John V. Baxley; R. O. Chapman

A criterion for discrete spectra of partial differential operators

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 3, 403–414

Persistent URL: http://dml.cz/dmlcz/128356

A CRITERION FOR DISCRETE SPECTRA OF PARTIAL DIFFERENTIAL OPERATORS

J. V. BAXLEY and R. O. CHAPMAN, Winston-Salem

(Received July 30, 1987)

1. Let τ be the formal differential operator

$$\tau u = -\frac{1}{m}\sum_{k=1}^{n}\frac{\partial}{\partial x_k}\left(p_k\frac{\partial u}{\partial x_k}\right) \tag{1.1a}$$

and let

$$\tau_q u = \tau u + qu. \tag{1.1b}$$

For a given domain Ω in $\mathbf{R}^n$, this formal operator τ_q may give rise to a variety of selfadjoint operators in the weighted Hilbert space $L^2_m(\Omega)$ consisting of all measurable complex-valued functions u defined on Ω for which

$$\|u\| = \left[\int_\Omega |u|^2 m \, dx\right]^{\frac{1}{2}} < \infty.$$

We are concerned with problems in which there are points on the boundary of Ω for which τ_q is singular, and we wish to obtain criteria which guarantee that particular selfadjoint realizations of τ_q have discrete spectra. In fact, we shall state conditions under which our operators have compact inverses. In order to minimize technical considerations, we shall treat the case that Ω is a Cartesian product of n bounded open intervals, and for convenience we take $\Omega = X^n_{k=1}(0,1)$.

In the general theory for the one-dimensional problem [4], one generally starts with the minimal operator $L_0 u = \tau_q u$ for $u \in c_0^\infty(0,1)$, the class of infinitely differentiable functions with compact support in $(0,1)$. Assuming this operator L_0 is symmetric, selfadjoint extensions are obtained by imposing boundary conditions on the domain of the adjoint operator L_0^* in such a way that the restriction of L_0^* to the functions satisfying the boundary conditions is selfadjoint. One of the beautiful central results of the theory is that every selfadjoint extension of L_0 is determined in this fashion. Another attractive result (again for one dimension) is that every selfadjoint extension of L_0 has the same essential spectrum. Hence if one such selfadjoint extension has a purely discrete spectrum, so does every other selfadjoint extension of L_0.

For dimensions higher than one, this last result is not true. Thus the character of the spectrum, whether it is discrete or not, must be considered for every selfadjoint realization of τ_q.

The basic idea of this paper is to use the extension method of Fredrichs [4, pp. 1240–1242]. By considering a variety of initial domains on each of which τ_q gives rise to a symmetric semibounded operator, the extension method of Friederichs gives generally a variety of selfadjoint operators, some of which may not be distinct. We shall describe conditions under which all these selfadjoint operators have compact inverses and hence discrete spectra.

These techniques were used earlier, in the one–dimensional case, in [1] and [2]; and later by Rollins [11] to obtain criteria close in spirit to those of Eastham [5]. More recent one-dimensional criteria, using other methods, were given by Hinton and Lewis [8]. Very interesting, albeit older criteria, were obtained by Friedrichs [6, 7]. The application of our present methods in the less complicated two-dimensional setting can be found in [3]. Related results using different methods have been obtained by Lewis [9, 10].

2. With $\Omega = X_{k=1}^{n}(0,1)$, let $\Gamma = \partial\Omega$. Let $\Gamma_1 \subset \Gamma$ be the points $x = (x_1, x_2, \ldots, x_n)$ of Γ with $x_k = 1$ for at least one k. Let $\Gamma_2 = \Gamma - \Gamma_1$. Singularities of the formal operator τ of (1.1) will be confined to Γ_2. We shall assume:

(i) q, $m \in C(\Omega \cup \Gamma_1)$; $p_k \in C'(\Omega \cup \Gamma_1)$, for $k = 1, 2, \ldots, n$.

(ii) m, p_k are strictly positive on $\Omega \cup \Gamma_1$, $k = 1, 2, \ldots, n$.

(iii) $\sup\{|q(x)| : x \in \Omega\} < \infty$.

Thus q, m, or any p_k may tend to 0, ∞, or oscillate as x approaches a point in Γ_2, so any or all points in Γ_2 are allowed to be singular.

(iv)
$$\int_{\Omega} [m(x)]^n \Big[\prod_{k=1}^{n} \int_{x_k}^{1} [p_k(x)]^{-1} dx_k\Big] dx < \infty.$$

Now let Γ_0 be an arbitrary subset of Γ_2. Corresponding to Γ_0, we define D_0 as follows: $u \in D_0$ if and only if

(a) $u \in C^\infty(\Omega \cup \Gamma_1)$,

(b) $u = 0$ on Γ_1,

(c) there exists $\delta_u > 0$ (depending on u) such that if $0 < x_k < \delta_u$ and

$$x_k \leqslant x_j < 1 \text{ for } j \neq k, \text{ then } \frac{\partial u}{\partial x_k}(x) = 0 \text{ if } P_k x \in \Gamma_0,\ u(x) = 0 \text{ if } P_k x \notin \Gamma_0,$$

where $P_k x$ is the natural projection of x onto the coordinate hyperplane $x_k = 0$.

For $0 < \delta < 1$, let $\Omega_\delta = X_{k=1}^n(\delta, 1) \subset \Omega$. If $u \in D_0$ and $\delta < \delta_u$ (see (c) above), then either u or the normal derivative of u is zero at each point of $\partial\Omega_\delta$.

3. Let $Lu = \tau_q u$, for $u \in D(L) \equiv D_0$. In order to use the Friedrichs' extension, we need the following lemma, where $\alpha = \inf\{q(x) \colon x \in \Omega\}$.

Lemma 3.1. *L is symmetric and semibounded below by α.*

Proof. Both assertions follow by integrating by parts (using Green's theorem) on Ω_δ for $\delta > 0$ sufficiently small and then letting $\delta \to 0$. Hypothesis (iii) is needed to guarantee the existence of (Lu, u). □

It follows from Lemma 1 that L has a Friedrichs' extension F. By varying the subset Γ_0 of Γ_2, many different initial domains D_0 will be obtained, giving rise to correspondingly different operators L. The corresponding extensions F will usually, though not necessarily, be distinct.

Lemma 3.2. *For $k = 1, 2, \ldots, n$, for $u \in D_0$ and for $x \in \Omega \cup \Gamma_1$,*

$$|u(x)|^2 \leqslant \int_{x_k}^1 [p_k(x)]^{-1} \mathrm{d}x_k \int_0^1 p_k(x) \left| \frac{\partial u}{\partial x_k}(x) \right|^2 \mathrm{d}x_k.$$

Proof. Using the fundamental theorem of calculus and the Schwarz inequality, we obtain

$$|u(x)|^2 = \left| \int_{x_k}^1 \frac{\partial u}{\partial x_k} \mathrm{d}x_k \right|^2 \leqslant \int_{x_k}^1 [p_k(x)]^{-1} \mathrm{d}x_k \int_{x_k}^1 p_k(x) \left| \frac{\partial u}{\partial x_k} \right|^2 \mathrm{d}x_k$$

and the desired result is immediate. □

Lemma 3.3. *For $\Omega^* \subset \Omega$, put $\alpha = \inf\{q(x) \colon x \in \Omega\}$ and*

$$M(\Omega^*) = \left\{ \int_{\Omega^*} [m(x)]^n \left[\prod_{k=1}^n \int_{x_k}^1 [p_k(x)]^{-1} \mathrm{d}x_k \right] \mathrm{d}x \right\}^{\frac{1}{n}}.$$

Then for $u \in D_0$,

$$\int_{\Omega^*} |u|^2 m\,\mathrm{d}x \leqslant M(\Omega^*)(\tau u, u),$$
$$(u,u) \leqslant M(\Omega)(\tau u, u), \quad \|u\| \leqslant M(\Omega)\|\tau u\|,$$
$$(Lu,u) \geqslant \Big(\frac{1}{M(\Omega)} + \alpha\Big)(u,u), \quad \|Lu\| \geqslant \Big(\frac{1}{M(\Omega)} + \alpha\Big)\|u\|.$$

(Note: If $\alpha \leqslant -\frac{1}{M(\Omega)}$, the last inequality above says nothing; indeed, this is the reason why, in our main result (Theorem 4.4 below), we need to assume that $\alpha > -\frac{1}{M(\Omega)}$.)

Proof. Put $Q_k(x) = \prod_{j=1}^{k} \int_0^1 p_j(x)\Big|\frac{\partial u}{\partial x_j}\Big|^2 \mathrm{d}x_j$. From Lemma 3.2, we have

$$|u(x)|^{\frac{2}{n}} \leqslant \Big(\int_{x_k}^1 [p_k(x)]^{-1}\mathrm{d}x_k\Big)^{\frac{1}{n}} \Big[\int_0^1 p_k(x)\Big|\frac{\partial u}{\partial x_k}\Big|^2 \mathrm{d}x_k\Big]^{\frac{1}{n}}$$

for each $k = 1, 2, \ldots, n$. Multiplying these inequalities and integrating gives

$$\int_{\Omega^*} |u|^2 m\,\mathrm{d}x \leqslant \int_{\Omega^*} m(x)\Big[\prod_{k=1}^{n} \int_{x_k}^1 [p_k(x)]^{-1}\mathrm{d}x_k\Big]^{\frac{1}{n}} Q_n^{\frac{1}{n}}(x)\mathrm{d}x$$
$$\leqslant M(\Omega^*)\Big[\int_{\Omega} Q_n^{\frac{1}{n-1}}(x)\mathrm{d}x\Big]^{\frac{n-1}{n}};$$

we used Hölder's inequality with $p = n$, $q = \frac{n}{n-1}$, $\frac{1}{p} + \frac{1}{q} = 1$ to obtain the last inequality and further replaced Ω^* by Ω in the final integral.

To expedite the remainder of the proof, we make the following conventions. For $k < n$, we put $\Omega_k = X_{j=1}^{k}(0,1)$ and in any integral of the form $\int_{\Omega_k} \ldots \mathrm{d}x$, we shall intend $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2 \ldots \mathrm{d}x_k$. On the other hand, in any integral of the form $\int_{\Omega_k} \ldots \hat{\mathrm{d}x}$, we use the caret to intend $\hat{\mathrm{d}x} = \mathrm{d}x_n \mathrm{d}x_{n-1} \ldots \mathrm{d}x_{n-k+1}$.

Returning to our argument, we observe that one factor of our last integrand is independent of x_n and so we iterate this last integral as an $n-1$ dimensional integral and a one dimensional integral to obtain

$$\int_{\Omega^*} |u|^2 m\,\mathrm{d}x \leqslant M(\Omega^*)\Big[\int_{\Omega_{n-1}} \Big[\int_0^1 p_n(x)\Big|\frac{\partial u}{\partial x_k}\Big|^2 \mathrm{d}x_n\Big]^{\frac{1}{n-1}} \Big[\int_0^1 Q_{n-1}^{\frac{1}{n-1}}(x)\mathrm{d}x_n\Big]\mathrm{d}x\Big]^{\frac{n-1}{n}}$$

$$\leqslant M(\Omega^*)\Big[\int_{\Omega_n} p_n(x)\Big|\frac{\partial u}{\partial x_k}\Big|^2 \mathrm{d}x\Big]^{\frac{1}{n}} \Big[\int_{\Omega_{n-1}} \Big(\int_0^1 Q_{n-1}^{\frac{1}{n-1}}(x)\mathrm{d}x_n\Big)^{\frac{n-1}{n-2}}\mathrm{d}x\Big]^{\frac{n-2}{n}}, \tag{3.1}$$

where now we have used Hölder's inequality with $p = n-1$, $q = \frac{n-1}{n-2}$ on the $n-1$ dimensional integral over Ω_{n-1}.

Our last integral has the form

$$(3.2) \qquad I_k = \Big[\int\limits_{\Omega_{n-k}} \Big(\int\limits_{\Omega_k} Q_{n-k}^{\frac{1}{n-k}}(x) \hat{\mathrm{d}x} \Big)^{\frac{n-k}{n-k-1}} \mathrm{d}x \Big]^{\frac{n-k-1}{n}}.$$

We factor $Q_{n-k}(x) = \big(\int_0^1 p_{n-k}(x) \big| \frac{\partial u}{\partial x_{n-k}} \big|^2 \mathrm{d}x_{n-k} \big) Q_{n-k-1}(x)$ and use Hölder's inequality on the inside integral (with $p = n-k$, $q = \frac{n-k}{n-k-1}$) to obtain

$$\int\limits_{\Omega_k} Q_{n-k}^{\frac{1}{n-k}} \hat{\mathrm{d}x} \leqslant \Big(\int\limits_{\Omega_{k+1}} p_{n-k}(x) \Big| \frac{\partial u}{\partial x_{n-k}} \Big|^2 \hat{\mathrm{d}x} \Big)^{\frac{1}{n-k}} \Big(\int\limits_{\Omega_k} Q_{n-k-1}^{\frac{1}{n-k-1}}(x) \hat{\mathrm{d}x} \Big)^{\frac{n-k-1}{n-k}}.$$

Thus

$$(3.3) \quad I_k \leqslant \Big[\int\limits_{\Omega_{n-k}} \Big(\int\limits_{\Omega_{k+1}} p_{n-k}(x) \Big| \frac{\partial u}{\partial x_{n-k}} \Big|^2 \hat{\mathrm{d}x} \Big)^{\frac{1}{n-k-1}} \Big(\int\limits_{\Omega_k} Q_{n-k-1}^{\frac{1}{n-k-1}}(x) \hat{\mathrm{d}x} \Big) \mathrm{d}x \Big]^{\frac{n-k-1}{n}}.$$

We now note that the integral over Ω_{k+1} is independent of x_{n-k} and interate the integral over Ω_{n-k} to obtain

$$I_k \leqslant \Big[\int\limits_{\Omega_{n-k-1}} \Big(\int\limits_{\Omega_{k+1}} p_{n-k}(x) \Big| \frac{\partial u}{\partial x_{n-k}} \Big|^2 \hat{\mathrm{d}x} \Big)^{\frac{1}{n-k-1}} \Big(\int\limits_{\Omega_{k+1}} Q_{n-k-1}^{\frac{1}{n-k-1}}(x) \hat{\mathrm{d}x} \Big) \mathrm{d}x \Big]^{\frac{n-k-1}{n}}.$$

Now Hölder's inequality with $p = n-k-1$, $q = \frac{n-k-1}{n-k-2}$ gives

$$(3.4) \qquad I_k \leqslant \Big(\int\limits_{\Omega_n} p_{n-k}(x) \Big| \frac{\partial u}{\partial x_{n-k}} \Big|^2 \mathrm{d}x \Big)^{\frac{1}{n}} I_{k+1}.$$

Returning to (3.1) and using (3.4) repeatedly a total of $n-3$ times, we arrive at (3.5)

$$\int_{\Omega^*} |u|^2 m \,\mathrm{d}x \leqslant M(\Omega^*) \Big(\prod_{k=3}^{n} \int_{\Omega} p_k(x) \Big| \frac{\partial u}{\partial x_k} \Big|^2 \mathrm{d}x \Big)^{\frac{1}{n}} \Big[\int\limits_{\Omega_2} \Big(\int\limits_{\Omega_{n-2}} Q_2^{\frac{1}{2}}(x) \hat{\mathrm{d}x} \Big)^2 \mathrm{d}x \Big]^{\frac{1}{n}}.$$

Nothing that $Q_2(x)$ has two factors and using the Schwarz inequality on the integral over Ω_{n-2}, we get

$$(3.6) \qquad \Big(\int\limits_{\Omega_{n-2}} Q_2^{\frac{1}{2}}(x) \hat{\mathrm{d}x} \Big)^2 \leqslant \int\limits_{\Omega_{n-2}} \Big(\int_0^1 p_1(x) \Big| \frac{\partial u}{\partial x_1} \Big|^2 \mathrm{d}x_1 \Big) \hat{\mathrm{d}x} \int\limits_{\Omega_{n-1}} p_2(x) \Big| \frac{\partial u}{\partial x_2} \Big|^2 \hat{\mathrm{d}x}.$$

Since the first factor on the right of (3.6) is independent of x_1 (it depends only on x_2), we iterate the integral over Ω_2 in (3.5) and get

$$(3.7)\qquad \int\limits_{\Omega_2}\Big(\int\limits_{\Omega_{n-2}} Q_2^{\frac{1}{2}}(x)\hat{\mathrm{d}}x\Big)^2\mathrm{d}x \leqslant \int_\Omega p_1(x)\Big|\frac{\partial u}{\partial x_1}\Big|^2\mathrm{d}x\int_\Omega p_2(x)\Big|\frac{\partial u}{\partial x_2}\Big|^2\mathrm{d}x.$$

Using (3.7) in (3.5) leads to

$$(3.8)\qquad \int_{\Omega^*}|u|^2m\,\mathrm{d}x \leqslant M(\Omega^*)\Big(\prod_{k=1}^{n}\int_\Omega p_k(x)\Big|\frac{\partial u}{\partial x_k}\Big|^2\mathrm{d}x\Big)^{\frac{1}{n}}.$$

For $u\in D_0$, integration by parts gives

$$(\tau u,u)=\sum_{k=1}^{n}\int_\Omega p_k(x)\Big|\frac{\partial u}{\partial x_k}\Big|^2\mathrm{d}x \geqslant \int_\Omega p_j(x)\Big|\frac{\partial u}{\partial x_j}\Big|^2\mathrm{d}x$$

for each $j=1,2,\ldots,n$. Thus all parts of Lemma 3 follow immediately from (3.8), and the observation that $(Lu,u)=(\tau u,u)+(qu,u)\geqslant\frac{1}{M(\Omega)}(u,u)+\alpha(u,u)=\left(\frac{1}{M(\Omega)}+\alpha\right)(u,u)$. □

Theorem 3.4. *For each u in the domain of the Friedrichs extension F, we have*

$$(Fu,u)\geqslant\Big(\frac{1}{M(\Omega)}+\alpha\Big)(u,u),\quad \|Fu\|\geqslant\Big(\frac{1}{M(\Omega)}+\alpha\Big)\|u\|$$

Proof. By construction of the Friedrichs extension [4, pp. 1240–1242], there exists a sequence $\{u_n\}$ in D_0 so that $\|u_n-u\|\to 0$ and $(Lu_n,u_n)\to(Fu,u)$ as $n\to\infty$. By Lemma 3.3, $(Lu_n,u_n)\geqslant\left(\frac{1}{M(\Omega)}+\alpha\right)(u_n,u_n)$. Letting $n\to\infty$ yields the first inequality; the second follows from the Schwarz inequality.

4. We continue to let $\Omega=X_{k=1}^{n}(0,1)$ and for $0<\delta<1$, $\Omega_\delta=X_{k=1}^{n}(\delta,1)$. In addition, for $x\in\Omega$, with $x_k\geqslant\delta$ for every k, we put $\Omega_x=X_{k=1}^{n}(\delta,x_k)$ and $\Omega_{x,j}=X_{k=1}^{j}(\delta,x_k)$, so that $\Omega_{x,n}=\Omega_x$. To simlify the notation, we have sublimated the dependence of Ω_x and $\Omega_{x,j}$ on δ, which will generally be fixed in our discussion. We also define for $v\in D_0$.

$$(4.1)\qquad W_j(x)=\int\limits_{\Omega_{x,j}} v(x)\mathrm{d}x,\quad j=1,2,\ldots,n-1$$

where, as before $\mathrm{d}x=\mathrm{d}x_1\ldots\mathrm{d}x_j$, and

$$(4.2)\qquad W_0(x)=v(x).$$

Note that

$$(4.3) \qquad W_j(x) = \int_{\delta}^{x_j} W_{j-1}(x) \mathrm{d}x_j$$

and therefore that

$$(4.4) \qquad \frac{\partial W_j}{\partial x_j} = W_{j-1}(x).$$

Finally, we put

$$(4.5) \qquad M_\delta = \max_{1 \leqslant k \leqslant n} \left[\max_{x \in \bar{\Omega}_\delta} [p_k(x)]^{-1} \right]$$

□

Lemma 4.1. *If $x, y \in \bar{\Omega}_\delta$ and differ only in the i^{th} coordinate, then for $v \in D_0$*

$$|W_{n-1}(y) - W_{n-1}(x)| \leqslant \left[M_\delta \|y - x\| (\tau v, v) \right]^{\frac{1}{2}}$$

where $\|y - x\|$ is the Euclidean norm of $y - x \in \mathbf{R}^n$.

Proof. First consider the case $i = n$. We may clearly assume $y_n > x_n$. Then from (4.1) and the fundamental theorem of calculus

$$W_{n-1}(y) - W_{n-1}(x) = \int_{\Omega_{x,n-1}} \Big(\int_{x_n}^{y_n} \frac{\partial v}{\partial x_n} \mathrm{d}x_n \Big) \mathrm{d}x.$$

Hence using the Schwarz inequality:

$$\begin{aligned} |W_{n-1}(y) - W_{n-1}(x)|^2 &\leqslant \int_{\Omega_{x,n-1}} \Big(\int_{x_n}^{y_n} p_n^{-1} \mathrm{d}x_n \Big) \mathrm{d}x \int_\Omega p_n \Big| \frac{\partial v}{\partial x_n} \Big|^2 \mathrm{d}x \\ &\leqslant M_\delta |y_n - x_n| (\tau v, v) \end{aligned}$$

which gives the lemma for $i = n$.

The case $1 \leqslant i < n$ is essentially the same for each such i. Suppose $i = n-1$ and $y_{n-1} > x_{n-1}$. Then from (4.3)

$$\begin{aligned}
W_{n-1}(y) - W_{n-1}(x) &= \int_{\delta}^{y_{n-1}} W_{n-2}(x)\mathrm{d}x_{n-1} - \int_{\delta}^{x_{n-1}} W_{n-2}(x)\mathrm{d}x_{n-1} \\
&= \int_{x_{n-1}}^{y_{n-1}} W_{n-2}(x)\mathrm{d}x_{n-1} = \int_{x_{n-1}}^{y_{n-1}} \Big[\int_{\Omega_{x,n-2}} v(x)\mathrm{d}x \Big] \mathrm{d}x_{n-1} \\
&= -\int_{x_n}^{1} \Big[\int_{x_{n-1}}^{y_{n-1}} \Big[\int_{\Omega_{x,n-2}} \frac{\partial v}{\partial x_n} \mathrm{d}x \Big] \mathrm{d}x_{n-1} \Big] \mathrm{d}x_n .
\end{aligned}$$

Hence, using the Schwarz inequality as before:

$$\begin{aligned}
|W_{n-1}(y) - W_{n-1}(x)|^2 &\leqslant (\tau v, v) \int_{x_n}^{1} \Big[\int_{x_{n-1}}^{y_{n-1}} \Big(\int_{\Omega_{x,n-2}} p_n^{-1} \mathrm{d}x \Big) \mathrm{d}x_{n-1} \Big] \mathrm{d}x_n \\
&\leqslant M_{\delta} |y_{n-1} - x_{n-1}| (\tau v, v)
\end{aligned}$$

which gives the lemma for $i = n-1$. □

Lemma 4.2. *If x, $y \in \bar{\Omega}_{\delta}$, then*

$$|W_{n-1}(y) - W_{n-1}(x)| \leqslant n \big[M_{\delta} \|y - x\| (\tau v, v) \big]^{\frac{1}{2}}.$$

P r o o f. Beginning with $k = 1$, and continuing to $k = n$, we may change x_k to y_k to get a pair of points in $\bar{\Omega}_{\delta}$ which differ only in the k^{th} coordinate. For each such pair, we may apply lemma 4.1; adding up the resulting inequalities and applying the triangle inequality gives the desired result. □

Lemma 4.3. *Suppose that $\{v_k(x)\}$ is a sequence in D_0 for which $(\tau v_k, v_k)$ is a bounded sequence of numbers. Then given δ with $0 < \delta < 1$, every subsequence of $\{v_k(x)\}$ has a further subsequence which is Cauchy in $L^2_m(\Omega_{\delta})$.*

P r o o f. Define

$$W_{k,j}(x) = \int_{\Omega_{x,j}} v_k(x)\mathrm{d}x \tag{4.6}$$

as in (4.1). By lemma 4.2, for $x, y \in \bar{\Omega}_\delta$,

$$|W_{k,n-1}(y) - W_{k,n-1}(x)| \leqslant n\big[M_\delta\|y - x\|(\tau v_k, v_k)\big]^{\frac{1}{2}}.$$

Since for $\Delta = (\delta, \delta, \ldots, \delta) \in \bar{\Omega}_\delta$, we have $W_{k,n-1}(\Delta) = 0$, it follows that the sequence $\{W_{k,n-1}\}$ is uniformly bounded and equicontinuous on $\bar{\Omega}_\delta$. Thus, by Ascoli's theorem, any subsequence of $\{W_{k,n-1}\}$ has a further subsequence which converges uniformly on $\bar{\Omega}_\delta$. Let us pass to such a subsequence, but for simplicity, we continue to use the same notation. Thus, we assume $\{W_{k,n-1}\}$ converges uniformly and also certainly is Cauchy in $L^2_m(\Omega_\delta)$. We claim that for each $j = 0, 1, \ldots, n-1$, $\{W_{k,j}\}$ is Cauchy in $L^2_m(\Omega_\delta)$. We proceed by (backwards) induction. Since our claim is already true for $j = n-1$, we assume it is true for $j = k > 1$ and prove it true for $j = k-1$. We shall show that there exists a constant c_δ, depending only on δ, so that

$$\|W_{j,k-1} - W_{\ell,k-1}\|^2 \leqslant c_\delta\|W_{j,k} - W_{\ell,k}\|, \tag{4.7}$$

from which our induction argument is finished. Letting $K_\delta = \max_{x\in\bar{\Omega}_\delta} m(x)$, we have from (4.4)

$$\|W_{j,k-1} - W_{\ell,k-1}\|^2 \leqslant K_\delta \int\limits_{\Omega_\delta} \Big(\frac{\partial W_{k,j}}{\partial x_j} - \frac{\partial W_{\ell,j}}{\partial x_j}\Big)(\bar{W}_{k,j-1} - \bar{W}_{\ell,j-1})\mathrm{d}x$$

and applying the divergence theorem to integrate by parts gives

$$\|W_{j,k-1} - W_{\ell,k-1}\|^2 \leqslant -K_\delta \int\limits_{\Omega_\delta} (W_{k,j} - W_{\ell,j})\Big(\frac{\partial \bar{W}_{k,j-1}}{\partial x_j} - \frac{\partial \bar{W}_{\ell,j-1}}{\partial x_j}\Big)\mathrm{d}x$$

because the boundary term is zero since $W_{k,j}(W_{\ell,j})$ vanishes on the face $x_j = \delta$ and $W_{k,j-1}(W_{\ell,j-1})$ vanishes on the face $x_j = 1$. From (4.6),

$$\frac{\partial W_{k,j-1}}{\partial k_j} = \int\limits_{\Omega_{x,j-1}} \frac{\partial v_k}{\partial x_j}\mathrm{d}x$$

and thus

$$\|W_{j,k-1} - W_{\ell,k-1}\|^2 \leqslant K_\delta \int\limits_{\Omega_\delta} |W_{k,j} - W_{\ell,j}|\Big(\int\limits_{\Omega_{x,j-1}} \Big|\frac{\partial v_k}{\partial x_j} - \frac{\partial v_\ell}{\partial x_j}\Big|\mathrm{d}x\Big)\mathrm{d}x.$$

Applying the Schwarz inequality first to the integral over $\Omega_{x,j-1}$ and then to the integral over Ω_δ, we get

$$\begin{aligned}
&\|W_{j,k-1} - W_{\ell,k-1}\|^2 \leqslant \\
&\leqslant K_\delta \|W_{k,j} - W_{\ell,j}\| \Big(\int_{\Omega_\delta} m^{-1} \Big[\int_{\Omega_{x,j-1}} p_j^{-1} \mathrm{d}x \Big] \Big[\int_{\Omega_{x,j-1}} p_j \Big| \frac{\partial v_k}{\partial x_j} - \frac{\partial v_\ell}{\partial x_j} \Big|^2 \mathrm{d}x \Big] \Big)^{\frac{1}{2}} \\
&\leqslant K_\delta \sqrt{L_\delta M_\delta} \|W_{k,j} - W_{\ell,j}\| \Big(\int_{\Omega_\delta} \Big[\int_{\Omega_{x,j-1}} p_j \Big| \frac{\partial v_k}{\partial x_j} - \frac{\partial v_\ell}{\partial x_j} \Big|^2 \mathrm{d}x \Big] \mathrm{d}x \Big)^{\frac{1}{2}}
\end{aligned}$$

where M_δ is defined in (4.5) and $L_\delta = \max_{x \in \bar{\Omega}_\delta} [m(x)]^{-1}$. Replacing $\Omega_{x,j-1}$ by $X_{i=1}^{j-1}(\delta, 1)$ in the last integral, we see easily that

$$\|W_{j,k-1} - W_{\ell,k-1}\|^2 \leqslant C_\delta \|W_{k,j} - W_{\ell,j}\| \big[(\tau v_k, v_k) + (\tau v_\ell, v_\ell) \big]^{\frac{1}{2}}$$

and (4.7) follows.

Thus, by induction $\{v_k\} = \{W_{k,0}\}$ is Cauchy in $L^2_m(\Omega)$. □

Theorem 4.4. *Suppose the coefficients of the formal differential operator τ_q satisfy* (i)–(iv) *and that $\alpha = \inf\{q(x) \colon x \in \Omega\} > -\frac{1}{M(\Omega)}$. Then the Friedrichs extension F has a compact inverse and hence a purely discrete spectrum.*

Proof. Suppose u_k is in the domain of F and $\|Fu_k\| = 1$ for each $k = 1, 2, \ldots$. We shall show that $\{u_k\}$ has a subsequence which is Cauchy in $L^2_m(\Omega)$ and the completeness of $L^2_m(\Omega)$ gives the desired conclusion.

By construction of the Friedrichs extension [4, pp. 1240–1242], we may choose $v_k \in D_0$ such that

$$\|u_k - v_k\| < \frac{1}{k}, \quad |(Fu_k, u_k) - (Lv_k, v_k)| < \frac{1}{k} \tag{4.8}$$

for $k = 1, 2, \ldots$. It follows from Theorem 3.4 and the Schwarz inequality that

$$|(Lv_k, v_k)| \leqslant \frac{M(\Omega)}{1 + \alpha M(\Omega)} + 1, \quad \text{for every } k. \tag{4.9}$$

For $u \in D_0$, $(Lu, u) = (\tau u, u) + (qu, u) \geqslant (\tau u, u) + \alpha(u, u) \geqslant (\tau u, u) - \frac{1}{M(\Omega)}(u, u)$; this inequality and Lemma 3.3 give $0 \leqslant (\tau v_k, v_k) \leqslant (Lv_k, v_k) + \frac{1}{M(\Omega)}(v_k, v_k) \leqslant (Lv_k, v_k) + \frac{1}{1+\alpha M(\Omega)}(Lv_k, v_k)$ and hence $\{(\tau v_k, v_k)\}$ is a bounded sequence of numbers. Let $\delta_j = \frac{1}{j+1}$ so that $\delta_j \to 0$ as $j \to \infty$. By Lemma 4.3, the sequence $\{v_k\}$ has a sub-sequence $\{v_k^{(1)}\}$ which is Cauchy in $L_m^2(\Omega_{\delta_1})$. This same lemma then gives a subsequence $\{v_k^{(2)}\}$ of $\{v_k^{(1)}\}$ which is Cauchy in $L_m^2(\Omega_{\delta_2})$. Continuing in this way, we get at the j^{th} stage a subsequence $\{v_k^{(j)}\}$ of $\{v_k^{(j-1)}\}$ which is Cauchy in $L_m^2(\Omega_{\delta_j})$.

We claim that the "diagonal" sequence $\{v_j^{(j)}\}$ is Cauchy in $L_m^2(\Omega)$. Let $\varepsilon > 0$ be given. For $0 < \delta < 1$, Lemma 3.3 gives

$$\begin{aligned}\int_{\Omega-\Omega_\delta} |v_j^{(j)} - v_k^{(k)}|^2 m\, dx &\leqslant 2\int_{\Omega-\Omega_\delta} |v_j^{(j)}|^2 m\, dx + 2\int_{\Omega-\Omega_\delta} |v_k^{(k)}|^2 m\, dx \\ &\leqslant 2M(\Omega-\Omega_\delta)\big[(\tau v_j^{(j)}, v_j^{(j)}) + (\tau v_k^{(k)}, v_k^{(k)})\big].\end{aligned}$$

Since $(\tau v_k, v_k)$ is bounded, we may thus choose N so large that for $\delta = \delta_N$

$$\int_{\Omega-\Omega_\delta} |v_j^{(j)} - v_k^{(k)}|^2 m\, dx < \frac{\varepsilon}{2}, \quad \text{for all } j, k. \tag{4.10}$$

The sequence $\{v_j^{(j)}\}$ is clearly Cauchy in $L_m^2(\Omega_\delta)$ for $\delta = \delta_N$. Thus, there exists N_1 such that

$$\int_{\Omega_\delta} |v_j^{(j)} - v_k^{(k)}|^2 m\, dx < \frac{\varepsilon}{2}, \quad \text{if } j, k \geqslant N_1. \tag{4.11}$$

Combining (4.10) and (4.11), we get

$$\|v_j^{(j)} - v_k^{(k)}\|^2 < \varepsilon \quad \text{for } j, k \geqslant N_1$$

and thus $\{v_j^{(j)}\}$ is Cauchy in $L_m^2(\Omega)$. Since $\{v_j^{(j)}\}$ is a subsequence of $\{v_k\}$, it follows from (4.8) that the corresponding subsequence of $\{u_k\}$ is Cauchy in $L_m^2(\Omega)$, and the proof is complete. □

Corollary. *If the coefficients of* τ_1 *satisfy* (i)–(iv), *then the Friedrichs extension* F *has a purely discrete spectrum.*

Proof. Add an appropriate constant to $q(x)$ and apply Theorem 4.4. □

References

[1] *J. V. Baxley*: The Friedrichs extension of certain singular differential operators, Duke Math. J. *35* (1968), 455–462.

[2] *J. V. Baxley*: Eigenvalues of singular differential operators by finite difference methods, I, II, J. Math. Anal. Appl. *37* (1972), 244–254, 257–275.

[3] *J. V. Baxley*: Some partial differential operators with discrete spectra, Spectral Theory of Differential Operators (Birmingham, AL, 1981), North-Holland Math. Studies 55, North-Holland, Amsterdam, 1981, pp. 53–59.

[4] *N. Dunford and J. T. Schwartz*: Linear Operators, Part II, Wiley (Interscience), New York, 1963.

[5] *M. S. P. Eastham*: The least limit point of the spectrum associated with singular differential operators, Proc. Camb. Phil. Soc. *67* (1970), 277–281.

[6] *K. O. Fredrichs*: Criteria for the discrete character of the spectra of ordinary differential operators, In Courant Anniversary Volume, Interscience, New York, 1948.

[7] *K. O. Friedrichs*: Criteria for discrete spectra, Comm. Pure Appl. Math. *3* (1950), 439–134.

[8] *D. B. Hinton and R. T. Lewis*: Singular differential operators with spectra discrete and bounded below, Proc. Royal Soc. Edinburgh Sect. *A 84* (1979), 117–134.

[9] *R. T. Lewis*: Singular elliptic operators of second order with purely discrete spectra, Trans. Amer. Math. soc. *271* (1982), 653–666.

[10] *R. T. Lewis*: The spectra of some singular elliptic operators of second order, Spectral Theory of Differential Operators (Birmingham, AL, 1981), North-Holland Math. Studies 55, North-Holland, Amsterdam, 1981, pp. 303–318.

[11] *L. W. Rollins*: Criteria for discrete spectrum of singular self-adjoint operators, Proc. Amer. Math. Soc. *34* (1972), 195–200.

Authors' address: J. V. Baxley and R. O. Chapman, Department of Mathematics and Computer Science, Wake Forest University, Winston-Salem, NC 27 109.