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ON OSCILLATORY PROPERTIES OF SOLUTIONS OF A CERTAIN NONLINEAR THIRD ORDER DIFFERENTIAL EQUATION

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1. We are interested in oscillatory solutions of a nonlinear differential equation of the third order

$$u''' + q(t)u' + p(t)u^{\alpha} = 0, \tag{1}$$

where $p(t)$, $q(t)$ and $q'(t)$ are continuous functions on the interval (a, ∞), $-\infty < a$, $\alpha > 1$ is a ratio of odd integers.

By a solution of (1) we mean a function $u(t)$ defined on an interval (T, ∞), $a \leqslant T$, with a continuous third derivative, which satisfies equation (1). By an oscillatory solution we mean a nontrivial solution u of (1) that has infinitely many null-points with a limit point at infinity. Otherwise the solution is called nonoscillatory.

The object of generalization are results in the papers [4] and [1] concerning oscillatory solutions of equation (1) in the case $q(t) \equiv 0$ on (a, ∞).

In the proofs of this paper some results of the paper [3] are applied.

2. N. Parhi and S. Parhi [4] proved the following theorem:

Theorem A. *Let $p(t) < 0$ for $t \in (a, \infty)$ and let $\int_{t_0}^{\infty} p(\tau)\,d\tau = -\infty$, $t_0 > a$. Then every bounded nontrivial solution u of the differential equation*

$$u''' + p(t)u^{\alpha} = 0 \tag{1_1}$$

defined on $\langle t_0, \infty)$ is oscillatory on $\langle t_0, \infty)$.

In the paper [1] the following theorem is proved:

Theorem B. *Let the assumptions of Theorem A be fulfilled. Then a necessary and sufficient condition for a solution u of (1_1) to be oscillatory for $t \geqslant t_0$ is that*

$$u(t)u''(t) - \frac{1}{2}u'^2(t) < 0$$

for $t \geqslant t_0$.

I. W. Heidel [3] proved several interesting results. Some of them are formulated in the following two theorems.

Theorem C. *Let $q(t) \leqslant 0$ and $p(t) \leqslant 0$ for $t \in (a, \infty)$. If $u(t)$ is a nontrivial nonoscillatory solution of (1) on $\langle t_0, \infty)$, $t_0 > a$, then there is a number $c \geqslant t_0$ such that either $u(t)u'(t) > 0$ for $t \geqslant c$, or $u(t)u'(t) \leqslant 0$ for $t \geqslant c$.*

Theorem D. *Let the supposition of Theorem C be fulfilled and let, moreover, $\int_{t_0}^{\infty} tq(t)\mathrm{d}t > -\infty$, or $-\frac{2}{t^2} \leqslant q(t) \leqslant 0$ for $t \geqslant t_0$, $t_0 > 0$. If $u(t)$ is a nontrivial nonoscillatory solution of (1) on $\langle t_0, \infty)$, then $u(t)u'(t) > 0$ for $t \in \langle t_0, \infty)$.*

In some proofs the two following lemmas will be used. They are special cases of Lemma 4 of [5].

Lemma A. *Let $u(t) \in C^2(\langle t_0, \infty))$. Then $u(t) > 0$, $u'(t) < 0$, $u''(t) \leqslant 0$ cannot hold for all $t \geqslant t_0$.*

Lemma B. *Let $u(t) \in C^3(\langle t_0, \infty))$. Then $u(t) > 0$, $u'(t) < 0$, $u'''(t) \geqslant 0$ cannot hold for all $t \geqslant t_0$.*

3. In this section we generalize Theorem A and Theorem B for equation (1) if $p(t) < 0$, $q(t) \leqslant 0$ and $q'(t) \geqslant 0$ for $t \in (a, \infty)$ and prove a corollary of Theorem D for the solutions of equation (1).

Lemma 1. *Let $p(t) < 0$, $q(t) \leqslant 0$ for $t \in (a, \infty)$ and let $u(t)$ be a solution of (1) with the properties $u(t_0) \geqslant 0$, $u'(t_0) \geqslant 0$, $u''(t_0) > 0$, $t_0 > a$. Then $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ for $t > t_0$ and $u(t) \to \infty$, $u'(t) \to \infty$ for $t \to \infty$.*

This lemma can be proved in a similar manner as Theorem 1 in [1] and therefore the proof is omitted.

In the paper [2] the following theorem is presented without proof and therefore we prove it.

Theorem 1. *Let the coefficients of equation (1) fulfil the suppositions of Lemma 1 and let, moreover, $\int_{t_0}^{\infty} p(\tau)\mathrm{d}\tau = -\infty$ and $q'(t) \geqslant 0$ for $t \geqslant t_0$. Then every nontrivial* *bounded solution u of* (1) *on* $\langle t_0, \infty)$, $t_0 > a$, *is either oscillatory on* $\langle t_0, \infty)$, *or converges monotonically to zero for* $t \to \infty$.

Proof. Without loss of generality suppose $u(t) > 0$ and bounded on $\langle t_0, \infty)$. We prove that this can occur only if it converges monotonically to zero for $t \to \infty$. By Lemma 1, $u'(t)$ cannot have on $\langle t_0, \infty)$ more than two zeros and then it does not change the sign. Then there exists a point $T \geqslant t_0$ such that for $u'(t)$, $t > T$ we have two possibilities. Let

(i) $$u'(t) > 0 \quad \text{for} \quad t > T.$$

Integrating equation (1), in this case we get the identity

(2) $$u''(t) + q(t)u(t) + \int_{t_0}^{t} \left[p(\tau)u(\tau)^{\alpha-1} - q'(\tau)\right] u(\tau)\mathrm{d}\tau = k.$$

The boundedness of $u(t)$ and the suppositions $q'(t) \geqslant 0$ for $t \geqslant t_0$ and $\int_{t_0}^{\infty} p(\tau)\mathrm{d}\tau = -\infty$ imply that there exists a point $T_1 \geqslant T$ such that

$$u''(t) > 0 \quad \text{for} \quad t > T_1.$$

From Lemma 1 we get a contradiction with the supposition that $u(t)$ is bounded.

Let

(ii) $$u'(t) < 0 \quad \text{for} \quad t > T.$$

Then $u(t)$ is decreasing for $t > T$. There are two cases for $u(t)$. Either $u(t) > K > 0$ and then the identity (2) implies that $u''(t) \to \infty$ for $t \to \infty$, which contradicts $u'(t) < 0$ for $t > T$, or $K = 0$ and $u(t)$ converges monotonically to zero. □

Lemma 2. *Let the coefficients of equation* (1) *fulfil the assumptions of Theorem* 1. *Let u be a solution of* (1) *with the property $u(t) > 0$ for $t \geqslant t_0$. Then there exists a point $t_1 \geqslant t_0$ such that either $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ for $t \geqslant t_1$, or $u'(t) < 0$ for $t \geqslant t_1$, and*

$$\lim_{t\to\infty} u(t) = 0, \quad \limsup_{t\to\infty} u'(t) = 0.$$

Proof. Suppose that $u(t) > 0$ for $t \geqslant t_0$. There are three possibilities for $u''(t)$.

1) $u''(t) > 0$ for $t > t_0$.

Then $u'(t)$ is increasing for $t > t_0$ and we have two cases:

(i) $u'(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. In this case $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ for $t \geqslant t_1$, and this is the assertion of Lemma 2.

(ii) $u'(t) < 0$ for $t \geqslant t_1 \geqslant t_0$ and then there exists $\lim_{t\to\infty} u'(t) = K \leqslant 0$. If $K < 0$ then $u(t) \leqslant K(t-t_1) + u(t_1)$ which is a contradiction with $u(t) > 0$ for large $t > t_1$. Therefore $\lim_{t\to\infty} u'(t) = 0$ and $\lim_{t\to\infty} u(t) = k \geqslant 0$. If $k > 0$, then the identity (2) implies that $u''(t) \to \infty$ for $t \to \infty$, but this is a contradiction with $u'(t) \to 0$ for $t \to \infty$ and therefore $\lim_{t\to\infty} u(t) = 0$.

2) $u''(t) < 0$ for $t < t_0$. By Lemma A the case $u'(t) < 0$ cannot occur for $t \geqslant t_1 \geqslant t_0$. If there exists $t_1 \geqslant t_0$ such that $u'(t) > 0$ for $t \geqslant t_1$, then from the identity (2) we obtain a contradiction.

3) $u''(t)$ has infinitely many null-points for $t \geqslant t_0$ at which it changes the sign ($u''(t)$ oscillates on $\langle t_0, \infty)$). For $u'(t)$ we have three possibilities:

(i) $u'(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Then $u(t)$ is increasing and from (2) we obtain a contradiction with the oscillatoricity of $u''(t)$.

(ii) $u'(t) < 0$ for $t \geqslant t_1 \geqslant t_0$. Then necessarily $\lim_{t\to\infty} \sup u'(t) = 0$ and $\lim_{t\to\infty} u(t) = 0$. In the opposite case (2) implies that $u''(t) \to \infty$ for $t \to \infty$ and then it cannot oscillate.

(iii) $u'(t)$ is oscillatory on $\langle t_1, \infty)$, $t_1 \geqslant t_0$. This case is in contradiction with the assertion of Theorem C. □

Lemma 3. *Let the supposition of Theorem* 1 *on the coefficients of equation* (1) *be fulfilled. Then for every solution u of* (1) *which converges monotonously to zero for $t \to \infty$ with the property $u(t)u'(t) < 0$ for $t > t_0$ there exists $T \geqslant t_0$ such that for $t \geqslant T$ the inequality*

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) < 0$$

holds.

Proof. Let $u(t) > 0$ and $u'(t) < 0$ for $t \geqslant t_0$. Let $t_0 < t_1 < t_2 < \ldots$ be an arbitrary sequence of points diverging to infinity if $u'(t)$ is a monotone function, or such a sequence of points for which $u'(t_i) \to 0$ if $t_i \to \infty$. By Lemma 2 such sequence of t_i, $i = 1, 2, \ldots$ exists.

Multiply equation (1) by the solution u and integrate from t_i to t. We obtain the integral identity

$$\text{(3)}\quad u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) + \int_{t_i}^{t}\Big[p(\tau)u(\tau)^{\alpha-1}(t) - \frac{1}{2}q'(\tau)\Big]u^2(\tau)\,d\tau =$$
$$= u(t_i)u''(t_i) - \frac{1}{2}u'^2(t_i) + \frac{1}{2}q(t_i)u^2(t_i), \quad i = 1, 2, \ldots.$$

For a solution $u(t)$ for which $u(t) > 0$, $u'(t) < 0$ for $t \geqslant t_0$ the identity (2) and Theorem 1 yield that $u''(t)$ must be bounded on $\langle t_0, \infty)$ the integral $\int_{t_0}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)] u(\tau) \mathrm{d}\tau$ exists and $u(t) \to 0$ for $t \to \infty$. Then there exists a point $T > t_0$ such that for $t > T$ we have

$$-\int_t^{\infty} \Big[p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}\, q'(\tau)\Big] u^2(\tau) \mathrm{d}\tau \leqslant -\int_t^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)] u(\tau) \mathrm{d}\tau.$$

If in the identity (3) $t_i \to \infty$ for $i \to \infty$ we obtain the relation

$$u(t)u''(t) - \frac{1}{2}\, u'^2(t) + \frac{1}{2}\, q(t)u^2(t) = \int_t^{\infty} \Big[p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}\, q'(\tau)\Big] u^2(\tau) \mathrm{d}\tau < 0$$

for $t \geqslant T$. □

Theorem 2. *Let $p(t) < 0$, $q(t) \leqslant 0$, $q'(t) \geqslant 0$ for $t \in (a, \infty)$ and let $\int_{t_0}^{\infty} p(\tau) \mathrm{d}\tau = -\infty$, $t_0 \geqslant a$. Then a necessary and sufficient condition for the solution u of (1) defined on $\langle t_0, \infty)$ to be oscillatory for $t \geqslant t_0$, or to be monotonously converging to zero is*

$$u(t)u''(t) - \frac{1}{2}\, u'^2(t) + \frac{1}{2}\, q(t)u^2(t) < 0 \tag{4}$$

for $t > t_1$, $t_1 \geqslant t_0$.

P r o o f. Sufficient condition. Let (4) hold for $t > t_1 \geqslant t_0$ and let e.g. $u(t) > 0$ for $t \geqslant t_0$. It follows from Lemma 2 that there exists $t_1 \geqslant t_0$ such that either $u(t) > 0$, $u'(t) > 0$, $u''(t) > 0$ for $t > t_1$, or $u(t) > 0$, $u'(t) < 0$ for $t > t_1$. In the latter case the solution u by Lemma 2 monotonously converges to zero (and by Lemma 3 fulfils the condition (4)). In the former case, by Lemma 1 $u(t) \to \infty$ for $t \to \infty$, and from the integral identity (3) for $t_i = T \geqslant t_1$ and from the suppositions of Theorem 2 it follows that for large t the inequality

$$u(t)u''(t) - \frac{1}{2}\, u'^2(t) + \frac{1}{2}\, q(t)u^2(t) > 0$$

holds and this is a contradiction with (4).

Necessary condition. By Lemma 3 we must prove that an oscillatory solution in $\langle t_0, \infty)$ fulfils the condition (4). Let $u(t)$ be an oscillatory solution of (1) on $\langle t_0, \infty)$ and let t_i $i = 1, 2, \ldots$ be its null-points on $\langle t_0, \infty)$. Then the identity (3) implies that the function $u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t)$ is increasing on $\langle t_1, \infty)$ and $u(t_i)u''(t_i) - \frac{1}{2}u'^2(t_i) + \frac{1}{2}q(t_i)u^2(t_i) < 0$ for $i = 1, 2, \ldots$. Consequently, (4) holds for $t \geqslant t_1$. □

Corollary 1. *Let the suppositions of Theorem 2 be fulfilled and let, moreover, the suppositions of Theorem D be fulfilled. Then a necessary and sufficient condition for a solution u of (1) to be oscillatory on $\langle t_0, \infty)$ is that the condition (4) is fulfilled for $t > T \geqslant t_0$, where T is sufficiently large.*

4. In this section we shall study equation (1) with $p(t) < 0$, $q(t) \geqslant 0$, $q'(t) \geqslant 0$ for $t \in (a, \infty)$.

Theorem 3. *Let $p(t) < 0$, $q(t) \geqslant 0$, $q'(t) \geqslant 0$ for $t \in (a, \infty)$, let $q(t)$ be bounded on (a, ∞) and $\int_{t_0}^{\infty} p(t)\mathrm{d}t = -\infty$, $t_0 > a$. Then every bounded solution of (1) on $\langle t_0, \infty)$ is oscillatory on this interval.*

P r o o f. Let e.g. $u(t) > 0$ be bounded on $\langle t_0, \infty)$, $t_0 > a$. Three cases for its first derivative $u'(t)$ are possible.

1) $u'(t) > 0$ for $t \geqslant T \geqslant t_0$. The identity (2) for $t \geqslant t_0$ implies that $u''(t) \to \infty$ for $t \to \infty$ and therefore $u(t)$ cannot be bounded on $\langle t_0, \infty)$, which is a contradiction.

2) $u'(t) \leqslant 0$ for $t \geqslant T \geqslant t_0$. In this case equation (1) implies that $u'''(t) > 0$ for $t \geqslant T$ and by Lemma B this is impossible.

3) $u'(t)$ has infinitely many null-points at which it changes the sign. If in this case $u(t) > K > 0$ for $t \geqslant t_0$, then we obtain from (2) that $u''(t) > 0$ for $t \geqslant T \geqslant t_0$ and therefore $u'(t)$ must be increasing for $t \geqslant T$, which is a contradiction with the oscillatoricity of $u'(t)$. Therefore $\liminf_{t\to\infty} u(t) = 0$. If we suppose that $\liminf_{t\to\infty} u(t) = 0$, we have the following two possibilities:

(i) $\int_{t_0}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)]u(\tau)\mathrm{d}\tau = \infty$. However, in this case we obtain from (2) that $u''(t) > 0$ for $t \geqslant T \geqslant t_0$ and this contradiction with the oscillatory of $u'(t)$.

(ii) $0 \leqslant -\int_{t_0}^{\infty} [p(\tau)u^{\alpha-1}(\tau) - q'(\tau)]u(\tau)\mathrm{d}\tau < \infty$.

In this case let $\{t_i\}_{i=1}^{\infty}$, $t_i \to \infty$ for $i \to \infty$, be a sequence of points at which $u'(t_i) = 0$ and $u''(t_i) > 0$. Clearly $u(t_i) \to 0$ for $i \to \infty$. It follows from (2) that $\{u''(t_i)\}$ is bounded on $\langle t_0, \infty)$.

Now if we write the identity (3) in the form

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) + \int_{t_1}^{t} [p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]u^2(\tau)\mathrm{d}\tau = k, \tag{5}$$

where $k = u(t_1)u''(t_1) - \frac{1}{2}u'^2(t_1) + \frac{1}{2}q(t_1)u^2(t_1) > 0$, we obtain for $u''(t_i)$, $i = 2, 3, \ldots$ the equality

$$u''(t_i) = \frac{k}{u(t_i)} - \frac{1}{2}q(t_i)u(t_i) - \frac{1}{u(t_i)}\int_{t_1}^{t_i} [p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)]u^2(\tau)\mathrm{d}\tau.$$

It follows from this relation for $t_i \to \infty$ that $u''(t_i) \to \infty$, which is a contradiction with the boudedness of $\{u''(t_i)\}$. □

Lemma 4. *Let the supposition of Theorem 3 be fulfilled. Then for every solution u of* (1) *with the property $u(t) > 0$ for $t \geqslant t_0$, there exists $T \geqslant t_0$ such that for all $t \geqslant T$ the inequality*

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + q(t)u^2(t) > 0 \tag{6}$$

holds.

Proof. Let $u(t) > 0$ for $t \geqslant t_0$. Then there are three possibilities for $u'(t)$.

(i) $u'(t) > 0$ for $t \geqslant t_0$. It follows from (5), where $t_1 = t_0$, that there exists $T \geqslant t_0$ such that for all $t \geqslant T$ the inequality (6) holds.

(ii) $u'(t) \leqslant 0$ for $t \geqslant t_0$. From equation (1) we obtain in this case that $u'''(t) > 0$ for $t \geqslant t_0$, but by Lemma B this is not possible.

(iii) $u'(t)$ has on $\langle t_0, \infty)$ at least two null-points at which it changes the sign.

At one of them we have $u''(t) \geqslant 0$. Let $t = T_1$. It follows from (5) with $t_1 = T_1$ that $k = u(T_1)u''(T_1) + \frac{1}{2}q(T_1)u^2(T_1) \geqslant 0$ and that there exists $T \geqslant T_1$ such that (6) holds for $t > T$. □

Theorem 4. *Let the suppositions of Theorem 3 be fulfilled. Then a necessary and sufficient condition for the solution u of* (1) *defined on $\langle t_0, \infty)$, $t_0 > a$ to be oscillatory on $\langle t_0, \infty)$ is that*

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) < 0 \tag{7}$$

for $t \geqslant T \geqslant t_0$.

Proof. Sufficient condition. Let u be a solution of (1) satisfying the condition (7) for $t \geqslant T \geqslant t_0$, and let e.g. $u(t) > 0$ for $t \geqslant T$. By Lemma 4 there exists $T_1 \geqslant t_0$ such that (6) holds for $t \geqslant T_1$, and this is a contradiction with (7). This proves that u must be oscillatory.

Necessary condition can be proved in the same manner as in Theorem 2. □

Remark 1. Let u be a solution of (1) with the property $u(t_0) = u'(t_0) = 0$, $u''(t_0) > 0$ and let the supposition of Theorem 2 or of Theorem 4 be fulfilled. Then $u(t) > 0$ for $t > t_0$.

This assertion follows from the identity (5), where $k = 0$.

5. In this section we shall discuss two cases of suppositions on the coefficients of equation (1), in which we do not prove a necessary and sufficient condition for the oscillatoricity of solutions of equation (1).

Theorem 5. *Let $p(t) < 0$, $q(t) \geqslant 0$, $q'(t) \leqslant 0$ for $t \in (a, \infty)$ and let $\lim_{t\to\infty} q(t) = 0$ and $\int_{t_0}^{\infty} p(\tau)\mathrm{d}\tau = -\infty$, $t_0 \geqslant a$. Then every bounded solution u of* (1) *defined on $\langle t_0, \infty)$ is either oscillatory on $\langle t_0, \infty)$, or $\lim_{t\to\infty} \inf |u(t)| = 0$.*

Proof. Let u be a bounded solution of (1) defined on $\langle t_0, \infty)$, $t_0 \geqslant a$. If we integrate (1) term by term for $t \geqslant t_0$ we have

$$u''(t) + \int_{t_0}^{t} q(\tau)u'(\tau)\mathrm{d}\tau + \int_{t_0}^{t} p(\tau)u^{\alpha}(\tau)\mathrm{d}\tau = u''(t_0). \tag{8}$$

Let $|u(t)| \leqslant K$, $K > 0$. The function $q(t)$ is nonincreasing and $q(t) \to 0$ for $t \to \infty$. For a given $\varepsilon/4K > 0$ there exists $T_0 \geqslant t_0$ such that $0 \leqslant q(t) \leqslant \frac{\varepsilon}{4K}$ for $t > T_0$. Let T_1, T_2 be such that $T_2 > T_1 > T_0$. By the second mean value theorem there exists c, $T_1 \leqslant c \leqslant T_2$ such that

$$\begin{aligned}\left|\int_{T_1}^{T_2} q(\tau)u'(\tau)\mathrm{d}\tau\right| &= \left|q(T_1)\int_{T_1}^{c} u'(\tau)\mathrm{d}\tau + q(T_2)\int_{c}^{T_2} u'(\tau)\mathrm{d}\tau\right| \\ &\leqslant |q(T_1)|\,|u(c) - u(T_1)| + |q(T_2)|\,|u(T_2) - u(c)| \\ &\leqslant \frac{\varepsilon}{4K}\,4K = \varepsilon.\end{aligned}$$

Then by the Cauchy-Bolzano criterion $\int_{t_0}^{\alpha} q(\tau)u'(\tau)\mathrm{d}\tau$ converges.

Suppose now that $u(t) > 0$ for $t > t_0$ for $t > t_0$ and u is bounded on $\langle t_0, \infty)$. For $u'(t)$ there are three posibilities on $\langle t_0, \infty)$.

(i) $u'(t) > 0$ for $t \geqslant t_1 \geqslant t_0$. Then it follows from (8) that $u''(t) \to \infty$ for $t \to \infty$, which is a contradiction with the boundedness of $u(t)$.

(ii) $u'(t) \leqslant 0$ for $t \geqslant t_1 \geqslant t_0$. In this case equation (1) implies that $u'''(t) > 0$ for $t \geqslant t_1$, but by Lemma B this is impossible.

(iii) $u'(t)$ changes its sign infinitely many times on $\langle t_0, \infty)$. If in this case $u(t) > K_1 > 0$ for $t \geqslant t_1 \geqslant t_0$, then $\int_{t_0}^{\infty} p(\tau)u^{\alpha}(\tau)\mathrm{d}\tau = -\infty$ and (8) yields $u''(t) \to \infty$ for $t \to \infty$, which is a contradiction with the boundedness of $u(t)$. Therefore $\lim_{t\to\infty} \inf u(t) = 0$. □

Theorem 6. *Let $p(t) < 0$, $q(t) \geqslant 0$, $q'(t) \leqslant 0$ for $t \in (a, \infty)$ and let $\lim_{t\to\infty} q(t) = 0$ and $-p(t) + \frac{1}{2}q'(t) \geqslant k > 0$ for $t \in (a, \infty)$. If u is a solution of* (1) *defined on $\langle t_0, \infty)$, $t_0 \geqslant a$ such that it fulfils the condition*

$$u(t)u''(t) - \frac{1}{2}u'^2(t) + \frac{1}{2}q(t)u^2(t) < 0 \tag{9}$$

for $t \geqslant t_1 \geqslant t_0$, then u is oscillatory on $\langle t_0, \infty)$.

Proof. The supposition $-p(t)+\frac{1}{2}q'(t)\geqslant k>0$ clearly implies the relation $\int_{t_0}^{\infty}p(\tau)\mathrm{d}\tau=-\infty$.

Suppose now that a solution $u(t)$ of (1) fulfils the condition (9) and that it is nonoscillatory. Let e.g. $u(t)>0$ for $t\geqslant t_0$. Then for $u'(t)$ there are two possibilities. The possibiblity $u'(t)\leqslant 0$ for $t\geqslant t_1\geqslant t_0$ is eliminated by Lemma B.

(i) $u'(t)>0$ for $t\geqslant t_1\geqslant t_0$. If $u(t)$ is bounded from above then it is oscillatory, or $\liminf_{t\to\infty}u(t)=0$ by Theorem 5. But $\liminf_{t\to\infty}u(t)=0$ is in contradiction with $u(t)>0$, $u'(t)>0$ for $t\geqslant t_1$. If $\lim_{t\to\infty}u(t)=\infty$, then there exists $T_1\geqslant t_1$ such that for $t\geqslant T_1$ we have $u(t)>1$ and $-\left[p(t)u^{\alpha-1}(t)-\frac{1}{2}q'(t)\right]u^2(t)\geqslant ku^2(t)>k>0$. The integral identity (5) implies that $u(t)u''(t)-\frac{1}{2}u'^2(t)+\frac{1}{2}q(t)u^2(t)>0$ for large t and this is a contradiction with (9).

(ii) $u'(t)$ changes its sign infinitely many times. Then there exists a sequence $\{t_k\}_{k=1}^{\infty}$, $t_k\to\infty$ for $k\to\infty$, such that $u'(t_k)=0$, $u''(t_k)\geqslant 0$ for $k=1,2,\ldots$. But at the points t_k we obtain a contradiction with (9). □

Theorem 7. *Let $p(t)<0$, $q(t)\leqslant 0$, $q'(t)\leqslant 0$ for $t\in(a,\infty)$, let $q(t)$ be bounded from below on (a,∞) and $\int_{t_0}^{\infty}p(\tau)\mathrm{d}\tau=-\infty$, $t_0\geqslant a$. Then every bounded solution u defined on $\langle t_0,\infty)$ is either oscillatory on $\langle t_0,\infty)$, or it converges monotonously to zero for $t\to\infty$.*

Proof. Let u be a solution of (1) defined and bounded on $\langle t_0,\infty)$, $t_0\geqslant a$ and let e.g. $u(t)>0$ for $t>t_0$. Then for $u'(t)$ we have three possibilities:

(i) $u'(t)>0$ for $t\geqslant t_1\geqslant t_0$. In this case we obtain from (1) that $u'''(t)>0$ for $t\geqslant t_1$. If $u''(t)>0$ for $t\geqslant t_1$ then Lemma 1 yields $u(t)\to\infty$ for $t\to\infty$, which is a contradiction with boudedness of $u(t)$.

If $u''(t)<0$ for $t\geqslant t_1\geqslant t_0$, then after integration of (1) term by term we obtain the relation (8) where $t_0=t_1$. Clearly $\lim_{t\to\infty}u(t)=k<\infty$, $\int_{t_1}^{\infty}pu^{\alpha}(\tau)\mathrm{d}\tau=-\infty$ and if $m\leqslant q(t)\leqslant 0$ then $\int_{t_1}^{t}q(\tau)u'(\tau)\mathrm{d}\tau\geqslant m\int_{t_1}^{t}u'(\tau)\mathrm{d}\tau\geqslant m[k-u(t_1)]$, and $\int_{t_1}^{t}q(\tau)u'(\tau)\mathrm{d}\tau\to l$, $0>l>-\infty$.

We see now from (8) that $u''(t)\to\infty$ for $t\to\infty$ and this is a contradiction with the boudedness of $u(t)$.

(ii) $u'(t)\leqslant 0$ for $t\geqslant t_1\geqslant t_0$. Then $u(t)$ is nonincreasing. Let $\lim_{t\to\infty}u(t)=k\geqslant 0$. If $k>0$ we obtain from (8) that $u''(t)\to\infty$ for $t\to\infty$ which is again a contradiction.

(iii) $u'(t)$ changes its sign infinitely many times and there exists a point $T_1\geqslant t_0$ such that $u(T_1)>0$, $u'(T_1)=0$, $u''(T_1)\geqslant 0$. By Lemma 1 $u(t)\to\infty$ for $t\to\infty$ and this is a contradiction. □

Theorem 8. *Let $-p(t)>k>0$, $q(t)\leqslant 0$, $q'(t)\leqslant 0$ for $t\in(a,\infty)$ and let $q'(t)\to 0$ for $t\to\infty$. Let $u(t)$ be a solution of (1) defined on $\langle t_0,\infty)$, $t_0\geqslant a$, which* *fulfils the condition* (9) *for* $t \geqslant t_1 \geqslant t_0$. *Then* $u(t)$ *is either oscillatory on* $\langle t_0, \infty)$, *or it converges monotonously to zero for* $t \to \infty$.

P r o o f. Let u be a soution of (1) defined on $\langle t_0, \infty)$ which fulfils (9) for $t \geqslant t_1$, and let u be nonoscillatory.

Let e.g. $u(t) > 0$ for $t \geqslant t_0$. $u'(t)$ has three possibilities:

(i) $u'(t) \geqslant 0$ for $t \geqslant T_1 \geqslant t_0$. Then the identity (5) with $t_1 = T_1$ contradicts (9), because $\int_{T_1}^{\infty} \left[p(\tau)u^{\alpha-1}(\tau) - \frac{1}{2}q'(\tau)\right] u^2(\tau)\mathrm{d}\tau = -\infty$.

(ii) $u'(t) < 0$ for $t \geqslant T_1 \geqslant t_0$. In this case, if $u(t) > L > 0$ we obtain a contradiction as in the case (i). Therefore $u(t)$ can converge monotonously to zero for $t \to \infty$.

(iii) $u'(t)$ changes its sign infinitely many times. In this case there exists a point $T > t_0$ at which $u(T) > 0$, $u'(T) = 0$, $u''(T) \geqslant 0$, and by Lemma 1 we obtain a contradiction with the property (iii) of $u'(t)$. □

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