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REPRESENTATIONS OF RIESZ SPACES AS SPACES OF MEASURES II

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1. INTRODUCTION

In this paper, the investigations of [7] (and partly of [5]) are continued. Therefore I assume familiarity with the results of [7].

In the elementary section 2, the notions which are necessary to describe measure representations of σ-hypercomplete Riesz spaces (see Theorem 5.5) are presented. In section 3, some Riesz subspaces of the π-space E, which appear in their measure representations as well-known spaces of measures, are introduced. The next section is devoted to the study of discrete representation spaces, and in connection with this, to give simple descriptions of the dual spaces of the Riesz subspaces of E considered in the preceding section. Finally in section 5, the σ-hypercomplete Riesz spaces are introduced and investigated; in particular the dual spaces of some of their Riesz subspaces are determined similarly to the results of section 4, and the question is solved whether $L^p(\mu)$-spaces are σ-hypercomplete.

Notations and terminology are adopted from [7]. In addition, I denote by δ_y the Dirac measure at the point y, and, for a measure μ, by $\mathfrak{N}(\mu)$ the set of all μ-null sets.

2. σ-RIESZ SUBSPACES AND σ-EMBEDDED RIESZ SUBSPACES

Let E be a Riesz space, and let F be a Riesz subspace of E. I call F σ-embedded in E if $\sup x_n$ exists in F for each sequence (x_n) from F_+ which is bounded in E. F is, by definition, a σ-Riesz subspace of E if for each sequence (x_n) from F_+ for which $\sup x_n$ exists in E, this supremum belongs to F.

For instance, $\{f \in \mathbf{R}^{\mathbf{N}} \colon \{f \neq 0\}$ is finite$\}$ is an order dense ideal of $\mathbf{R}^{\mathbf{N}}$ (hence in particular σ-Dedekind complete) which is neither σ-embedded in $\mathbf{R}^{\mathbf{N}}$ nor a σ-Riesz subspace of $\mathbf{R}^{\mathbf{N}}$, while $\{\alpha 1_{\mathbf{N}} : \alpha \in \mathbf{R}\}$ is σ-embedded in $\mathbf{R}^{\mathbf{N}}$ and a σ-Riesz subspace of $\mathbf{R}^{\mathbf{N}}$ (without being order dense or an ideal).

The following assertions can easily be verified:

Proposition 2.1. *Let E, E_1, E_2 be Riesz spaces, and let F be a Riesz subspace of all of them. Then:*

(a) *If F is σ-embedded in E, then F is σ-Dedekind complete.*
(b) *E is σ-embedded in itself iff E is σ-Dedekind complete.*
(c) *If E_1 is a Riesz subspace of E_2 and F is σ-embedded in E_2, then F is also σ-embedded in E_1.*
(d) *E is a σ-Riesz subspace of itself.*
(e) *If E_1 is an ideal or an order dense Riesz subspace of E_2 and F is a σ-Riesz subspace of E_2, then F is also a σ-Riesz subspace of E_1.*

Proposition 2.2. *For a Riesz subspace F of E we have:*

(a) *If E is σ-Dedekind complete and F is a σ-Riesz subspace of E, then F is σ-embedded in E.*
(b) *If F is an ideal or order dense in E and F is σ-embedded in E, then F is a σ-Riesz subspace of E.*

That the assumption of σ-Dedekind completeness cannot be dropped in (a), is a consequence of Proposition 2.1 (b),(d). Also (b) does not hold anymore if F is not an ideal or not order dense in E; for this claim, consider $E := \{f \in \mathbf{R}^Y : f \text{ is bounded}\}$ and $F := C(Y)$, for a compact Stonian space Y. Then, by the Dedekind completeness of F, F is σ-embedded in E, but in general F is not a σ-Riesz subspace of E (take $Y := \beta\mathbf{N}$, and consider the sequence $(1_{\{n\}})_{n\in\mathbf{N}}$).

The only non-trivial result in this section is

Proposition 2.3. *If E is σ-Dedekind complete and F is an order dense σ-Riesz subspace of E, then the map*

$$\varphi : E^\pi \to F^\pi, \qquad \xi \mapsto \xi|_F$$

is a Riesz isomorphism.

P r o o f. Obviously φ is linear. Let $\xi \in E^\pi$ and $x \in F_+$. The order continuity of ξ implies

$$\begin{aligned}(\xi|_F)^+(x) &= \sup\{(\xi|_F)(z) : z \in F,\, 0 \leqslant z \leqslant x\} \\ &= \sup\{\xi(u) : u \in E,\, 0 \leqslant u \leqslant x\} = (\xi^+)(x).\end{aligned}$$

We conclude $(\xi|_F)^+ = \xi^+|_F$. This relation, together with the order continuity of ξ, also implies that φ is injective.

To prove that φ is surjective, take $\eta \in (F^\pi)_+$, and let $x \in E_+$. Assume that $\sup\{\eta(z)\colon z \in F, 0 \leqslant z \leqslant x\} = \infty$. Then there is an increasing sequence (z_n) in F_+ with $z_n \leqslant x$ for all n and $\sup \eta(z_n) = \infty$. There exists $z := \sup z_n$ in E, and since F is a σ-Riesz subspace of E, we get $z \in F$ which gives the contradiction $\eta(z) = \infty$. The assertion now follows from [8; 17B]. □

3. Bounded elements of a π-space

In this section, let E be a π-space, let e be a weak unit of E^ϱ, and let R be a subset of R_e which is closed under the formation of finite suprema and satisfies $e = \sup R$. Let further $(Y, \hat{u}, v)$ be an e-mr of E [7; 2.16].

Let me introduce some Riesz subspaces of E which correspond to spaces of bounded or σ-bounded measures:

$$E_c := \{x \in E\colon \text{ there exists } g \in R \text{ with } (e-g)(|x|) = 0\};$$
$$E_b := E[e];$$
$$E_{\sigma b} := \{x \in E\colon \text{ there exists a countable family } (g_\iota) \text{ from } R_e \text{ with } (e - \sup g_\iota)(|x|) = 0\}.$$

Observe that the definition of E_c depends on e and R, while E_b and $E_{\sigma b}$ are introduced only with the aid of e. On the other hand, it is also possible to define E_b and $E_{\sigma b}$ in terms of R as the following propositions will show.

Proposition 3.1. *For $x \in E$, the following are equivalent:*

(a) $x \in E_b$;

(b) $\sup\limits_{g \in R_e} g(|x|) < \infty$;

(c) *for each* $\varepsilon > 0$, *there exists* $g \in R_e$ *with* $(e-g)(|x|) < \varepsilon$;

(d) $\sup\limits_{g \in R} g(|x|) < \infty$;

(e) *for each* $\varepsilon > 0$, *there exists* $g \in R$ *with* $(e-g)(|x|) < \varepsilon$.

If (a)–(e) *are satisfied, then we have also*

(f) *There exists a disjoint sequence* (g_n) *in* R_e *with* $e(|x|) = \sum g_n(|x|)$.

Proof. All follows easily from the relation $e = \sup R_e = \sup R$. □

Proposition 3.2. *For $x \in E$, the following are equivalent:*

(a) $x \in E_{\sigma b}$;

(b) *there exists a countable family* (g_ι) *from* R *with* $(e - \sup g_\iota)(|x|) = 0$.

Proof. If $g \in R_e$, then $g = \sup_{h \in R} \inf(g, h)$; hence there exists a sequence (h_n) from R with $g(|x|) \leqslant \sup h_n(|x|)$. This observation implies the non-trivial implication of the proposition. ☐

I denote by $\mathscr{M}_{\sigma b}(Y)$ the set of σ-bounded elements of $\mathscr{M}(Y)$, i.e. the set of all $\mu \in \mathscr{M}(Y)$ for which there exists a sequence (K_n) of open-compact subsets of Y such that $Y \setminus \bigcup K_n \in \mathfrak{N}(\mu)$.

Here is the description of the spaces E_c, E_b and $E_{\sigma b}$ in terms of a measure representation:

Proposition 3.3. *We have $v(E_b) = \mathscr{M}_b(Y) \cap vE$ and $v(E_{\sigma b}) = \mathscr{M}_{\sigma b}(Y) \cap vE$; if R is associated to $(Y, \hat{u}, v)$, then also $v(E_c) = \mathscr{M}_c(Y) \cap vE$ holds.*

Proof. The first identity is obvious.

Let $x \in E_{\sigma b}$ and let $(g_\iota)_{\iota \in I}$ be a countable family from R_e with $(e - \sup g_\iota)(|x|) = 0$. For each ι there exists a sequence $(K_{\iota n})$ of open-compact subsets of Y with $U_{g_\iota} \setminus \bigcup_{n \in \mathbb{N}} K_{\iota n} \in \mathfrak{N}(vx)$. Then $Y \setminus \bigcup_{\iota \in I} \bigcup_{n \in \mathbb{N}} K_{\iota n} \in \mathfrak{N}(vx)$ which implies $vx \in \mathscr{M}_{\sigma b}(Y)$. If $x \in E$ with $vx \in \mathscr{M}_{\sigma b}(Y)$, then there exists a sequence (K_n) of open-compact subsets of Y with $Y \setminus \bigcup K_n \in \mathfrak{N}(vx)$. With $g_n := \hat{u}^{-1} 1_{K_n}$ we get $(e - \sup_{n \in \mathbb{N}} g_n)(|x|) = 0$, which shows that $x \in E_{\sigma b}$ holds.

It is clear that $x \in E_c$ implies $vx \in \mathscr{M}_c(Y)$, provided all U_g are compact ($g \in R$). If $vx \in \mathscr{M}_c(Y)$ for some $x \in E$ and if R is associated to $(Y, \hat{u}, v)$, then $(U_g)_{g \in R}$ is an open cover of $\operatorname{supp} vx$; hence there exists $g \in R$ with $\operatorname{supp} vx \subset U_g$, which implies $x \in E_c$. ☐

Corollary 3.4. *E_c, E_b, $E_{\sigma b}$ are order dense ideals of E with $E_c \subset E_b \subset E_{\sigma b}$, and $(E_c)^\pi$, $(E_b)^\pi$, $(E_{\sigma b})^\pi$ are order dense ideals of E^ϱ with $E^\pi \subset (E_{\sigma b})^\pi \subset (E_b)^\pi \subset (E_c)^\pi$.*

Corollary 3.5. *For each $x \in (E_{\sigma b})_+$ there exists a sequence (x_n) from $(E_c)_+$ with $x_n \uparrow x$.*

Proof. By Proposition 3.1, there is a sequence (K_n) of open-compact subsets of Y with $vx = \sup(1_{K_n} \cdot vx)$. Since $v(E_c)$ is order dense in $\mathscr{M}(Y)$ and since K_n is compact, there exists, for each n, a sequence $(z_{nm})_{m \in \mathbb{N}}$ in $(E_c)_+$ with $v(z_{nm}) \uparrow (1_{K_n} \cdot vx)$. Set $x_n := \sup_{k,j \leqslant n} z_{kj}$, for each $n \in \mathbb{N}$. ☐

Since $E_{\sigma b}$ is obviously a σ-Riesz subspace of E, we get immediately

Corollary 3.6. *$E_{\sigma b}$ is the smallest σ-Riesz subspace of E containing E_c.*

This result, together with Proposition 2.3, implies

Corollary 3.7. *If E is σ-Dedekind complete, then E^π and $(E_{\sigma b})^\pi$ are canonically Riesz isomorphic.*

Another condition implying $E^\pi = (E_{\sigma b})^\pi$ will be given in Proposition 4.9.

4. Discrete representations

Let E, e and R be as in the preceding section. I call an e-mr $(Y, \hat{u}, v)$ of E discrete if Y carries the discrete topology.

Recall that the element x of the arbitrary Riesz space F is called discrete if the ideal of F generated by x equals the vector subspace of F generated by x, and that F is called discrete if F possesses a maximal disjoint system consisting of discrete elements [2; p. 17].

Theorem 4.1. *E possesses a discrete e-mr iff E is discrete.*

P r o o f. If $(Y, \hat{u}, v)$ is a discrete e-mr of E, then $(v^{-1}\delta_y)_{y \in Y}$ is a maximal disjoint system of E.

Conversely, if E is discrete, then by [2; 2.17] there exists a set Y such that E can be identified with an order dense Riesz subspace of $\mathbf{R}^Y$. Then $E^\varrho = \mathbf{R}^Y$ and $Y = \{e \neq 0\}$. We endow Y with the discrete topology and get $\mathscr{M}(Y) = \mathbf{R}^Y$. Then $(Y, \hat{u}, v)$ is an e-mr of E, where

$$\hat{u}\colon E^\varrho \to \mathbf{R}^Y, \qquad \xi \mapsto \left(Y \to \mathbf{R},\, y \mapsto \frac{\xi(y)}{e(y)}\right),$$
$$v\colon E \to \mathbf{R}^Y, \qquad x \mapsto \left(Y \to \mathbf{R},\, y \mapsto x(y)e(y)\right).$$

□

But for discrete E the e-mr of E to which R is associated [7; 2.16] need not be discrete, as the following example shows:

Example 4.2. Let $E := \ell^1$. Then E is discrete, and $E^\varrho = \mathbf{R}^{\mathbf{N}}$. For each $f \in \mathbf{R}^{\mathbf{N}}$ let $\bar{f} \in C_\infty(\beta\mathbf{N})$ denote the natural extension of f to $\beta\mathbf{N}$. Set

$$\hat{u}\colon E^\varrho \to C_\infty(\beta\mathbf{N}), \qquad f \mapsto \bar{f},$$
$$v\colon E \to \mathscr{M}(\beta\mathbf{N}), \qquad x \mapsto \left(\mathfrak{B}_c(\beta\mathbf{N}) \to \mathbf{R},\, A \mapsto \sum_{n \in A \cap \mathbf{N}} x_n\right).$$

Then $(\beta\mathbf{N}, \hat{u}, v)$ is the $1_{\mathbf{N}}$-mr of E to which $\{1_{\mathbf{N}}\}$ is associated.

The next pair of theorems will clarify the phenomen observed in the last example.

Theorem 4.3. *The following are equivalent:*

(a) *Each e-mr of E is discrete;*
(b) *the e-mr of E to which R_e is associated is discrete;*
(c) *for each $g \in R_e$ and each disjoint family $(x_\iota)_{\iota \in I}$ from E_+, the set $\{\iota \in I: g(x_\iota) \neq 0\}$ is finite;*
(d) *for each $g \in R_e$ and each disjoint sequence $(x_n)_{n \in \mathbb{N}}$ from E_+, the set $\{n \in \mathbb{N}: g(x_n) \neq 0\}$ is finite.*

Proof. (a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are trivial.
(b)$\Rightarrow$(c): Let $(Y, \hat{u}, v)$ be the e-mr of E to which R_e is associated [7; 2.16]. Then each U_g (where $g \in R_e$) is compact, hence finite, which implies the assertion.
(d)$\Rightarrow$(a): Let $(Y, \hat{u}, v)$ be an e-mr of E. Assume that there exists an infinite open-compact subset U of Y. It is easy to construct a sequence (U_n) of pairwise disjoint non-empty open-compact subsets of U. For each n, there exists $x_n \in E_+ \setminus \{0\}$ with $\operatorname{supp} vx_n \subset U_n$. But $g := \hat{u}^{-1} 1_U \in R_e$, and we get the contradiction $g(x_n) = (vx_n)(U) > 0$ for all $n \in \mathbb{N}$. Hence each open-compact subset of Y is finite, and since Y is locally compact, it must be discrete. □

Theorem 4.4. *The following are equivalent:*

(a) *The e-mr of E to which R is associated is discrete;*
(b) *for each $g \in R$ and each disjoint family $(x_\iota)_{\iota \in I}$ from E_+, the set $\{\iota \in I: g(x_\iota) \neq 0\}$ is finite;*
(c) *for each $g \in R$ and each disjoint sequence $(x_n)_{n \in \mathbb{N}}$ from E_+, the set $\{n \in \mathbb{N}: g(x_n) \neq 0\}$ is finite.*

Proof. The proof is completely analogous to the proof of Theorem 4.3; in the implication (c)$\Rightarrow$(a) observe that one can find $g \in R$ with $\hat{u}g \geqslant 1_U$. □

The next proposition presents a condition which implies the equivalent assertions of Theorem 4.3.

Proposition 4.5. *If E has an order dense Riesz subspace F such that each disjoint sequence from F_+ is bounded in E, then E satisfies the equivalent conditions of Theorem 4.3; in particular, E is discrete.*

Proof. Let $g \in R_e$, let (x_n) be a disjoint sequence from E_+, and set $M := \{n \in \mathbb{N}: g(x_n) \neq 0\}$. For each $n \in M$ there exists $z_n \in F_+$ such that $z_n \leqslant x_n$ and

$g(z_n) > 0$. By assumption, there is $z \in E_+$ with $z \geqslant \frac{1}{g(z_n)} z_n$ for all $n \in M$. For each finite subset A of M, we get

$$g(z) \geqslant g\Big(\sum_{n \in A} \frac{1}{g(z_n)} z_n\Big) = \operatorname{card} A,$$

which shows that M is finite. □

The assumption in Proposition 4.5 is in particular satisfied if E is σ-laterally complete, but also in other cases, as the following example shows:

Example 4.6. Let X be an uncountable set, and let $\mathfrak{F}$ be an ultrafilter on X which contains the complements of the countable subsets of X and which contains a decreasing sequence (B_n) such that $\bigcap B_n = \emptyset$. Set

$$E := \{f \in \mathbf{R}^X : \exists \lim_{\mathfrak{F}} f \text{ in } \mathbf{R}\},$$
$$F := \{f \in \mathbf{R}^X : \{f \neq 0\} \text{ is finite}\}.$$

For each disjoint sequence (f_n) from F_+ we have obviously $\sup f_n \in E$. Now consider $g_n := n 1_{B_n \setminus B_{n+1}}$. Since $g_n \equiv 0$ on B_{n+1}, we have $g_n \in E$. But if $g \in E$ were an upper bound of (g_n), then fix $k > \lim_{\mathfrak{F}} g$ and observe that $g \geqslant k$ on B_k, a contradiction.

A similar example shows that the assumption of Proposition 4.5 is not equivalent to the conditions given in Theorem 4.3:

Example 4.7. Let $\mathfrak{F}$ be an ultrafilter on $\mathbf{N}$, set

$$E := \{f \in \mathbf{R}^{\mathbf{N}} : \exists \lim_{\mathfrak{F}} f \text{ in } \mathbf{R}\}$$

and $e := 1_{\mathbf{N}}$.

To show that 4.3(c) is satisfied, let $g \in R_e$ and let $(f_\iota)_{\iota \in I}$ be a disjoint family from E_+. If $\{\iota \in I : g(f_\iota) \neq 0\}$ were infinite, it would contain an infinite subset J such that $A := \bigcup_{\iota \in J} \{f_\iota \neq 0\} \notin \mathfrak{F}$. Then $f := \sum_{\iota \in J} \frac{1}{g(f_\iota)} f_\iota \in E$, and we get the contradiction $g(f) = \infty$.

But if $\mathfrak{F}$ contains the complements of the finite subsets of $\mathbf{N}$ and F is an order dense Riesz subspace of E, then the sequence $(n 1_{\{n\}})$ from F is not bounded in E.

As an application of the preceding investigations, I give some sufficient conditions under which the spaces $(E_c)^\pi$, $(E_b)^\pi$, $(E_{\sigma b})^\pi$ and E^π can easily be described in terms of representations.

Proposition 4.8. *If E possesses a discrete e-mr $(Y, \hat{u}, v)$ to which R is associated, then $\hat{u}((E_c)^\pi) = C(Y)$.*

Proof. By Proposition 3.3 we have $v(E_c) \subset \mathscr{M}_c(Y)$, and thus $C(Y) \subset \hat{u}((E_c)^\pi)$. By [3; 1.6.1a)] we get $\hat{u}((E_c)^\pi) \subset C_\infty(Y) = C(Y)$. □

Proposition 4.9. *If E possesses a σ-laterally complete order dense Riesz subspace F and $(Y, \hat{u}, v)$ is an e-mr of E, then $\hat{u}(E^\pi) = \hat{u}\big((E_{\sigma b})^\pi\big) = C_c(Y)$; in particular, E^π and $(E_{\sigma b})^\pi$ are canonically Riesz isomorphic.*

Proof. Obviously $C_c(Y) \subset \hat{u}(E^\pi) \subset \hat{u}\big((E_{\sigma b})^\pi\big)$, and by Proposition 4.5 and Theorem 4.3, we have $C_\infty(Y) = C(Y)$. Let $\xi \in \big((E_{\sigma b})^\pi\big)_+$, and assume that $\operatorname{supp} \hat{u}\xi$ is not compact. Then there exists a disjoint sequence (x_n) in F_+ such that, for each n, $\operatorname{supp} vx_n$ consists of one point y_n, and $(vx_n)(\{y_n\}) = 1/\hat{u}\xi(y_n)$. By assumption, $x := \sum x_n$ exists in F. We have $x \in E_{\sigma b}$ and $\xi(x) = \infty$, a contradiction. Thus $\hat{u}\big((E_{\sigma b})^\pi\big) \subset C_c(Y)$. □

The following example of a σ-laterally complete π-space which is not σ-Dedekind complete (perhaps of interest in its own right) shows that the assumptions of Proposition 4.9 and Corollary 3.7 are indeed different:

Example 4.10. Let E be the set of all $f \in \mathbf{R}^{\mathbf{R}}$ which can be written in the form $f = \sum_{\iota\in I} \alpha_\iota 1_{A_\iota} + g$, where $(A_\iota)_{\iota\in I}$ is a countable family of pairwise disjoint intervals with length > 0, $(\alpha_\iota)_{\iota\in I}$ is a family of reals, and $\{g \neq 0\}$ is countable. Obviously E is a σ-laterally complete order dense Riesz subspace of $\mathbf{R}^{\mathbf{R}}$ and thus a π-space. But if C denotes the Cantor set, then $1_C \notin E$, while 1_C is the pointwise infimum of a decreasing sequence from E; thus E is not σ-Dedekind complete.

Proposition 4.11. *Assume that E possesses an order dense Riesz subspace F such that for each disjoint sequence (x_n) from F_+ with $\sup e(x_n) \leqslant 1$ there exists $x \in E_b$ with $x \geqslant (1/n^2)x_n$ for all n. Then, if $(Y, \hat{u}, v)$ is an e-mr of E, we have $\hat{u}\big((E_b)^\pi\big) = C_b(Y)$.*

The assumption of this proposition is in particular satisfied if

(1) *E has a σ-laterally complete order dense Riesz subspace F,*

or if

(2) *E is Dedekind complete and has an order dense Riesz subspace F such that each disjoint sequence from F_+ is bounded in E.*

Proof. By Proposition 3.3, we have $C_b(Y) \subset \hat{u}\big((E_b)^\pi\big)$. If $\hat{u}\xi$ were unbounded for some $\xi \in \big((E_b)^\pi\big)_+$, then there exist disjoint open-compact subsets U_n of Y and $x_n \in F_+$ such that $\hat{u}\xi \geqslant n$ on U_n, $\operatorname{supp} vx_n \subset U_n$, $(vx_n)(U_n) = 1$, and we get the contradiction $\xi(x) = \infty$ (where $x \in E_b$ with $x \geqslant (1/n^2)x_n$ for all n).

That (1) implies the assumption of the proposition, is obvious. Now suppose that (2) holds. Then there exists $z \in E_+$ with $z \geqslant x_n$ for all n. Since vE is an ideal of $\mathscr{M}(Y)$ [7; 2.8 (a)], $\mu := \sup(1/n^2)vx_n$ belongs to vE. Keeping in mind Proposition 3.3, we see that $x := v^{-1}\mu$ satisfies the requirements. □

Observe that (1) or (2) in the preceding proposition imply that Y is discrete (Proposition 4.5 and Theorem 4.3), but that the assumption of this proposition does not, as can be seen by considering $E := \mathscr{M}(\beta\mathbf{N})$ and $e := 1_{\mathbf{N}}$. But even if Y is discrete, the assumption of Proposition 4.11 is weaker than (1) or (2): To verify this claim, consider $E := \ell^2$ and $e := 1_{\mathbf{N}}$ (for each $g \in R_e$, the set $\{g \neq 0\}$ is finite, so that each e-mr of E is discrete).

5. σ-HYPERCOMPLETENESS

The notion of hypercompleteness was introduced and investigated by the author in [4], [5], [6], and studied by Abramovich in [1]. In this section, the countable analogon will be considered.

Let E be a Riesz space. A subset R of $(E^\pi)_+$ satisfies, by definition, the σ-hc-condition if $\sup x_n$ exists in E for each increasing sequence (x_n) from E_+ which satisfies $\sup g(x_n) < \infty$ for each $g \in R$. The pair (e, R) is called σ-hc-pair of E if e is a weak unit of E^ϱ and R is a subset of R_e which satisfies the σ-hc-condition. The space E is called σ-hypercomplete if there exists a σ-hc-pair of E.

Obviously E is σ-hypercomplete iff there exists a weak unit e of E^ϱ such that (e, R_e) is a σ-hc-pair.

Recall that the definition of hypercompleteness requires that the condition above is satisfied for arbitrary upward-directed families (x_ι). Hence each hypercomplete Riesz space is σ-hypercomplete.

Let us consider some easy examples:

If X is an infinite set, then $E := \{f \in \mathbf{R}^X : \{f \neq 0\} \text{ is finite}\}$ is a Dedekind complete π-space which is not σ-hypercomplete (given a weak unit e of $E^\varrho = \mathbf{R}^X$, consider the sequence $f_n := \sum_{k=1}^{n} \frac{1}{k^2 e(x_k)} 1_{\{x_k\}}$, where (x_k) is an infinite sequence in X).

$E := \{f \in \mathbf{R}^X : \{f \neq 0\} \text{ is countable}\}$ is a σ-hypercomplete Dedekind complete π-space (see Theorem 5.3), which is not hypercomplete provided X is uncountable: For the last claim consider the family (1_A) where A runs through the finite subsets of X, and observe that for every weak unit e of $E^\varrho = \mathbf{R}^X$ and for each $g \in R_e$ the set $\{g \neq 0\}$ is finite.

That two weak units of E^ϱ (even of E^π) may behave differently with respect to the question whether they may serve as first component of a σ-hc-pair, was shown in [5; 3.1].

The property described in Proposition 4.5 and σ-hypercompleteness as well as σ-lateral completeness and σ-hypercompleteness are independent properties, as can be seen by regarding the space ℓ^1 and Example (iii) following Theorem 2.10 of [5].

Theorem 5.1. *Each σ-hypercomplete Riesz space E is a σ-Dedekind complete π-space, and $e = \sup R$ for each σ-hc-pair (e, R) of E.*

Proof. It is obvious that E is σ-Dedekind complete. Now let $x \in E_+$ such that $\xi(x) = 0$ for all $\xi \in E^\pi$; then $\sup_{n\in\mathbb{N}} nx$ exists in E which implies $x = 0$ since E is Archimedean. By [10; 88.3], E^π separates E. The proof of the last claim is the same as the proof of [5; 2.4]. □

That a Dedekind complete π-space need not be σ-hypercomplete, was observed in the first example above; that σ-hypercompleteness does not imply Dedekind completeness can be seen by considering $E := \{f \in \mathbf{R}^X : \exists \alpha_f \in \mathbf{R}$ such that $\{f \neq \alpha_f\}$ is countable$\}$, for uncountable X.

In some special cases, σ-hypercompleteness implies hypercompleteness:

Proposition 5.2. *If $(e, \{e\})$ is a σ-hc-pair of E, then it is also an hc-pair of E.*

Proof. Let $0 \leqslant x_\iota \uparrow$ such that $\alpha := \sup e(x_\iota) < \infty$. There exists a sequence (ι_n) such that (x_{ι_n}) increases and satisfies $\alpha = \sup_{n\in\mathbb{N}} e(x_{\iota_n})$. By assumption, $x := \sup x_{\iota_n}$ exists in E, and $e(x) = \alpha$. Let $(Y, \hat{u}, v)$ be the e-mr of E to which $\{e\}$ is associated [7; 2.16]. Assume $(vx)(A) < \sup(vx_\iota)(A)$ for some $A \in \mathfrak{B}_c(Y)$; since $(vx)(Y) = \sup(vx_\iota)(Y)$, we get $(vx)(Y \setminus A) > \sup(vx_\iota)(Y \setminus A)$ which contradicts the relation $x = \sup x_{\iota_n}$. Hence $vx = \sup vx_\iota$, and thus $x = \sup x_\iota$. □

Theorem 5.3. *For a π-space E, the following are equivalent:*

(a) *E is σ-universally complete;*

(b) *(e, R) is a σ-hc-pair of E for each weak unit e of E^ϱ and each subset R of R_e satisfying $e = \sup R$.*

Proof. (a) $\Rightarrow$ (b): By Proposition 4.5 and Theorem 4.3 there exists a set X such that $\mathbf{R}^X$ is the universal completion of E.

Let $0 \leqslant f_n \uparrow$ in E with $\sup g(f_n) < \infty$ for all $g \in R$. Let $x \in X$. Then $1_{\{x\}} \in E$, and there is $g \in R$ with $g(1_{\{x\}}) > 0$. We get

$$\sup f_n(x) = \frac{1}{g(1_{\{x\}})} \sup g\big(f_n(x) 1_{\{x\}}\big) \leqslant \frac{1}{g(1_{\{x\}})} \sup g(f_n) < \infty.$$

Hence $\{f_n : n \in \mathbf{N}\}$ is bounded in $\mathbf{R}^X$. [2; 23.22 and 23.23] yield the existence of $\sup f_n$ in E.

(b)$\Rightarrow$(a): By Theorem 5.1, only σ-lateral completeness need to be checked. This can be done using the same arguments as in [5; 2.10 d) $\Rightarrow$a)]. □

The following observation is very easy to verify:

Proposition 5.4. *Let F be a σ-Riesz subspace of E which is order dense or an ideal in E. Then we have:*

(a) *If $R \subset (E^\pi)_+$ satisfies the σ-hc-condition for E, then $\{g|_R : g \in R\}$ satisfies the σ-hc-condition for F.*

(b) *If (e, R) is a σ-hc-pair of E, then $\left(e|_{E_b \cap F}, \{g|_F : g \in R\}\right)$ is a σ-hc-pair of F.*

Now I want to give the description of the measure representations which arise from σ-hypercomplete spaces:

Theorem 5.5. *For each e-mr $(Y, \hat{u}, v)$ of the π-space E, the following are equivalent:*

(a) *There exists $R \subset R_e$ such that R is associated to $(Y, \hat{u}, v)$ and (e, R) is a σ-hc-pair of E;*

(b) *vE is σ-embedded in $\mathscr{M}(Y)$;*

(c) *vE is a σ-Riesz subspace of $\mathscr{M}(Y)$.*

Proof. (a) $\Rightarrow$ (b): Let $0 \leqslant x_n \uparrow$ in E such that there exists $\mu \in \mathscr{M}(Y)$ with $vx_n \leqslant \mu$ for all $n \in \mathbb{N}$. Then $\sup g(x_n) \leqslant \mu(U_g) < \infty$ for all $g \in R$, which implies the existence of $\sup x_n$ in E and consequently of $\sup vx_n$ in vE.

(b) $\Rightarrow$ (c) follows from Proposition 2.2(b).

(c) $\Rightarrow$ (a): Set $R := \{\hat{u}^{-1}1_U : U \subset Y,\ U \text{ open-compact}\}$. Let $0 \leqslant x_n \uparrow$ in E with $\sup g(x_n) < \infty$ for all $g \in R$. Since $\mathscr{M}(Y)$ is hypercomplete, $\sup vx_n$ exists in $\mathscr{M}(Y)$ and therefore also in vE, which yields the existence of $\sup x_n$ in E. □

Let me now continue the investigations which were started in section 4.

Proposition 5.6. *If (e, R) is a σ-hc-pair of E, then E satisfies the assumption of Proposition 4.11 (with $F := E$).*

Proof. Let (x_n) be a disjoint sequence from E_+ with $\sup e(x_n) \leqslant 1$. By σ-hypercompleteness, $x := \sup \sum_{k=1}^{n} \frac{1}{k^2} x_k$ exists in E, and obviously $x \in E_b$. □

The example $E = \ell^2$ shows that the assumption of Proposition 4.11 does not imply σ-hypercompleteness (see Theorem 5.11 below).

The next two propositions generalize results of Constantinescu [3; 1.6.1b)].

Proposition 5.7. *If (e, R) is a σ-hc-pair of E and $(Y, \hat{u}, v)$ is an e-mr of E, then:*

(a) $\hat{u}\left((E_b)^\pi\right) = C_b(Y)$.

(b) $\hat{u}(E^\pi) = \hat{u}\big((E_{\sigma b})^\pi\big)$ *is an ideal of* $C_b(Y)$ *which contains* $C_i(Y)$.

(c) *If* R *is associated to* $(Y, \hat{u}, v)$ *and* Y *is paracompact, then*

$$\hat{u}(E^\pi) = \hat{u}\big((E_{\sigma b})^\pi\big) = C_i(Y) = C_c(Y).$$

Proof. (a) follows from Propositions 5.6 and 4.11.

(b) follows from (a) and Corollary 3.7.

(c). The inclusions $C_c(Y) \subset C_i(Y) \subset \hat{u}\big((E_{\sigma b})^\pi\big) \subset \hat{u}(E^\pi)$ follow from (b). Now let $\xi \in (E^\pi)_+$ and assume that $K := \operatorname{supp} \hat{u}\xi$ is not compact. Since Y is paracompact, there exists a family $(Y_\iota)_{\iota \in I}$ of open-compact pairwise disjoint subsets of Y with $Y = \bigcup Y_\iota$. Hence there is a countably infinite subset J of I with $K \cap Y_\iota \neq \emptyset$ for all $\iota \in J$. For each $\iota \in J$ there exist an open-compact non-empty subset U_ι of $K \cap Y_\iota$ and $\alpha_\iota > 0$ with $\hat{u}\xi \geqslant \alpha_\iota$ on U_ι, and further a $\mu_\iota \in vE_+$ with $\operatorname{supp} \mu_\iota \subset U_\iota$ and $\mu_\iota(U_\iota) = 1/\alpha_\iota$. Then $\mu := \sum_{\iota \in J} \mu_\iota$ belongs to $\mathscr{M}(Y)$, and Theorem 5.5(a) $\Rightarrow$ (c) implies $\mu \in vE$. We have arrived at the contradiction $\xi(v^{-1}\mu) \geqslant \sum_{\iota \in J} \mu(U_\iota) = \infty$. □

Remark. The above proof shows that in paracompact hyperstonian spaces Y the identity $C_i(Y) = C_c(Y)$ holds.

Proposition 5.8. *If* (e, R) *is a* σ*-hc-pair of* E *and* $(Y, \hat{u}, v)$ *is the e-mr of* E *to which* R *is associated, then* $\hat{u}\big((E_c)^\pi\big) = C(Y)$ *(where* R *is assumed to be closed under the formation of finite suprema).*

Proof. The inclusion $C(Y) \subset \hat{u}\big((E_c)^\pi\big)$ follows from Proposition 3.3. Now let $\xi \in \big((E_c)^\pi\big)_+$, and assume the existence of $y \in Y$ with $\hat{u}\xi(y) = \infty$. Then there are $g \in R$ and a disjoint sequence (U_n) of open-compact non-empty subsets of U_g with $\hat{u}\xi \geqslant n$ on U_n for all n, and further $x_n \in (E_c)_+$ with $\operatorname{supp} vx_n \subset U_n$ and $(vx_n)(U_n) = 1$. Since $\mathscr{M}(Y)$ is hypercomplete, $\mu := \sup(1/n^2)vx_n$ exists in $\mathscr{M}(Y)$, and from Theorem 5.5(a)$\Rightarrow$(c) we conclude $\mu \in vE$. Moreover $\mu \in v(E_c)$ since $\operatorname{supp} \mu \subset U_g$, and we get the contradiction $\xi(v^{-1}\mu) = \infty$. □

In the last part of this section I want to investigate the L^p-spaces for σ-hypercompleteness and hypercompleteness. In the following let X be a set, $\mathfrak{R}$ a δ-ring of subsets of X and μ a positive measure on $\mathfrak{R}$.

Theorem 5.9. $L^1(\mu)$ *is hypercomplete;* $\big(1_X, \{1_X\}\big)$ *is an hc-pair of* $L^1(\mu)$.

Proof. Apply Proposition 5.2 and the Monotone Convergence Theorem. □

I need the following

Lemma 5.10. *If the space $L^0(\mu)$ of (equivalence classes of) μ-measurable almost finite functions is infinite-dimensional, then there exists a disjoint sequence (A_n) in $\mathfrak{R}$ with $\mu(A_n) > 0$ for all n.*

Proof. Let ν be the atomfree part of μ [9; 5.5.22]. If $\nu \neq 0$,then the assertion follows from Liapounov's Theorem [9; 5.6.2]. If $\nu = 0$, then μ is atomical, and, by Zorn's Lemma, there exists a family (A_ι) of pairwise disjoint μ-atoms with $X \setminus \bigcup A_\iota \in \mathfrak{N}(\mu)$. Since each μ-measurable function is almost constant on an atom (this is obvious for step functions and follows in the general case by approximation with step functions), I must be infinite. $\square$

Theorem 5.11. *For $1 < p \leqslant \infty$, the following are equivalent:*

(a) *$L^p(\mu)$ is finite-dimensional;*
(b) *$L^p(\mu)$ is hypercomplete;*
(c) *$L^p(\mu)$ is σ-hypercomplete.*

Proof. (a)$\Rightarrow$(b)$\Rightarrow$(c) is trivial.

($\neg$a)$\Rightarrow$($\neg$c): We have $(L^p(\mu))^\pi = L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$. In the following, I write $(f, g) := \int fg \,\mathrm{d}\mu$ for the canonical duality of $L^p(\mu)$ and $L^q(\mu)$. Let e be a weak unit of $(L^p(\mu))^\varrho$.

First case: μ is atomical.

By Lemma 5.10, there exists a disjoint sequence (A_n) of μ-atoms; set $\alpha_n := \mu(A_n)$. Suppose first that for each $g \in R_e$ the set $\{n \in \mathbf{N}\colon (1_{A_n}, g) > 0\}$ is finite. Set

$$f_n := \begin{cases} \sum_{k=1}^{n} \alpha_k^{-\frac{1}{p}} 1_{A_k} & \text{if } p < \infty \\ \sum_{k=1}^{n} k 1_{A_k} & \text{if } p = \infty. \end{cases}$$

Then (f_n) is unbounded in $L^p(\mu)$, but for all $g \in R_e$ we have $\sup(f_n, g) < \infty$.

Suppose now that there exists $g \in R_e$ such that $M := \{n \in \mathbf{N}\colon (1_{A_n}, g) > 0\}$ is infinite; we may assume $M = \mathbf{N}$. There exists $\gamma_n \in \mathbf{R}$ with $g = \gamma_n$ μ-a.e. on A_n. Since $g \in L^q(\mu)$, we have $\sum \gamma_n^q \alpha_n < \infty$. By [5; 2.8b)] there exists $(\beta_n) \in \mathbf{R}_+^{\mathbf{N}} \setminus \ell^p$ with $(\gamma_n \alpha_n^{\frac{1}{q}} \beta_n) \in \ell^1$. Then the sequence (f_n), given by $f_n := \sum_{k=1}^{n} \beta_k \alpha_k^{-\frac{1}{p}} 1_{A_k}$, is unbounded in $L^p(\mu)$, but for each $h \in R_e$ we have $h \leqslant g$ μ-a.e. on $\bigcup A_n$ and therefore

$$\sup_{n \in \mathbf{N}} (f_n, h) \leqslant \sup_{n \in \mathbf{N}} \sum_{k=1}^{n} \beta_k \alpha_k^{1-\frac{1}{p}} \gamma_k < \infty.$$

Second case: μ is not atomical.

There exist $g \in R_e$ and $\alpha > 0$ with $\{0 < g \leqslant \alpha\} \notin \mathfrak{N}(\nu)$, where ν denotes the atomfree component of μ [9; 5.5.22]. There exist $A \in \mathfrak{R}$ and, by Liapounov's Theorem [9; 5.6.2], a disjoint sequence (A_n) in $\mathfrak{R}$ such that $A_n \subset A \subset \{0 < g \leqslant \alpha\}$ and $\alpha_n := \mu(A_n) > 0$ for all n. If $p < \infty$, there exists, by [5; 2.8a)], $(\gamma_n) \in \mathbf{R}_+^{\mathbf{N}}$ such that $\beta := \sum \alpha_n \gamma_n < \infty$ and $\sum \alpha_n \gamma_n^p = \infty$; if $p = \infty$, there exists, by [5; 2.8b)], $(\gamma_n) \in \mathbf{R}_+^{\mathbf{N}} \setminus \ell^\infty$ with $\beta := \sum \alpha_n \gamma_n < \infty$. Then the sequence (f_n), defined by $f_n := \sum_{k=1}^{n} \gamma_k 1_{A_k}$, is unbounded in $L^p(\mu)$, but for each $h \in R_e$ we have $h \leqslant g$ μ-a.e. on $\bigcup A_n$ and therefore

$$\sup_{n \in \mathbf{N}} (f_n, h) \leqslant \alpha \sup_{n \in \mathbf{N}} \int f_n \mathrm{d}\mu = \alpha\beta < \infty.$$

$\square$

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