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MODULAR BASES IN A HILBERT A-MODULE

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Summary. Following Ozawa [4] we introduce the concept of a modular base in a Hilbert A-module and prove that the cardinalities of any two such bases are the same.

Keywords: H^*-algebra, primitive projection, projection base, Hilbert A-module, modular base, modular dimension

AMS classification: 46H25

INTRODUCTION

Throughout this paper A denotes a proper H^*-algebra with an inner product and norm $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively ([1]). A nonzero selfadjoint idempotent in A is called a projection. If a projection cannot be expressed as a sum of two pairwise orthogonal projections, then it is said to be primitive. A maximal family of pairwise orthogonal primitive projections is called a projection base. Denote by $\tau(A)$ the trace class of A, i.e. let $\tau(A) = \{xy\colon x, y \in A\}$ and let tr be the trace functional on $\tau(A)$. tr has the following properties: $\operatorname{tr} xy = \langle y, x^*\rangle = \langle x, y^*\rangle = \operatorname{tr} yx$ $(x, y \in A)$. For each $a \in A$ there exists a unique positive element $[a] \in A$ (i.e. such that $\langle [a]x, x\rangle \geqslant 0$ $(x \in A)$) such that $[a]^2 = a^*a$, moreover $a \in \tau(A)$ if and only if $[a] \in \tau(A)$. Then a norm can be defined on $\tau(A)$ by setting $\tau(a) = \operatorname{tr}[a]$ $\big(a \in \tau(A)\big)$, for which the following relations hold: $|\operatorname{tr}(.)| \leqslant \tau(.)$, $\|\cdot\| \leqslant \tau(.)$ and $\tau(xy) \leqslant \|x\|\,\|y\|$ $(x, y \in A)$ ([6]). It was shown in [7] that $\tau(A)$ is a Banach *-algebra. In [8] Smith proved that every nonzero positive element $a \in A$ has a unique spectral representation $a = \sum\limits_n \lambda_n e_n$, where the λ_n-s are positive real numbers with $\lambda_i > \lambda_j$ if $i < j$, and the e_n-s are mutually orthogonal projections.

Now let H be a (right) A-module on which there is a generalized inner product $[\cdot,\cdot]$, i.e. $[\cdot,\cdot]\colon H \times H \to \tau(A)$ such that
(1) $[f,f] \geqslant 0$ and $[f,f] = 0$ if and only if $f = 0$;
(2) $[f, g + h] = [f,g] + [f,h]$;
(3) $[f, ga] = [f,g]a$;
(4) $[f,g]^* = [g,f]$
holds for every $f, g, h \in H$ and $a \in A$. $[\cdot,\cdot]$ satisfies the so called strong Schwartz inequality, i.e.

$$(\tau[f,g])^2 \leqslant \tau[f,f]\tau[g,g] \quad (f,g \in H).$$

For a more general statement cf. [3].

In the rest of the paper let H be a Hilbert A-module, i.e. suppose that H is complete in the metric d defined by

$$d(f,g) = \sqrt{\tau[f-g, f-g]} \quad (f,g \in H).$$

As Saworotnow showed in [5], on H a linear structure can be introduced such that $\lambda(fa) = (\lambda f)a = f(\lambda a)$ $(\lambda \in \mathbf{C},, a \in A,, f \in H)$ and

$$\langle f,g \rangle = \mathrm{tr}[g,f] \quad (f,g \in H)$$

defines an inner product on H. Denote by $\|\cdot\|$ the norm corresponding to this inner product.

It is easy to see that A is a Hilbert A-module if we define the generalized inner product by $[x,y] = x^*y$ $(x,y \in A)$. Similar considerations can be performed for every eA, where $e \in A$ is a projection. The norms arising from these generalized inner products are equal to the original one.

If H_1 and H_2 are Hilbert A-modules, then a mapping $U\colon H_1 \to H_2$ is called an A-unitary operator if it is surjective and
(1) $U(f+g) = Uf + Ug$,
(2) $U(fa) = (Uf)a$,
(3) $[Uf, Ug] = [f,g]$
for every $f,g \in H_1$ and $a \in A$. In this case U is a unitary operator between the Hilbert spaces H_1 and H_2. Finally, it was also proved in [4] that

$$f = \sum_\alpha f e_\alpha$$

holds for every $f \in H$ and projection base $\{e_\alpha\}_{\alpha \in \Lambda}$.

We begin with the following basic lemma.

Lemma 1. *Let $f \in H$ be such that $[f, f]$ is a projection. Then the submodule fA is isomorphic an isometric to $[f, f]A$, consequently fA is closed. Moreover, we have $f[f, f] = f$.*

Proof. Let $f \in H$ and consider the function $T(fa) = [f, fa] = [f, f]a$ $(a \in A)$. Then T is a linear operator preserving the module operation with the range $[f, f]A$. Since

$$[fa, fa] = a^*[f, f]^*[f, f]a = \big[[f, f]a, [f, f]a\big] \quad (a \in A),$$

taking traces we get that T is an isometry. Since $[f, f]A$ is closed so is fA. Now let $[f, f] = e_1 + \ldots + e_n$ be the decomposition of $[f, f]$ into pairwise orthogonal primitive projections (cf. [1, Theorem 3.2]). Extend the set $\{e_1, \ldots, e_n\}$ by $\{e'_\alpha\}_{\alpha \in \Lambda}$ to a projection base. Then

$$f = f[f, f] + \sum_\alpha fe'_\alpha.$$

Since $[fe'_\alpha, fe'_\alpha] = e'_\alpha[f, f]e'_\alpha = 0$ $(\alpha \in \Lambda)$, it follows that $f[f, f] = f$. □

Definition. The family $\{f_\alpha\}_{\alpha \in \Lambda} \subset H$ is said to be *modular orthonormal* if

(1) $[f_\alpha, f_\beta] = 0$ if $\alpha \neq \beta$;

(2) $[f_\alpha, f_\alpha]$ is primitive projection in A for every $\alpha \in \Lambda$.

A maximal modular orthonormal family is called a *modular base*.

Remark 1. If $\{f_\alpha\}_{\alpha \in \Lambda} \subset H$ is a modular orthonormal family, $a_\alpha \in A$ $(\alpha \in \Lambda)$ and $F \subset \Lambda$ is a finite set, then, using the above lemma, simple calculation shows that $[f - \sum_{\alpha \in F} f_\alpha a_\alpha, f - \sum_{\alpha \in F} f_\alpha a_\alpha]$ equals

$$[f, f] + \sum_{\alpha \in F}([f_\alpha, f] - [f_\alpha, f_\alpha]a_\alpha)^*([f_\alpha, f] - [f_\alpha, f_\alpha]a_\alpha) - \sum_{\alpha \in F}[f, f_\alpha][f_\alpha, f].$$

As a consequence we have

$$[f, f] \geqslant \sum_{\alpha \in F}[f, f_\alpha][f_\alpha, f].$$

Theorem 1. *Let $\{f_\alpha\}_{\alpha \in \Lambda}$ be a modular orthonormal family in H. Then the following assertions are equivalent:*

(i) *$\{f_\alpha\}_{\alpha \in \Lambda}$ is a modular base.*

(ii) *If $f \in H$ is such that $[f_\alpha, f] = 0$ $(\alpha \in \Lambda)$, then $f = 0$.*

(iii) *The orthogonal sum (in the Hilbert space sense) of the closed subspaces* $H_\alpha = f_\alpha A$ $(\alpha \in \Lambda)$ *is* H.

(iv) $f = \sum_\alpha f_\alpha [f_\alpha, f]$ *for every* $f \in H$.

(v) $[f, g] = \sum_\alpha [f, f_\alpha][f_\alpha, g]$ *holds for any* $f, g \in H$, *where the sum is unconditionally convergent in the norm* τ.

(vi) $\|f\|^2 = \sum_\alpha \|[f_\alpha, f]\|^2$ *for every* $f \in H$.

Proof. (i) $\Rightarrow$ (ii). Suppose that $f \in H$ and $[f_\alpha, f] = 0$ $(\alpha \in \Lambda)$. If $f \neq 0$, then let $[f, f] = \sum_n \lambda_n e_n$ be the spectral representation of $[f, f]$. Now for $f' = \frac{1}{\sqrt{\lambda_1}} f e_1$ we have $[f', f'] = e_1$ and $[f_\alpha, f'] = 0$ $(\alpha \in \Lambda)$, which is a contradiction.

(ii) $\Rightarrow$ (iii). By the previous lemma H_α is a closed submodule which is a subspace as well $(\alpha \in \Lambda)$. Now the implication follows from [5, Lemma 3].

(iii) $\Rightarrow$ (iv). If $f \in H$, then for every $\alpha \in \Lambda$ there exists an $a_\alpha \in A$ such that $f = \sum_\alpha f_\alpha a_\alpha$. This implies that

$$[f_\alpha, f] = [f_\alpha, f_\alpha] a_\alpha \quad (\alpha \in \Lambda).$$

Since $f_\alpha [f_\alpha, f_\alpha] = f_\alpha$ $(\alpha \in \Lambda)$, we have (iv).

(iv) $\Rightarrow$ (v). We have to prove only the unconditional convergence. By the properties of the norm τ we have

$$\tau([f, f_\alpha][f_\alpha, g]) \leqslant \|[f_\alpha, f]\| \, \|[f_\alpha, g]\| \quad (\alpha \in \Lambda).$$

But from the proof of Lemma 1 we know that

$$\|[f_\alpha, f]\|^2 = \|f_\alpha [f_\alpha, f]\|^2 \quad \text{and} \quad \|[f_\alpha, g]\|^2 = \|f_\alpha [f_\alpha, g]\|^2 \quad (\alpha \in \Lambda).$$

Now (v) follows.

(v) $\Rightarrow$ (vi). Let $f \in H$. Then

$$[f, f] = \sum_\alpha [f, f_\alpha][f_\alpha, f].$$

By the above remark, using the fact that τ is additive on the positive elements of $\tau(A)$, we have

$$\tau\Big([f, f] - \sum_{\alpha \in F} [f, f_\alpha][f_\alpha, f]\Big) = \tau[f, f] - \sum_{\alpha \in F} \tau[f, f_\alpha][f_\alpha, f] = \|f\|^2 - \sum_{\alpha \in F} \|[f_\alpha, f]\|^2$$

for every $F \subset \Lambda$, which implies (vi).

The implications (vi) $\Rightarrow$ (ii) $\Rightarrow$ (i) are trivial. □

R e m a r k 2. In Corollary 1 below which can be called a generalized Bessel inequality we need the following simple statement.

If $(e_\varepsilon)_{\varepsilon\in\mathscr{E}}$ is a net of selfadjoint elements of $\tau(A)$ converging in the norm τ to an $a \in \tau(A)$ such that there is an $x \in A$ for which $x = x^*$ and

$$a_\varepsilon \leqslant x \quad (\varepsilon \in \mathscr{E}),$$

then $a \leqslant x$.

To prove it we note that the convergence in τ implies the convergence in $\|\cdot\|$.

Corollary 1. *Let $\{f_\alpha\}_{\alpha\in\Lambda}$ be a modular orthonormal family in H. Then*

$$[f,f] \geqslant \sum_\alpha [f,f_\alpha][f_\alpha,f],$$

where the sum is unconditionally convergent in $\tau(A)$.

P r o o f. By Theorem 1 (vi) we have

$$\sum_\alpha \tau([f,f_\alpha][f_\alpha,f]) = \sum_\alpha \|[f_\alpha,f]\|^2 < \infty.$$

Now the statement follows from Remarks 1 and 2. □

In the proof of our main theorem we use

Lemma 2. *Let $n,m \in \mathbf{N}$ be such that $n \neq m$. Suppose that $e_1, \ldots, e_{n+m}$ are primitive projections in A. Then*

$$e_1 + \ldots + e_n \neq e_{n+1} + \ldots + e_{n+m}.$$

P r o o f. Using the second structure theorem for H^*-algebras ([1, Theorem 4.2 and 4.3]) A can be identified with the direct sum of Hilbert-Schmidt operator algebras $\bigoplus_{\gamma\in\Gamma} \mathbf{HS}(\mathscr{H}_\gamma)$, where the $\mathscr{H}_\gamma$-s are suitably chosen Hilbert spaces and the inner product on $\mathbf{HS}(\mathscr{H}_\gamma)$ may differ from the standard one at most by a real constant which is not less than 1. In this representation every e_j can be considered as a vector $(P^j_\gamma)_{\gamma\in\Gamma}$ such that there is exactly one $\gamma \in \Gamma$ for which $P^j_\gamma \neq 0$ and for this γ P^j_γ is one dimensional projection on $\mathscr{H}_\gamma$. Now suppose that $e_1 + \ldots + e_n = e_{n+1} + \ldots + e_{n+m}$. It is easy to see that there is a $\gamma_0 \in \Gamma$ such that

$$\operatorname{card}\{k \in \{1,\ldots,n\} : P^k_{\gamma_0} \neq 0\} \neq \operatorname{card}\{l \in \{n+1,\ldots,n+m\} : P^l_{\gamma_0} \neq 0\}.$$

If we take the trace corresponding to the Hilbert space $\mathscr{H}_{\gamma_0}$ in the equation

$$\sum_{k=1}^{n} P_{\gamma_0}^{k} = \sum_{l=n+1}^{n+m} P_{\gamma_0}^{l},$$

we arrive at a contradiction. □

Theorem 2. *If* $\{f_\alpha\}_{\alpha\in\Lambda}$ *and* $\{g_i\}_{i\in I}$ *are modular bases in* H, *then* $\operatorname{card}\Lambda = \operatorname{card} I$.

Proof. If Λ and I are infinite sets, then the proof is standard. In fact, for every $\alpha \in \Lambda$ consider the set

$$S_\alpha = \{i \in I\colon [f_\alpha, g_i] \neq 0\}.$$

By Theorem 1 (vi) S_α is countable. (ii) of the same theorem implies that every $i \in I$ belongs to at least one set S_α ($\alpha \in \Lambda$). Then we have

$$\operatorname{card} I \leqslant \operatorname{card}\Lambda \cdot \aleph_0 = \operatorname{card}\Lambda.$$

Changing the role of Λ and I we get the other inequality.

Now we prove that if one of these bases is finite, then so is the other. To this end suppose that Λ is finite and I is infinite. Since $|\operatorname{tr}(.)| \leqslant \tau(.)$, thus, by Theorem 1 (v), we have

$$\begin{aligned}\infty > \operatorname{tr}\sum_\alpha [f_\alpha, f_\alpha] &= \operatorname{tr}\sum_\alpha\sum_i [f_\alpha, g_i][g_i, f_\alpha]\\ &= \sum_\alpha\sum_i \operatorname{tr}[f_\alpha, g_i][g_i, f_\alpha]\\ &= \sum_i\sum_\alpha \operatorname{tr}[g_i, f_\alpha][f_\alpha, g_i]\\ &= \sum_i \operatorname{tr}[g_i, g_i] = \infty,\end{aligned}$$

where we have used the fact that the trace of a projection is not less than 1.

Finally, assume that Λ and I are finite. Then we have

$$\sum_\alpha [f_\alpha, f_\alpha] = \sum_i [g_i, g_i]$$

and Lemma 2 implies that $\operatorname{card}\Lambda = \operatorname{card} I$. □

As a consequence we can state

Corollary 2. *All projection bases in A have the same cardinality.*

Proof. Consider A as a Hilbert A-module. The only thing which has to be proved is that every projection base $\{e_\alpha\}_{\alpha\in\Lambda}$ is a modular base in A. By Theorem 1 (ii) we have to show that $e_\alpha x = 0$ ($\alpha \in \Lambda$) implies that $x = 0$. But this follows from the first structure theorem for H^*-algebras ([1, Theorem 4.1]). □

Remark 3. By the second structure theorem for H^*-algebras it is to see that the relation between $\operatorname{Dim} A$ and $\dim A$ (the Hilbert space dimension of A) is quite complicated. However, it is easy to see that $\operatorname{Dim} A < \infty$ if and only if $\dim A < \infty$.

Just as in [4], card Λ occuring in Theorem 2 is called *the modular dimension* of H and denoted by $\operatorname{Dim} H$.

Remark 4. It is natural to ask whether any two Hilbert A-modules H_1 and H_2 are A-unitarily equivalent (i.e. there is an A-unitary operator between H_1 and H_2) if and only if $\operatorname{Dim} H_1 = \operatorname{Dim} H_2$. The "only if" part is obvious while the "if" part does not hold in general. To show it let $A = \mathbf{C} \oplus \mathbf{C} \oplus \mathbf{M}_{2\times 2}(\mathbf{C})$ (where $\mathbf{M}_{2\times 2}(\mathbf{C})$ is the algebra of 2×2-type complex matrices) with the natural operations and inner product. Let

$$e_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 0 \\ \mathbf{I} \end{pmatrix},$$

where $I \in \mathbf{M}_{2\times 2}(\mathbf{C})$ is the identity matrix. Then $H_1 = e_1 A$ and $H_2 = e_2 A$ can be considered Hilbert A-modules. It is trivial that $\operatorname{Dim} H_1 = \operatorname{Dim} H_2 = 2$, but, if H_1 and H_2 were A-unitarily equivalent, then they would be unitarily equivalent Hilbert spaces as well which is a contradiction.

As for our final result we need the following lemma which shows that the topological simplicity of A is a necessary and sufficient condition of the validity of the statement formulated in the above remark.

Lemma 3. *The minimal right ideals of A are A-unitarily equivalent if and only if A is topologically simple.*

Proof. In the proof we use [2, Proposition 7 and Theorem 8 on pp. 47–48].

To prove the necessity let $I_1 = \overline{Ae_1A}$, $I_2 = \overline{Ae_2A}$ be two different minimal closed ideals of A, where $e_1, e_2 \in A$ are primitive projections. Then $R_1 = e_1 A \subset I_1$ and $R_2 = e_2 A \subset I_2$ are minimal right ideals for which $R_1^* R_1 \subset I_1$, $R_2^* R_2 \subset I_2$ since I_1, I_2 are selfadjoint. But $I_1 \neq I_2$ implies that $I_1 \perp I_2$, consequently we get that there is no A-unitary operator between R_1 and R_2. Now it follows that A is topologically simple.

To prove the sufficiency we may assume that $A = \mathbf{HS}(\mathscr{H})$, where $\mathscr{H}$ is a Hilbert space and the inner product on $\mathbf{HS}(\mathscr{H})$ is the standard one. Let P_1 and P_2 be one dimensional projections on $\mathscr{H}$. Suppose that φ_1 and φ_2 are vectors from $\mathscr{H}$ of norm 1 generating the range of P_1 and P_2, respectively. If S is the operator defined by $Sx = \langle x, \varphi_1 \rangle \varphi_2$ $(x \in \mathscr{H})$, then let

$$U(P_1 T) = S P_1 T \quad \big(T \in \mathbf{HS}(\mathscr{H})\big).$$

Simple calculation shows that U is an $\mathbf{HS}(\mathscr{H})$-unitary operator from $P_1\mathbf{HS}(\mathscr{H})$ onto $P_2\mathbf{HS}(\mathscr{H})$. □

From this lemma, by Lemma 1 and Theorem 1 (iii) and (iv), we have

Theorem 3. *Let A be topologically simple. If H_1 and H_2 are Hilbert A-modules, then H_1 and H_2 are A-unitarily equivalent if and only if* $\operatorname{Dim} H_1 = \operatorname{Dim} H_2$.

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