$

(vi) The proof of Theorem 4.5 (vii) (c) holds here verbatim.

(iv) This is immediate from (vi).

(v) The proof of the $\mathscr{D}(\mathscr{K})$-regularity of $|\mu_\theta||\mathscr{D}(\mathscr{K})$ in Theorem 4.5 (vi) holds here verbatim.

(vii) Let $E \in M_\theta$. Since $\mu_{\theta_j^+}(E) < \infty$ and $\mu_{\theta_j^-}(E) < \infty$ for $j = 1, 2$, by Theorem 2.2 there exist A, $B \in \mathscr{B}_c(X) \cap M_\theta$ such that $A \subset E \subset B$, A σ-compact and $\mu_{\theta_j^+}(B \setminus A) = \mu_{\theta_j^-}(B \setminus A) = 0$ for $j = 1, 2$. Thus $|\mu_\theta|(B \setminus A) = 0$. As by (vi) $|\mu_\theta|(F) = v(\mu_\theta|\mathscr{R}, \mathscr{R})(F)$ for $F \in \mathscr{R}$, the result holds.

(viii) The proof is the same as that of Theorem 4.6 (ii).

(ix) The first part is due to Theorem 4.1. As in the proof of Theorem 4.6 (iv) we have

$$\begin{aligned}|\theta(f)| &= \left|\int_K f\,\mathrm{d}(\mu_{\theta_1}|\mathscr{B}(K)) + i\int_K f\,\mathrm{d}(\mu_{\theta_2}|\mathscr{B}(K))\right| \\ &\leqslant \int_K |f|\,\mathrm{d}v(\mu_\theta|\mathscr{B}(K),\mathscr{B}(K)) \\ &\leqslant \int_X |f|\,\mathrm{d}\hat{\nu} = \int_X |f|\,\mathrm{d}\breve{\mu}_\psi = \psi(|f|)\end{aligned}$$

for $f \in C_c(X)$ with $\operatorname{supp} f = K$ since $v(\mu_\theta|\mathscr{B}(K),\mathscr{B}(K)) \leqslant v(\mu_\theta|\mathscr{D}(\mathscr{K}),\mathscr{D}(\mathscr{K})) = |\mu_\theta||\mathscr{D}(\mathscr{K})$ by (iv). Therefore, $|\theta| \leqslant \psi$. □

Lemma 4.8. *If $\theta \in \mathscr{K}(X)^*$, then $|\mu_\theta(K)| \leqslant \mu_{|\theta|}(K)$, $K \in \mathscr{K}$.*

Proof. Let $\theta_1 = \operatorname{Re}\theta$ and $\theta_2 = \operatorname{Im}\theta$. Then, by Theorem 2.2, $\mu_{\theta_j^+}|M_\theta$ and $\mu_{\theta_j^-}|M_\theta$ are M_θ-regular for $j = 1, 2$. Therefore, given $n \in \mathbb{N}$, there exists $U_n \in \mathscr{U} \cap M_\theta$ such that $K \subset U_n$, $\mu_{\theta_j^+}(U_n \setminus K) < \frac{1}{n}$, $\mu_{\theta_j^-}(U_n \setminus K) < \frac{1}{n}$ for $j = 1, 2$. Let $W_n = \bigcap_{k=1}^{n} U_k$ and $W = \bigcap_{1}^{\infty} W_n$. Then $W \in M_\theta$, as M_θ is a δ-ring. Besides, $K \subset W$ and $\mu_{\theta_j^+}(W \setminus K) = \mu_{\theta_j^-}(W \setminus K) = 0$ for $j = 1, 2$. By Urysohn's lemma there exists $f_n \in C_c^+(X)$ with $\chi_K \leqslant f_n \leqslant \chi_{W_n}$. If $g_n = \bigwedge_1^n f_i$, then $g_n \downarrow \chi_K$ a.e. with respect to $\mu_{\theta_j^+}$ and $\mu_{\theta_j^-}$ for $j = 1, 2$. If $K_1 = \operatorname{supp} g_1$, then by the Lebesgue dominated convergence Theorem we have $\lim_n \int_X g_n\,\mathrm{d}\breve{\mu}_{\theta_j^+} = \lim_n \int_{K_1} g_n\,\mathrm{d}\breve{\mu}_{\theta_j^+} = \mu_{\theta_j^+}(K)$ and $\lim_n \int_X g_n\,\mathrm{d}\breve{\mu}_{\theta_j^-} = \lim_n \int_{K_1} g_n\,\mathrm{d}\breve{\mu}_{\theta_j^-} = \mu_{\theta_j^-}(K)$ for $j = 1, 2$, where we consider the restriction $\breve{\mu}_{\theta_j^+}|\mathscr{B}(K_1) = \mu_{\theta_j^+}|\mathscr{B}(K_1)$ and $\breve{\mu}_{\theta_j^-}|\mathscr{B}(K_1) = \mu_{\theta_j^-}|\mathscr{B}(K_1)$. Then as $g_n \to \chi_K$ $\mu_{|\theta|}$-a.e., by the same theorem

$$\begin{aligned}|\mu_\theta(K)| &= \left|\lim_n \int_{K_1} g_n\,\mathrm{d}\mu_\theta\right| = |\lim_n \theta(g_n)| \leqslant \lim_n |\theta|(g_n) \\ &= \lim_n \int_X g_n\,\mathrm{d}\breve{\mu}_{|\theta|} = \mu_{|\theta|}(K).\end{aligned}$$

□

Lemma 4.9. *For $\theta \in \mathscr{K}(X)^*$ and $A \in \mathscr{B}(X) \cap M_\theta$, $|\mu_\theta|(A) = 0$ if and only if $|\mu_\theta|(A) = 0$. Consequently, $M_\theta \subset M_{|\theta|}$.*

Proof. Suppose $\mu_{|\theta|}(A) = 0$. Let $\theta_1 = \operatorname{Re}\theta$ and $\theta_2 = \operatorname{Im}\theta$. Thus, by Theorem 4.6 (vi) and by proposition 15, §1, Chapter IV of [1] we have

$$\begin{aligned}
\mu_{|\theta|}(A) &\leqslant (\mu_{|\theta_1|} + \mu_{|\theta_2|})(A) = v(\mu_{\theta_1}, M_{\theta_1})(A) + v(\mu_{\theta_2}, M_{\theta_2})(A) \\
&= \sup\{v(\mu_{\theta_1}, M_{\theta_1})(C) : C \subset A,\ C \in \mathscr{K}\} \\
&\quad + \sup\{v(\mu_{\theta_2}, M_{\theta_2})(C) : C \subset A,\ C \in \mathscr{K}\} \\
&= \sup\{v(\mu_{\theta_1}|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))(C) : C \subset A, C \in \mathscr{K}\} \\
&\quad + \sup\{v(\mu_{\theta_2}|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))(C) : C \subset A, C \in \mathscr{K}\} \\
&\leqslant 2\sup\{v(\mu_{\theta}|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K}))(C) : C \subset A, C \in \mathscr{K}\} = 2|\mu_\theta|(A) = 0
\end{aligned}$$

since $v(\mu_{\theta_j}, M_{\theta_j})$ is M_{θ_j}-regular and $v(\mu_{\theta_j}|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K})) = v(\mu_{\theta_j}, M_{\theta_j})|\mathscr{D}(\mathscr{K})$ by (iv) and (v) of Theorem 4.5 for $j = 1, 2$ and $v(\mu_\theta|\mathscr{D}(\mathscr{K}), \mathscr{D}(\mathscr{K})) = |\mu_\theta||\mathscr{D}(\mathscr{K})$ and $|\mu_\theta|$ is M_θ-regular by (iv) and (iii) of Theorem 4.7.

Conversely, let $|\mu_\theta|(A) = 0$. Then $\mu_{|\theta_j|}(A) \leqslant \mu_{|\theta|}(A)$ for $j = 1, 2$ by the said proposition of [1] and consequently, by Theorem 4.6 (vi) we have

$$|\mu_\theta|(A) \leqslant v(\mu_{\theta_1}, M_{\theta_1})(A) + v(\mu_{\theta_2}, M_{\theta_2})(A) = \mu_{|\theta_1|}(A) + \mu_{|\theta_2|}(A) = 0$$

and hence $|\mu_\theta|(A) = 0$.

Now, let $E \in M_\theta$. Then, by Theorem 4.7 (vii), $E = A \cup (E \setminus A)$ with $A \subset E \subset B$, A, $B \in \mathscr{B}(X) \cap M_\theta$ and $|\mu_\theta|(B \setminus A) = 0$. Consequently, $\mu_{|\theta|}(B \setminus A) = 0$ and thus $\mu^*_{|\theta|}(E \setminus A) = 0$. This shows that $E \setminus A \in M_{|\theta|}$ and hence $E \in M_{|\theta|}$ since $A \in \mathscr{B}(X)$ and $\mu_{\theta_j^+}(A) < \infty$ and $\mu_{\theta_j^-}(A) < \infty$ for $j = 1, 2$. □

Lemma 4.10. *For $\theta \in \mathscr{K}(X)^*$, $M_\theta = M_{|\theta|}$.*

Proof. In fact, in the light of Lemma 4.9 it suffices to show that $M_{|\theta|} \subset M_\theta$. Let $E \in M_{|\theta|}$. Then by Theorem 2.2 there exist A, $B \in \mathscr{B}(X)$ such that $A \subset E \subset B$ and $\mu_{|\theta_j|}(B \setminus A) = 0$. Thus $\mu_{|\theta|}(E) = \mu_{|\theta|}(A)$. If $\theta_1 = \operatorname{Re}\theta$ and $\theta_2 = \operatorname{Im}\theta$, then by Proposition 15, §1, Chapter IV of [1], $\mu_{|\theta_j|}(B \setminus A) = 0$, $\mu_{|\theta_j|}(A) \leqslant \mu_{|\theta|}(A) < \infty$ and

$$\mu^*_{|\theta_j|} = \mu^*_{\theta_j^+} + \mu^*_{\theta_j^-} \text{ for } j = 1, 2.$$

Thus $\mu_{\theta_j^+}(B \setminus A) = \mu_{\theta_j^-}(B \setminus A) = 0$ for $j = 1, 2$. Consequently, $\mu_{\theta_j^+}(E \setminus A) = \mu_{\theta_j^-}(E \setminus A) = 0$ for $j = 1, 2$ and hence $E \in M_\theta$. □

Theorem 4.11. *If $\theta \in \mathscr{K}(X)^*$, then $M_\theta = M_{|\theta|}$ and $|\mu_\theta| = \mu_{|\theta|}$.*

Proof. Let $E \in M_\theta$ and $\varepsilon > 0$. By Theorem 4.7 (vi) there exists $\{E_i\}_1^n \subset \mathscr{D}(\mathscr{K})$ with $E_i \cap E_j = \emptyset$ for $i \neq j$ and $\bigcup_i^n E_i \subset E$ such that

$$\sum_1^n |\mu_\theta(E_i)| > |\mu_\theta|(E) - \tfrac{1}{2}\varepsilon.$$

As $\mu_\theta|\mathscr{D}(\mathscr{K})$ is $\mathscr{D}(\mathscr{K})$-regular by Theorem 4.7 (v), there exists $C_i \in \mathscr{K}$ such that $C_i \subset E_i$ and $|\mu_\theta(E_i) - \mu_\theta(C_i)| < \frac{\varepsilon}{2n}$ for $i = 1, \ldots, n$. Consequently, by Lemma 4.8 we have

$$\mu_{|\theta|}(E) \geqslant \mu_{|\theta|}\Big(\bigcup_1^n C_i\Big) = \sum_1^n \mu_{|\theta|}(C_i) \geqslant \sum_1^n |\mu_\theta(C_i)|$$
$$> \sum_1^n |\mu_\theta(E_i)| - \tfrac{1}{2}\varepsilon > |\mu_\theta|(E) - \varepsilon.$$

Thus

$$\mu_{|\theta|}(E) \geqslant |\mu_\theta|(E) \text{ for } E \in M_\theta. \tag{1}$$

If ψ is as in Theorem 4.7 (ix), then as $|\theta| \leqslant \psi$, from (viii) and (ix) of Theorem 4.7 we have

$$\mu_{|\theta|}(E) \leqslant \check{\mu}_\psi(E) = |\mu_\theta|(E)$$

for $E \in \mathscr{B}_c(X) \cap M_\theta$. Consequently, by (1)

$$\mu_{|\theta|}(E) = |\mu_\theta|(E) \tag{2}$$

for $E \in \mathscr{B}_c(X) \cap M_\theta$.

If $E \in M_\theta$, then by Theorem 2.2 there exist A, $B \in \mathscr{B}(X)$ with $A \subset E \subset B$, A σ-compact, $|\mu_\theta|(B \setminus A) = 0$ and $|\mu_\theta|(E) = |\mu_\theta|(A)$ (vide the proof of Theorem 4.7 (vii)). Hence by (2) and by Lemma 4.9 we have

$$|\mu_\theta|(E) = |\mu_\theta|(A) = \mu_{|\theta|}(A) = \mu_{|\theta|}(A) + \mu_{|\theta|}(E \setminus A) = \mu_{|\theta|}(E).$$

Since $M_\theta = M_{|\theta|}$ by Lemma 4.10, the theorem is established. □

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