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ON MEAN VALUE THEOREMS FOR SMALL GEODESIC SPHERES IN RIEMANNIAN MANIFOLDS

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1. INTRODUCTION

In this paper we study to what extent the mean value theorems in a Riemannian manifold (M, g) characterize the structure of the manifold itself. The *mean value theorems* stand for various relations about the first, the second mean values and the stochastic mean values for small geodesic spheres at center $m \in M$ with radius $\varepsilon > 0$. The works on this subject are recently studied by many authors ([6], [9], [10], [12], [17], [22]), characterizing the harmonic, the Einstein and the super-Einstein spaces by expanding up to order $+\infty$, 4 and 6 the above three mean values respectively (Theorem A below).

Our results are stated as follows. We first obtain a higher order precision of Theorem A, i.e., by expanding the above three mean values up to order 8, we characterize the particular classes of 2-stein spaces which should be located between the harmonic and the super-Einstein spaces (Theorem 1). In particular for $3 \leqslant \dim M \leqslant 6$, the manifolds (M, g) are spaces satisfying simpler curvature conditions (Theorem 2). Theorems 1 and 2 give a partial answer to Kowalski's conjecture given in [10] and [11]. We also introduce three new conditions $(S2)_k$–$(S4)_k$ (see Section 2 for the definitions) stated on the mean value theorems and prove: (1) for each $k = 3$, 4, the condition $(S3)_k$ is equivalent to $(M3)_{k-1}$; (2) each of the conditions $(S2)_3$ and $(S4)_3$ characterizes the space of constant scalar curvature, and each of the conditions $(S2)_4$ and $(S4)_4$ characterizes the quasi-super-Einstein space (Theorem 4). We further show that the condition $(S2)_k$ is closely related to the independence of the first exit time and the first exit position of a Brownian motion from a geodesic ball at center m with radius $\varepsilon > 0$ (Theorem 3). This independence property is only recently studied by M. Kôzaki and Y. Ogura [13], M. Liao [15] and M. Pinsky [19].

In Section 2, we state our results precisely. Our main results are stated in Theorems 1, 2, 3 and 4. We denote by $M_{m,i}f(m)$ and $L_{m,i}f(m)$ the coefficients of order ε^{2i} in the asymptotic expansions for the first mean value $M_m(\varepsilon, f)$ and the second one $L_m(\varepsilon, f)$ respectively. In Section 3, we calculate the difference $M_{m,4}f(m) - L_{m,4}f(m)$ for the super-Einstein space and give the proof of Theorem 1 in part. Sections 4 and 5 are for preparation of the proof of the rest of Theorem 1. Section 5 is also for preparation of the proof of Theorem 4. In Section 4, we calculate $L_{m,4}f(m)$ for the super-Einstein space. In Section 5, we calculate the stochastic mean value $E_m f(X(T_\varepsilon))$ and the mean exit time $E_m T_\varepsilon$ up to order ε^8 for the manifold. In Sections 6 and 7, we will prove the rest of Theorem 1 and Theorem 2 respectively. In the final Section 8, we will prove Theorems 3 and 4.

2. Statement of results

Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geqslant 2$ and $B_m(\varepsilon)$ be the geodesic ball in M at center $m \in M$ with small radius $\varepsilon > 0$. The *first mean value* $M_m(\varepsilon, f)$ for a real valued continuous function f is defined by

$$M_m(\varepsilon, f) = \left(\operatorname{vol}\left(\partial B_m(\varepsilon)\right)\right)^{-1} \int\limits_{\partial B_m(\varepsilon)} f(\omega)\, \mathrm{d}\sigma(\omega),$$

where $\mathrm{d}\sigma$ stands for the volume element on the geodesic sphere $\partial B_m(\varepsilon)$. Similarly, the *second mean value* $L_m(\varepsilon, f)$ for an f is defined by

$$L_m(\varepsilon, f) = \left(\operatorname{vol}\left(S^{n-1}(1)\right)\right)^{-1} \int\limits_{S^{n-1}(1)} \left(f \circ \exp_m(\varepsilon u)\right) \mathrm{d}u,$$

where $\exp_m$ is the exponential map at $m \in M$ and $\mathrm{d}u$ is the usual volume element on the $(n-1)$-dimensional unit sphere $S^{n-1}(1)$.

In [10] and [11], O. Kowalski conjectured the next

Conjecture. *For an analytic Riemannian manifold (M, g), the following conditions are mutually equivalent:*

(i)$_k$ *for each $m \in M$, the mean value formula*

$$M_m(\varepsilon, f) = f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all harmonic functions f near m;

$(\mathrm{ii})_k$ *for each* $m \in M$, *the mean value formula*

$$L_m(\varepsilon, f) = f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all harmonic functions f *near* m;

$(\mathrm{iii})_k$ *for each* $m \in M$, *the estimate*

$$M_m(\varepsilon, f) = L_m(\varepsilon, f) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all harmonic functions f *near* m;

$(\mathrm{iv})_k$ *for each* $m \in M$, *the estimate*

$$M_m(\varepsilon, f) = L_m(\varepsilon, f) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all functions f *of class* C^{2k+2} *near* m.

In the above, k is a natural number or $+\infty$ and, in the case of $k = +\infty$, the formulae are understood to hold without remainder terms.

Let $X = \big(X(t), P_m\big)$ $(m \in M)$ be a Brownian motion on (M, g), i.e., the diffusion process on (M, g) whose infinitesimal operator is the Laplacian Δ on (M, g). Let also T_ε be the first exit time from the geodesic ball $B_m(\varepsilon)$, i.e., $T_\varepsilon = \inf\{t > 0 : X(t) \notin B_m(\varepsilon)\}$. The *stochastic mean value* for an f and the *mean exit time* from $B_m(\varepsilon)$ are defined by $E_m f\big(X(T_\varepsilon)\big)$ and $E_m T_\varepsilon$ respectively, where E_m denotes the expectation with respect to the probability measure P_m.

Also we set $A_m(\varepsilon) = \mathrm{vol}\big(\partial B_m(\varepsilon)\big)$ the volume of the geodesic sphere $\partial B_m(\varepsilon)$ and

$$\Phi_m(\varepsilon) = \int_0^\varepsilon A_m^{-1}(s) \int_0^s A_m(t)\, \mathrm{d}t\, \mathrm{d}s.$$

Finally a function f is called *bi-harmonic* near m if it is defined and smooth in a neighbourhood of m and Δf is harmonic there.

In [12], we also introduced the following conditions:

$(\mathrm{M1})_k$ *for each* $m \in M$, *the estimate*

$$M_m(\varepsilon, f) = E_m f\big(X(T_\varepsilon)\big) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all functions f *of class* C^{2k+2} *near* m;

$(\mathrm{M2})_k$ *for each* $m \in M$, *the mean value formula*

$$M_m(\varepsilon, f) = f(m) + (E_m T_\varepsilon)\Delta f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all bi-harmonic functions f *near* m;

$(\mathrm{M3})_k$ *for each* $m \in M$, *the mean value formula*

$$M_m(\varepsilon, f) = f(m) + \Phi_m(\varepsilon)\Delta f(m) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all bi-harmonic functions f ***near*** m;

$(\mathrm{M4})_k$ *there exists a sequence of polynomials* p_j, $j = 1, 2, \ldots, k$ *without constant terms such that, for each* $m \in M$, *the expansion*

$$M_m(\varepsilon, f) = f(m) + \sum_{j=1}^{k} p_j(\Delta) f(m) \varepsilon^{2j} + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0)$$

holds for all functions f ***of class*** C^{2k+2} ***near*** m.

The conditions $(\mathrm{L1})_k$–$(\mathrm{L4})_k$ are defined in the same way as $(\mathrm{M1})_k$–$(\mathrm{M4})_k$ are done respectively with the first mean value $M_m(\varepsilon, f)$ replaced by the second one $L_m(\varepsilon, f)$. The conditions $(\mathrm{M4})_\infty$ and $(\mathrm{L4})_\infty$ are understood to hold for all analytic functions f at m.

For an $m \in M$, let $(U; x^1, x^2, \ldots, x^n)$ be a normal coordinate system around m, and denote by (g_{ij}) and (R_{ijkl}) the metric tensor and the curvature tensor with respect to the normal frame $(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n})$, respectively. Throughout we exploit Einstein's convention as well as the extended one, i.e., the summation convention for repeated indices. The Ricci tensor and the scalar curvature are denoted by (ϱ_{ij}) and τ respectively; $\varrho_{ij} = R^u{}_{iuj}$, $\tau = \varrho^u_u$. We also denote the length of a tensor $T = (T_{ij})$ by $|T|$, i.e., $|T|^2 = T_{ij}T^{ij}$. Finally, we denote the covariant derivative by ∇_i and set $\Delta = \nabla^p \nabla_p$.

We call an Einstein space *super-Einstein* if $|R|^2$ is constant and $\dot{R}_{ij} \equiv R_{ipqr} R_j^{pqr} = \frac{|R|^2}{n} g_{ij}$. We also call an Einstein space 2-*stein* if

$$(\bar{R} \circ \bar{R})_{ijkl} = \frac{3n|R|^2 + 2\tau^2}{n^2(n+2)} (g_{ij}g_{kl} + g_{ik}g_{jl} + g_{il}g_{jk}),$$

where

$$\begin{aligned}(\bar{R} \circ \bar{R})_{ijkl} = \bar{R}_{ij}{}^{pq}(\bar{R}_{klpq} + \bar{R}_{lkpq}) + \bar{R}_{ik}{}^{pq}(\bar{R}_{jlpq} + \bar{R}_{ljpq}) \\ + \bar{R}_{il}{}^{pq}(\bar{R}_{jkpq} + \bar{R}_{kjpq}) \quad (\bar{R}_{ijkl} \equiv R_{ikjl}).\end{aligned}$$

Further we call a 2-stein space 2^*-*stein* if $|R|^2$ is constant (or equivalently, if the space is super-Einsteinian). Similarly we call a space *quasi-super-Einstein* if τ and $|R|^2 - |\varrho|^2$ are constants, and if

$$\dot{R}_{ij} = \frac{|R|^2 - |\varrho|^2}{n} g_{ij} - \varrho^{pq} R_{ipjq} + 2\varrho_{ip}\varrho^p_j - \frac{3}{2}\Delta \varrho_{ij}. \tag{2.1}$$

Finally, we call the space (M, g) *harmonic* if, for each $m \in M$, there exist an $\varepsilon > 0$ and a function $F : (0, \varepsilon) \to \mathbf{R}$ such that the function $f(n) = F(d(m, n))$ is harmonic in $B_m(\varepsilon) \setminus \{m\}$, where d is the distance function defined by the Riemannian metric.

For the proof of our Theorem 1 mentioned below, we use the next theorems.

Theorem A ([6], [9], [10], [12], [17], [22]). *Let (M, g) be an n-dimensional connected C^ω Riemannian manifold with $n \geqslant 3$. Then the following assertions hold.*

(1) *Each of the conditions* $(\mathrm{i})_\infty$–$(\mathrm{iv})_\infty$, $(\mathrm{M1})_\infty$ – $(\mathrm{M4})_\infty$ *and* $(\mathrm{L1})_\infty$–$(\mathrm{L4})_\infty$ *is necessary and sufficient in order that (M, g) be a harmonic space.*

(2) *Each of the conditions* $(\mathrm{i})_2$–$(\mathrm{iv})_2$, $(\mathrm{M1})_2$ – $(\mathrm{M4})_2$ and $(\mathrm{L1})_2$–$(\mathrm{L4})_2$ *is necessary and sufficient in order that (M, g) be an Einstein space.*

(3) *Each of the conditions* $(\mathrm{i})_3$–$(\mathrm{iv})_3$, $(\mathrm{M1})_3$ – $(\mathrm{M4})_3$ and $(\mathrm{L1})_3$–$(\mathrm{L4})_3$ *is necessary and sufficient in order that (M, g) be a super-Einstein space.*

Theorem B ([11], [12]). *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geqslant 3$ and fix a $k \in \{1, 2, \dots, \infty\}$. Then the following assertions hold.*

(1) *The condition* $(\mathrm{i})_k$ *is necessary and sufficient for* $(\mathrm{M1})_k$.

(2) *The condition* $(\mathrm{ii})_k$ *is necessary and sufficient for* $(\mathrm{L1})_k$.

(3) *The condition* $(\mathrm{iii})_k$ *is necessary and sufficient for* $(\mathrm{iv})_k$.

Remark. (1) Notice that due to [12], the assertions in Theorem A are valid for C^∞ Riemannian manifolds, except for the sufficiency of $(\mathrm{M4})_\infty$ and $(\mathrm{L4})_\infty$. (2) In [11], O. Kowalski proved the assertion (3) of Theorem B for C^ω Riemannian manifolds.

We also use the following notation.

$$\check{R}_{ij} = R_{iupq} R_{rs}{}^{pq} R_j^{urs}, \quad \check{R} = \check{R}_k^k,$$
$$\check{\bar{R}}_{ij} = \bar{R}_{iupq} \bar{R}_{rs}{}^{pq} \bar{R}_j^{urs}, \quad \check{\bar{R}} = \check{\bar{R}}_k^k.$$

Our main objective of this paper is the following

Theorem 1. *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geqslant 3$. Then the following assertion holds. Each of the conditions* $(\mathrm{i})_4$–$(\mathrm{iv})_4$, $(\mathrm{M1})_4$–$(\mathrm{M4})_4$ and $(\mathrm{L1})_4$ – $(\mathrm{L4})_4$ *is necessary and sufficient in order that (M, g) be a 2^*-stein space and satisfy*

$$3\nabla_i R_{abcd} \nabla_j R^{abcd} - 20\check{R}_{ij} + 16\check{\bar{R}}_{ij} = \lambda g_{ij}, \tag{2.2}$$

$$\lambda = \frac{1}{n}(3|\nabla R|^2 - 20\check{R} + 16\check{\bar{R}}) = \mathit{constant}. \tag{2.3}$$

R e m a r k . We divide the assertion of Theorem 1 into following three parts (a), (b), (c) and prove (c) in Section 3 and (a)–(b) in Section 6: each of the conditions, $(\mathrm{i})_4$ and $(\mathrm{M1})_4$–$(\mathrm{M4})_4$ in (a), $(\mathrm{ii})_4$ and $(\mathrm{L1})_4$–$(\mathrm{L4})_4$ in (b), and $(\mathrm{iii})_4$–$(\mathrm{iv})_4$ in (c), is necessary and sufficient in order that (M, g) be a 2^*-stein space and satisfy (2.2)–(2.3). Note that we can also give a simple proof of the assertion (c) by using Theorem 1 in [11], which was suggested by O. Kowalski (private communication).

A lower dimensional case of Theorem 1 is the following

Theorem 2. *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $3 \leqslant n \leqslant 6$. Then each of the conditions $(\mathrm{i})_4$–$(\mathrm{iv})_4$, $(\mathrm{M1})_4$–$(\mathrm{M4})_4$ and $(\mathrm{L1})_4$–$(\mathrm{L4})_4$ is necessary and sufficient in order that the following assertions hold:*

(1) *if $n = 3$, 4, then (M, g) is locally flat or locally isometric to a symmetric space of rank one;*

(2) *if $n = 5$, then (M, g) is a 2^*-stein space and, satisfies $|\nabla R|^2 =$ constant and*

$$\nabla_i R_{abcd} \nabla_j R^{abcd} = \frac{|\nabla R|^2}{n} g_{ij}; \tag{2.4}$$

(3) *if $n = 6$, then (M, g) is a 2^*-stein space and satisfies (2.3)–(2.4).*

In this paper, we also introduce three new conditions, i.e., the conditions $(\mathrm{S2})_k$–$(\mathrm{S4})_k$ are defined in the same way as $(\mathrm{M2})_k$–$(\mathrm{M4})_k$ are done respectively with the first mean value $M_m(\varepsilon, f)$ replaced by the stochastic mean value $E_m f(X(T_\varepsilon))$. These conditions are motivated by the fact that, if (M, g) is a harmonic space, then the conditions $(\mathrm{S2})_\infty$–$(\mathrm{S4})_\infty$ follow from Theorem A (1).

Now following [13], we define the following condition:

$(\mathrm{MI})_k$ *for each $m \in M$, the asymptotically mean independence formula*

$$E_m T_\varepsilon f(X(T_\varepsilon)) = (E_m T_\varepsilon)(E_m f(X(T_\varepsilon))) + O(\varepsilon^{2k+2}) \quad (\varepsilon \to 0) \tag{2.5}$$

holds for all functions f of class C^{2k+2} near m.

Then we have the following equivalence theorem, which we also use for the proof of Theorem 4.

Theorem 3. *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geqslant 2$. Then, for each $k = 1, 2, \ldots, +\infty$, the condition $(\mathrm{S2})_k$ is equivalent to the independence condition $(\mathrm{MI})_k$.*

Finally we prove the following

Theorem 4. *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geqslant 2$. Then the following assertions hold.*

(1) *Each of the conditions* $(S2)_3$ *and* $(S4)_3$ *is necessary and sufficient in order that* (M, g) *be of constant scalar curvature.*

(2) *Each of the conditions* $(S2)_4$ *and* $(S4)_4$ *is necessary and sufficient in order that* (M, g) *be a quasi-super-Einstein space.*

(3) *The conditions* $(S3)_3$ *and* $(S3)_4$ *are necessary and sufficient in order that* (M, g) *be an Einstein and a super-Einstein spaces respectively.*

Corollary. *Let* (M, g) *be an* n*-dimensional Einstein space with* $n \geqslant 3$. *Then each of the conditions* $(S2)_4$ *and* $(S4)_4$ *is equivalent to that the space* (M, g) *is a super-Einstein space.*

3. Proof of Theorem 1

Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold and an $m \in M$. Let $(U\,;\, x^1, x^2, \ldots, x^n)$ be a normal coordinate system around m. Let ∇ be the Levi-Civita connection of the Riemannian manifold (M, g) and $R(X, Y)$ its curvature tensor, i.e., $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$. We set $R_{ijk\ell} = g\big(R(\partial_i, \partial_j)\partial_k, \partial_\ell\big)$, $g^{ij} = (g_{ij})^{-1}$ and $g = \det(g_{ij})$, where $\partial_i = \frac{\partial}{\partial x^i}$. We also denote $\nabla_i = \nabla_{\partial_i}$ and $\nabla^r_{i_r \ldots i_2 i_1} = \nabla_{i_r} \cdots \nabla_{i_2} \nabla_{i_1}$ ($= I$ if $r = 0$). For a tensor $T = (T_{i_1 \ldots i_p})$, we denote $T_{i_1 \ldots i_p; j_1 \ldots j_r} = \nabla^r_{j_r \ldots j_1} T_{i_1 \ldots i_p}$ and $\nabla T = (T_{i_1 \ldots i_p; j})$. The inner product $S_{i_1 \ldots i_p} T_{i_1 \ldots i_p}$ of two tensors $S = (S_{i_1 \ldots i_p})$ and $T = (T_{i_1 \ldots i_p})$ is denoted by $\langle S, T \rangle$.

We also use the convention

$$x^{i_1 i_2 \ldots i_r} = x^{i_1} x^{i_2} \ldots x^{i_r}, \quad i_1, i_2, \ldots, i_r = 1, 2, \ldots, n.$$

Lemma 3.1. *It holds that*

$$\begin{aligned}
(3.1)\quad g_{ij} = {} & \delta_{ij} - \frac{1}{3} R_{kihj}(m) x^{kh} - \frac{1}{3!} R_{kihj;p}(m) x^{khp} \\
& + \frac{1}{5!}\Big\{ - 6 R_{kihj;pq} + \frac{16}{3} R_{kihu} R_{pjqu} \Big\}(m) x^{khpq} \\
& + \frac{1}{6!}\{ - 8 R_{kihj;pqr} + 16 R_{kihu} R_{pjqu;r} + 16 R_{kjhu} R_{piqu;r} \}(m) x^{khpqr} \\
& + \frac{1}{7!}\{ - 10 R_{kihj;pqrs} + 34 R_{kihu;pq} R_{rjsu} + 34 R_{kjhu;pq} R_{risu} \\
& + 55 R_{kihu;p} R_{qjru;s} - 16 R_{kihu} R_{pjqv} R_{rusv} \}(m) x^{khpqrs} \\
& + \frac{4}{3 \cdot 8!}\{ - 9 R_{kihj;pqrs\alpha} + 46 R_{kihu;pqr} R_{sj\alpha u} + 46 R_{kjhu;pqr} R_{si\alpha u} \\
& + 99 R_{kihu;pq} R_{rjsu;\alpha} + 99 R_{kjhu;pq} R_{risu;\alpha} \\
& - 55 R_{kihu;p} R_{qjrv} R_{su\alpha v} - 55 R_{kjhu;p} R_{qirv} R_{su\alpha v} \\
& - 34 R_{kuhv;p} R_{qirv} R_{sj\alpha u} \}(m) x^{khpqrs\alpha} + O(|x|^8).
\end{aligned}$$

Proof. This follows from the same arguments as in [20]. □

Corollary 3.2. *It holds that*

$$\sqrt{g} = 1 + \sum_{p=2}^{8} S_{i_1 i_2 \dots i_p}(m) x^{i_1 i_2 \dots i_p} + O(|x|^9), \tag{3.2}$$

where

$$\begin{aligned}
S_{kh} &= -\frac{1}{6}\varrho_{kh}, \\
S_{khp} &= -\frac{1}{12}\varrho_{kh;p}, \\
S_{khpq} &= \frac{1}{4!}\Big\{-\frac{3}{5}\varrho_{kh;pq} + \frac{1}{3}\varrho_{kh}\varrho_{pq} - \frac{2}{15}R_{kuhv}R_{puqv}\Big\}, \\
S_{khpqr} &= \frac{1}{5!}\Big\{-\frac{2}{3}\varrho_{kh;pqr} + \frac{5}{3}\varrho_{kh}\varrho_{pq;r} - \frac{2}{3}R_{kuhv}R_{puqv;r}\Big\}, \\
S_{khpqrs} &= \frac{1}{6!}\Big\{-\frac{5}{7}\varrho_{kh;pqrs} + 3\varrho_{kh}\varrho_{pq;rs} - \frac{8}{7}R_{kuhv}R_{puqv;rs} \\
&\quad + \frac{5}{2}\varrho_{kh;p}\varrho_{qr;s} - \frac{15}{14}R_{kuhv;p}R_{qurv;s} - \frac{5}{9}\varrho_{kh}\varrho_{pq}\varrho_{rs} \\
&\quad + \frac{2}{3}\varrho_{kh}R_{puqv}R_{rusv} - \frac{16}{63}R_{kuhv}R_{pvqw}R_{rwsu}\Big\}, \\
S_{khpqrs\alpha} &= \frac{1}{7!}\Big\{-\frac{3}{4}\varrho_{kh;pqrs\alpha} + \frac{14}{3}\varrho_{kh}\varrho_{pq;rs\alpha} + \frac{21}{2}\varrho_{kh;p}\varrho_{qr;s\alpha} \\
&\quad - \frac{35}{6}\varrho_{kh}\varrho_{pq}\varrho_{rs;\alpha} + \frac{7}{3}\varrho_{kh;p}R_{qurv}R_{su\alpha v} + \frac{14}{3}\varrho_{kh}R_{puqv}R_{rus\cdot} \\
&\quad - \frac{5}{3}R_{kuhv}R_{puqv;rs\alpha} - \frac{9}{2}R_{kuhv;pq}R_{rusv;\alpha} \\
&\quad - \frac{8}{3}R_{kuhv}R_{pvqw}R_{rwsu;\alpha}\Big\}.
\end{aligned}$$

In the sequel, we define $\sum_{i=k}^{\ell} a_i = 0$ whenever $\ell < k$.

Lemma 3.3. *It holds that*

$$\begin{aligned}
\sqrt{g}f = f(m) + (\nabla_i f)(m)x^i + \sum_{p=2}^{8}\Big\{&\frac{1}{p!}(\nabla^p_{i_1 i_2 \dots i_p} f)(m) \\
&+ (S \circ f)(i_1 i_2 \dots i_p)(m) + S_{i_1 i_2 \dots i_p}(m) f(m)\Big\} x^{i_1 i_2 \dots i_p} + O(|x|^9),
\end{aligned} \tag{3.3}$$

where

$$(S \circ f)(i_1 i_2 \dots i_p) = \sum_{r=2}^{p-1} \frac{1}{(p-r)!} S_{i_1 i_2 \dots i_r} \nabla^{p-r}_{i_{r+1} \dots i_p} f. \tag{3.4}$$

Proof. Due to [9], the expansion for f is represented as

$$f = f(m) + \sum_{p=1}^{8} \frac{1}{p!} (\nabla^p_{i_1 i_2 \dots i_p} f)(m) x^{i_1 i_2 \dots i_p} + O(|x|^9). \tag{3.5}$$

Hence (3.3) follows from (3.2) and (3.4)–(3.5). □

Let $\{M_{m,j}\}_{j=1,2,\dots,k}$, $\{L_{m,j}\}_{j=1,2,\dots,k}$ and $\{E_{m,j}\}_{j=1,2,\dots,k}$ denote the sequences of linear differential operators satisfying the formulae respectively:

$$M_m(\varepsilon, f) = f(m) + \sum_{j=1}^{k} M_{m,j} f(m) \varepsilon^{2j} + O(\varepsilon^{2k+2}),$$

$$L_m(\varepsilon, f) = f(m) + \sum_{j=1}^{k} L_{m,j} f(m) \varepsilon^{2j} + O(\varepsilon^{2k+2}),$$

$$E_m f(X(T_\varepsilon)) = f(m) + \sum_{j=1}^{k} E_{m,j} f(m) \varepsilon^{2j} + O(\varepsilon^{2k+2}),$$

for a function f of class C^{2k+2} near m (see [9], [11], [17]).

In order to calculate $(M_{m,k} - L_{m,k}) f(m)$ for $k = 1, 2, 3, 4$, we prepare some notations.

Due to [8], the volume $A_m(\varepsilon)$ of the geodesic sphere $\partial B_m(\varepsilon)$ satisfies

(3.6)

$$\begin{aligned} A_m(\varepsilon) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} \varepsilon^{n-1} \Big\{ & 1 - \frac{\tau}{6n}\varepsilon^2 + \frac{\varepsilon^4}{3 \cdot 5!\, n(n+2)} (-18\Delta\tau + 5\tau^2 + 8|\varrho|^2 - 3|R|^2) \\ & + \frac{\varepsilon^6}{6!\, n(n+2)(n+4)(n+6)} \Big(-\frac{5}{9}\tau^3 - \frac{8}{3}\tau|\varrho|^2 + \tau|R|^2 + \frac{64}{63}\check{\varrho} \\ & - \frac{64}{21}\langle \varrho \otimes \varrho, \overline{R} \rangle + \frac{32}{7} \langle \varrho, \dot{R} \rangle - \frac{110}{63}\check{R} - \frac{200}{63}\check{\check{R}} + \frac{45}{14}|\nabla\tau|^2 + \frac{45}{7}|\nabla\varrho|^2 \\ & + \frac{45}{7}\alpha(\varrho) - \frac{45}{14}|\nabla R|^2 + 6\tau\Delta\tau + \frac{48}{7}\langle \Delta\varrho, \varrho \rangle + \frac{54}{7}\langle \nabla^2\tau, \varrho \rangle - \frac{30}{7}\langle \Delta R, R \rangle \\ & - \frac{45}{7}\Delta^2\tau \Big) \Big\}(m) + O(\varepsilon^8), \end{aligned}$$

where

$$\check{\varrho} = \varrho_{ij}\varrho_{jk}\varrho_{ki}, \quad \langle \varrho \otimes \varrho, \overline{R} \rangle = \varrho_{ij}\varrho_{k\ell}\overline{R}_{ijk\ell},$$
$$\alpha(\varrho) = \nabla_i \varrho_{jk} \nabla_k \varrho_{ij}, \quad \langle \Delta\varrho, \varrho \rangle = \varrho_{ij}\nabla^2_{pp}\varrho_{ij},$$
$$\langle \Delta R, R \rangle = R_{ijk\ell}\nabla^2_{pp}R_{ijk\ell}.$$

We further set the inverse of $A_m(\varepsilon)$

$$A_m(\varepsilon)^{-1} = \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}}\varepsilon^{1-n}\Big\{1+\sum_{j=1}^{k} C_j(m)\varepsilon^{2j}\Big\} + O(\varepsilon^{2k+2}). \tag{3.7}$$

Also we use the following symbol; for a natural number r,

$$\mathscr{C}(i_1 i_2 \dots i_{2r}) = \frac{1}{(2r)!!}\sum_{\sigma} \delta_{i_{\sigma(1)}i_{\sigma(2)}}\delta_{i_{\sigma(3)}i_{\sigma(4)}}\dots\delta_{i_{\sigma(2r-1)}i_{\sigma(2r)}},$$

where σ runs over all permutations and $(2r)!! = 2\cdot 4\cdot \dots \cdot (2r)$.

Lemma 3.4. *It holds that, for $k = 1,2,3,4$,*

(3.8)

$$\begin{aligned}(M_{m,k} - L_{m,k})f(m) &= \alpha_{n,k}\mathscr{C}(i_1 i_2 \dots i_{2k})(S\circ f)(i_1 i_2 \dots i_{2k})(m)\\ &\quad + \sum_{s=1}^{k-2} C_s(m)\{L_{m,k-s}f\\ &\quad + \alpha_{n,k-s}\mathscr{C}(i_1 i_2 \dots i_{2k-2s})(S\circ f)(i_1 i_2 \dots i_{2k-2s})\}(m)\\ &\quad + C_{k-1}(m)L_{m,1}f(m), \quad (C_0 \equiv 0)\end{aligned}$$

where $\alpha_{n,k} = \Gamma(\frac{n}{2})\{2^k\Gamma(\frac{n}{2}+k)\}^{-1}$.

Proof. We first note the following formulae given in [9] and [10] respectively;

$$\begin{aligned}\tilde{\Delta}^k_m f(m) &\equiv \sum_{i_1,i_2,\dots,i_k}^{n} \partial^2_{i_1}\partial^2_{i_2}\dots\partial^2_{i_k}f(m) \quad (\partial^2_{i_r} \equiv \partial_{i_r}\partial_{i_r})\\ &= \frac{1}{1\cdot 3\cdot \dots \cdot (2k-1)}\mathscr{C}(i_1 i_2 \dots i_{2k})\nabla^{2k}_{i_1 i_2 \dots i_{2k}}f(m),\end{aligned} \tag{3.9}$$

$$L_{m,k}f(m) = \frac{\alpha_{n,k}}{2^k k!}\tilde{\Delta}^k_m f(m) = \frac{(\tilde{\Delta}^k_m f)(m)}{2^k\cdot k!\, n(n+2)\dots(n+2k-2)}. \tag{3.10}$$

On the other hand, the same technique in [8: Lemma 3.2] yields

$$M_m(\varepsilon, f) = A_m(\varepsilon)^{-1}\varepsilon^{n-1}\int_{S^{n-1}(1)} f\sqrt{g}(\exp_m \varepsilon u)\,du. \tag{3.11}$$

Substituting (3.3) and (3.7) into (3.11) and using (3.10), we obtain (3.8). □

Lemma 3.5. *Let (M,g) be a super-Einstein space. Then it holds that*

$$\mathscr{C}(i_1i_2i_3i_4)(S\circ f)(i_1i_2i_3i_4)(m) = -\frac{n+2}{12n}\tau(m)\Delta f(m), \tag{3.12}$$

$$\begin{aligned}\mathscr{C}(i_1i_2\dots i_6)&(S\circ f)(i_1i_2\dots i_6)(m)\\ &= -\frac{n+4}{6!}\Big\{\frac{15}{n}\tau\tilde{\Delta}_m^2 f - \frac{5n+8}{n^2}\tau^2\Delta f + \frac{3}{n}|R|^2\Delta f\Big\}(m),\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathscr{C}(i_1i_2\dots i_8)(S\circ f)(i_1i_2\dots i_8)(m) &= -\frac{1}{6\cdot 6!}\Big\{-\frac{15(n+6)}{n}\tau\tilde{\Delta}_m^3 f\\
&+\frac{15(n+4)(n+6)}{2n^2}\tau^2\tilde{\Delta}_m^2 f - \frac{3(n+8)}{2n^2}(3n|R|^2+2\tau^2)\Delta^2 f\\
&-4(\bar{R}\circ\bar{R})_{ijk\ell}\nabla^4_{ijk\ell}f\Big\}(m) + \frac{1}{6\cdot 7!}\Big\{\frac{7(n^2+9n+16)}{n^3}(3n\tau|R|^2+2\tau^3)\Delta f\\
&-\frac{(n+6)(35n^2+210n+296)}{3n^3}\tau^3\Delta f - \frac{24(n+10)}{n^2}\tau|R|^2\Delta f\\
&+\Big(\frac{9}{2}|\nabla R|^2 - \frac{56}{3}\check{R} + \frac{16}{3}\check{\bar{R}}\Big)\Delta f\\
&+(15\nabla_i R_{abcd}\nabla_j R_{abcd} - 36\nabla_p R_{iabc}\nabla_p R_{jabc} - 64\check{R}_{ij} + 224\check{\bar{R}}_{ij})\nabla^2_{ij}f\\
&+\frac{5}{3}\nabla_r(9|\nabla R|^2 + 2\check{R} - 36\check{\bar{R}})\nabla_r f\Big\}(m).\end{aligned} \tag{3.14}$$

Proof. (3.12)–(3.14) follow from (3.4). But the details of the proof of (3.14) are too long to be written down here, and will be omitted. □

Now we set, for simplicity

$$\bar{\mathscr{R}}f = 56\Big\{\frac{1}{3}(\bar{R}\circ\bar{R})_{ijk\ell}\nabla^4_{ijk\ell}f - \frac{3n|R|^2+2\tau^2}{n^2(n+2)}\Big(\Delta^2 f + \frac{2}{3n}\tau\Delta f\Big)\Big\}. \tag{3.15}$$

Proposition 3.6. *Let (M,g) be a super-Einstein space. Then it holds that*

$$\begin{aligned}(M_{m,4}-L_{m,4})f(m) &= \frac{1}{8!\,n(n+2)(n+4)(n+6)}\Big[-2\bar{\mathscr{R}}f + \frac{20}{3}\{(3\nabla_i R_{abcd}\nabla_j R_{abcd}\\
&-20\check{R}_{ij} + 16\check{\bar{R}}_{ij})\nabla^2_{ij}f - \lambda\Delta f\} + \frac{20n}{9}\nabla_i\lambda\nabla_i f\Big](m).\end{aligned} \tag{3.16}$$

Proof. Since (M, g) is super-Einsteinian, by (3.6)–(3.7) we have

$$C_1 = \frac{\tau}{6n}, \tag{3.17}$$
$$C_2 = \frac{1}{3 \cdot 5!\, n(n+2)}\Big(5\tau^2 + \frac{12}{n}\tau^2 + 3|R|^2\Big),$$
$$C_3 = \frac{1}{6!\, n(n+2)(n+4)(n+6)}\Big(\frac{5}{9}\tau^3 + \frac{4}{n}\tau^3 + \frac{464}{63n^2}\tau^3 + \frac{12}{n}\tau|R|^2 + \frac{45}{14}|\nabla R|^2 - \frac{160}{63}\check{R} - \frac{880}{63}\check{\bar{R}}\Big).$$

Also due to [9] and (3.10), we have

$$L_{m,1}f = \frac{1}{2n}\Delta f, \quad L_{m,2}f = \frac{1}{8n(n+2)}\Big(\Delta^2 f + \frac{2}{3n}\tau\Delta f\Big). \tag{3.18}$$

Now substituting (3.12)–(3.14) and (3.17)–(3.18) into (3.8) with $k = 4$ and using (3.19)–(3.20) in the sequel, we obtain (3.16). □

Lemma 3.7. *Let (M, g) be a super-Einstein space. Then it holds that*

$$\nabla_p R_{iabc}\nabla_p R_{jabc} = -\frac{2}{n^2}\tau|R|^2 g_{ij} + \check{R}_{ij} + 4\check{\bar{R}}_{ij}, \tag{3.19}$$

$$|\nabla R|^2 = -\frac{2}{n}\tau|R|^2 + \check{R} + 4\check{\bar{R}}. \tag{3.20}$$

Proof. By $\Delta\dot{R}_{ij} = 0$, we have

$$\begin{aligned}\nabla_p R_{iabc}\nabla_p R_{jabc} &= -\frac{1}{2}(\Delta R_{iabc}R_{jabc} + R_{iabc}\Delta R_{jabc})\\ &= -\nabla^2_{k\ell}R_{iabk}R_{jab\ell} - \nabla^2_{k\ell}R_{jabk}R_{iab\ell}\\ &= -\frac{2}{n^2}\tau|R|^2 g_{ij} + \check{R}_{ij} + 4\check{\bar{R}}_{ij}.\end{aligned}$$

Hence (3.19)–(3.20) follow. □

Proof of Theorem 1(c). In the following proof, we assume that (M, g) is a super-Einstein space due to Theorem A (3). Note also that $(\mathrm{iii})_4$ is equivalent to $(\mathrm{iv})_4$ by Theorem B (3).

Sufficiency. Suppose that $(\mathrm{iv})_4$ holds. For the normal coordinate system $(U; x^1, x^2, \ldots, x^n)$ around m, setting first $f(x) = x^i x^j x^k x^\ell$ into

$$(M_{m,4} - L_{m,4})f(m) = 0, \tag{3.21}$$

it follows from (3.16) that (M,g) is a 2^*-stein space. Then by (3.15), we have $\overline{\mathscr{R}}f(m) = 0$. Setting further $f(x) = x^i x^j$ into (3.21), we obtain (2.2) from (3.16) with $\overline{\mathscr{R}}f(m) = 0$. Setting also $f(x) = x^i$ into (3.21), we have (2.3), completing the proof.

Necessity. Suppose that (M,g) is a 2^*-stein space and satisfies (2.2)–(2.3). Then we have easily (3.21) from (3.16). □

4. Calculation of $L_{m,4}f(m)$

Let (M,g) be an n-dimensional C^∞ Riemannian manifold with $n \geqslant 2$ and f be any smooth function on (M,g). To calculate $L_{m,4}f(m)$, we use the following notation.

$$D^1_{jk} = \nabla^4_{iijk}, \quad D^2_{jk} = \nabla^4_{ijik},$$
$$D^3_{jk} = \nabla^4_{ijki}, \quad D^4_{jk} = \nabla^4_{jiik},$$
$$D^5_{jk} = \nabla^4_{jiki}, \quad D^6_{jk} = \nabla^4_{jkii}.$$

Now due to (3.9)–(3.10), we have

$$(4.1) \qquad L_{m,4}f(m) = \frac{(\tilde{\Delta}^4_m f)(m)}{2^4 \cdot 4!\, n(n+2)(n+4)(n+6)} = \frac{\mathscr{C}(i_1 i_2 \ldots i_8)\nabla^8_{i_1 i_2 \ldots i_8} f(m)}{8!\, n(n+2)(n+4)(n+6)}.$$

We note that the computation of $\mathscr{C}(i_1 i_2 \ldots i_8)\nabla^8_{i_1 i_2 \ldots i_8} f(m)$ is reduced to that of

$$(4.2) \qquad \mathscr{C}(i_1 i_2 \ldots i_8)\nabla^8_{i_1 i_2 \ldots i_8} f(m) = K_1 + K_2 + K_3,$$

where

$$K_1 = \mathscr{C}(i_1 i_2 \ldots i_6)\nabla^6_{i_1 i_2 \ldots i_6}\Delta f(m) = 15\tilde{\Delta}^3_m \Delta f(m),$$
$$K_2 = 2\Big\{\sum_{p=1}^{3} D^p_{jj} A_{kk} f(m) + \sum_{p=1}^{6} D^p_{jk}(A_{jk} + A_{kj} + B_{jk} + B_{kj} + C_{jk}) f(m)\Big\},$$
$$K_3 = \text{ the sum of 24 remainder terms.}$$

In the formula (4.2), the first term K_1 is obtained in [9]. The second one K_2 is computed via Lemmas 4.1–4.2 mentioned below. The third one K_3 is also computed as in Lemma 4.3 in the sequel.

Lemma 4.1 ([9]). *It holds that*

$$A_{jk} f \equiv D^4_{jk} f = \nabla^2_{jk}\Delta f + \nabla_j \varrho_{k\ell}\nabla_\ell f + \varrho_{k\ell}\nabla^2_{j\ell} f,$$
$$B_{jk} f \equiv D^2_{jk} f = A_{jk} f + \varrho_{j\ell}\nabla^2_{k\ell} f + R_{ijk\ell}\nabla^2_{i\ell} f,$$
$$C_{jk} f \equiv D^1_{jk} f = B_{jk} f + (\nabla_k \varrho_{j\ell} - \nabla_\ell \varrho_{jk})\nabla_\ell f + R_{ijk\ell}\nabla^2_{i\ell} f.$$

Lemma 4.2. *Let $T_{jk}f$ denote $A_{jk}f$, $B_{jk}f$ and $C_{jk}f = C_{kj}f$ generically. Then it holds that*

$$\begin{aligned}
D^2_{jj}A_{kk}f &= D^3_{jj}A_{kk}f = D^1_{jj}A_{kk}f + \nabla_i(\varrho_{ia}\nabla_a A_{kk}f),\\
D^1_{jk}T_{jk}f &= D^1_{jk}T_{kj}f = \Delta\nabla^2_{jk}T_{jk}f,\\
D^2_{jk}T_{jk}f &= D^4_{jk}T_{jk}f = D^1_{jk}T_{jk}f + \nabla_i(\varrho_{ij}\nabla_k T_{jk}f),\\
D^2_{jk}T_{kj}f &= D^4_{jk}T_{kj}f = D^1_{jk}T_{kj}f + \nabla_i(\varrho_{ij}\nabla_k T_{kj}f),\\
D^3_{jk}T_{jk}f &= D^5_{jk}T_{jk}f = D^2_{jk}T_{jk}f + \nabla^2_{ij}(\varrho_{ik}T_{jk}f) + \nabla^2_{ia}(R_{jaik}T_{jk}f),\\
D^3_{jk}T_{kj}f &= D^5_{jk}T_{kj}f = D^2_{jk}T_{kj}f + \nabla^2_{ij}(\varrho_{ik}T_{kj}f) + \nabla^2_{ia}(R_{jaik}T_{kj}f),\\
D^6_{jk}T_{jk}f &= D^6_{jk}T_{kj}f = D^3_{jk}T_{jk}f + \nabla_i(R_{jiak}\nabla_a T_{jk}f).
\end{aligned}$$

P r o o f. All formulae above can be verified using the Ricci identity. □

Lemma 4.3. *It holds that*

$$\begin{aligned}
\nabla^8_{ijk\ell ijk\ell}f &= \nabla^8_{ijk\ell ij\ell k}f = \nabla^8_{ijk\ell jik\ell}f = \nabla^8_{ijk\ell ji\ell k}f\\
&= D^3_{ik}B_{ik}f + \nabla^3_{ijk}\{\varrho_{ai}\nabla^3_{jka}f + R_{aki\ell}\nabla^3_{ja\ell}f + R_{aji\ell}\nabla^3_{ak\ell}f\},\\
\nabla^8_{ijk\ell jki\ell}f &= \nabla^8_{ijk\ell jk\ell i}f = \nabla^8_{ijk\ell ikj\ell}f = \nabla^8_{ijk\ell ik\ell j}f\\
&= D^3_{ik}B_{ki}f + \nabla^3_{ijk}\{\varrho_{ai}\nabla^3_{kja}f + R_{aji\ell}\nabla^3_{ka\ell}f + R_{aki\ell}\nabla^3_{aj\ell}f\},
\end{aligned}$$

$$\begin{aligned}
(*) \equiv \nabla^8_{ijk\ell j\ell ik}f &= \nabla^8_{ijk\ell j\ell ki}f = \nabla^8_{ijk\ell i\ell jk}f = \nabla^8_{ijk\ell i\ell kj}f\\
&= D^3_{ik}C_{ik}f + \nabla^3_{ijk}\{\varrho_{ai}\nabla^3_{ajk}f + R_{aji\ell}\nabla^3_{ak\ell}f + R_{aki\ell}\nabla^3_{\ell ja}f\},\\
\nabla^8_{ijk\ell kij\ell}f &= \nabla^8_{ijk\ell ki\ell j}f = \nabla^8_{ijk\ell kji\ell}f = \nabla^8_{ijk\ell kj\ell i}f,\\
&= D^6_{ik}B_{ik}f + \nabla^3_{ijk}\{\varrho_{ak}\nabla^3_{jia}f + R_{aki\ell}\nabla^3_{ja\ell}f + R_{aki\ell}\nabla^3_{\ell ja}f\},
\end{aligned}$$

$$\begin{aligned}
(**) \equiv \nabla^8_{ijk\ell k\ell ij}f &= \nabla^8_{ijk\ell k\ell ji}f\\
&= D^6_{ik}C_{ik}f + \nabla^3_{ijk}\{\varrho_{ak}\nabla^3_{aij}f + 2R_{aki\ell}\nabla^3_{aj\ell}f\},\\
\nabla^8_{ijk\ell\ell ijk}f &= \nabla^8_{ijk\ell\ell ikj}f = \nabla^8_{ijk\ell\ell jik}f = \nabla^8_{ijk\ell\ell jki}f\\
&= (*) + 2\nabla^4_{ijk\ell}(R_{aki\ell}\nabla^2_{ja}f) + \nabla_i\{\nabla_\ell(R_{akp\ell}\nabla^2_{ja}f)R_{pijk}\},\\
\nabla^8_{ijk\ell\ell kij}f &= \nabla^8_{ijk\ell\ell kji}f\\
&= (**) + 2\nabla^4_{ijk\ell}(R_{aki\ell}\nabla^2_{ja}f) + 2\nabla_i\{\nabla_\ell(R_{akp\ell}\nabla^2_{ja}f)R_{jipk}\}.
\end{aligned}$$

P r o o f. All formulae are deformed as in the above, using the Ricci identity. □

Proposition 4.4. *Let (M,g) be a super-Einstein space. Then it holds that*

$$(4.3)\, L_{m,4}f(m) = \frac{1}{8!\, n(n+2)(n+4)(n+6)}\Big\{105\Delta^4 f + \frac{420}{n}\tau\Delta^3 f + \frac{588}{n^2}\tau^2\Delta^2 f + \frac{112}{n}|R|^2\Delta^2 f + \frac{56}{3}(\bar{R}\circ\bar{R})_{ijkl}\nabla^4_{ijkl}f + \frac{272}{n^3}\tau^3\Delta f + \frac{168}{n^2}\tau|R|^2\Delta f - \frac{5}{3}(3\nabla_i R_{abcd}\nabla_j R_{abcd} - 20\check{R}_{ij} + 16\check{\check{R}}_{ij})\nabla^2_{ij}f + (82\varphi_i - \frac{5n}{18}\nabla_i\lambda)\nabla_i f\Big\}(m),$$

where $\varphi_i = \nabla_j\{(\check{R}_{ij} - 2\check{\check{R}}_{ij}) - \frac{1}{6}(\check{R} - 2\check{\check{R}})g_{ij}\}$.

Proof. The formula (4.3) follows from (4.1)–(4.2) via Lemmas 4.1–4.3 and [9: Lemma 3.6]. But the details of the calculation are too long to be written down here, and will be omitted. □

5. Stochastic mean value and mean exit time

In this section, we review some results in [13] for computation of the stochastic mean value $E_m f(X(T_\varepsilon))$ and the mean exit time $E_m T_\varepsilon$ (Lemmas 5.1–5.2 below) and obtain the expansion for them up to order 8 (Proposition 5.4).

Let (M,g) be an n-dimensional C^∞ Riemannian manifold with $n \geqslant 2$. Note first that the Laplacian Δ is given by

$$\Delta = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, g^{ij}\partial_j).$$

Following [17] and [18], we define the operator τ_ε by $\tau_\varepsilon f(x) = f(\frac{x}{\varepsilon})$ for each $\varepsilon > 0$, and denote by $\mathscr{P}_r$ the space of all homogeneous polynomials of degree r for each nonnegative integer r. It then follows that for each nonnegative integer k and f of class C^{k+1}

$$(5.1)\qquad \tau_\varepsilon^{-1}\Delta\tau_\varepsilon f(x) = \varepsilon^{-2}\Delta_{-2}f(x) + \sum_{j=0}^{k}\varepsilon^j\Delta_j f(x) + O(\varepsilon^{k+1})$$

as $\varepsilon \downarrow 0$, where $\Delta_{-2} = \sum_{i=1}^n \partial_i^2$ and Δ_j are second order elliptic differential operators with $\Delta_j(\mathscr{P}_r) \subset \mathscr{P}_{j+r}$ for all nonnegative integers r (see [17]). We also denote as

$$q^{i_1 i_2 \dots i_r}(x) = x^{i_1}x^{i_2}\dots x^{i_r}, \quad i_1, i_2, \dots, i_r = 1, 2, \dots, n$$

(= 1 if $r = 0$), for each nonnegative integer r.

Lemma 5.1 ([13]). *Let r be a nonnegative integer and k be a natural number. Suppose further that the functions $U_\mu^{i_1 i_2 \ldots i_r}$ ($= U_\mu$ if $r = 0$), $\mu = 0, 2, 3, \ldots, 2k-1$ ($U_1^{i_1 i_2 \ldots i_r}(x) = 0$ by convention) satisfy*

$$\text{(5.2)} \qquad \Delta_{-2} U_0^{i_1 i_2 \ldots i_r}(x) = -q^{i_1 i_2 \ldots i_r}(x), \quad |x| < 1,$$

$$\sum_{\mu=0}^{\nu-2} \Delta_{\nu-\mu-2} U_\mu^{i_1 i_2 \ldots i_r}(x) + \Delta_{-2} U_\nu^{i_1 i_2 \ldots i_r}(x) = 0, \quad |x| < 1, \ \nu = 2, 3, \ldots, 2k-1,$$

$$U_\nu^{i_1 i_2 \ldots i_r}(\xi) = 0, \quad |\xi| = 1, \quad \nu = 0, 2, 3, \ldots, 2k-1.$$

Then it holds that

$$\text{(5.3)} \quad E_p \int_0^{T_\varepsilon} q^{i_1 i_2 \ldots i_r}(X(t))\, dt$$

$$= \varepsilon^{r+2} U_0^{i_1 i_2 \ldots i_r}\left(\frac{x}{\varepsilon}\right) + \sum_{\mu=2}^{2k-1} \varepsilon^{\mu+r+2} U_\mu^{i_1 i_2 \ldots i_r}\left(\frac{x}{\varepsilon}\right) + O(\varepsilon^{r+2k+2})$$

uniformly in $p \in B_m(\varepsilon)$ as $\varepsilon \to 0$.

We next consider the boundary value problem

$$\text{(5.4)} \qquad \Delta_{-2} u(x) = -f(x), \quad |x| < 1,$$
$$u(\xi) = 0, \quad |\xi| = 1.$$

We denote the solution of (5.4) by $G_0 f(x)$.

Lemma 5.2 ([13]). *For each nonnegative integer r and polynomial $p \in \mathcal{P}_r$, it holds that*

$$\text{(5.5)} \qquad G_0 p(x) = \sum_{k=0}^{[\frac{r}{2}]} (-1)^k \frac{(\Delta_{-2}^k p)(x)(1 - |x|^{2(k+1)}) + G_0(\Delta_{-2}^{k+1} p)(x)}{c_r(0) c_r(1) \ldots c_r(k)},$$

where $c_r(k) = 2(k+1)(n + 2r - 2k)$. Especially, if r ($= 2s$) is even, then

$$\text{(5.6)} \qquad G_0 p(0) = \frac{\Delta_{-2}^s p(0)}{2^{s+1}(s+1)! \cdot n(n+2) \ldots (n+2s)},$$

and if r is odd, then

$$\text{(5.7)} \qquad G_0 p(0) = 0.$$

Finally, we list Δ_j appeared in (5.1). The formulae (5.8)–(5.10) in the following were first obtained by A. Gray and M. Pinsky [7] and (5.11) was obtained by [13].

Lemma 5.3. *The following formulae hold.*

$$\Delta_0 = \frac{1}{3}R_{kihj}(m)x^{kh}\partial_i\partial_j - \frac{2}{3}\varrho_{kj}(m)x^k\partial_j, \tag{5.8}$$

$$\Delta_1 = \frac{1}{6}R_{kihj;p}(m)x^{khp}\partial_i\partial_j + \left(-\frac{1}{2}\varrho_{kj;h} + \frac{1}{12}\varrho_{kh;j}\right)(m)x^{kh}\partial_j, \tag{5.9}$$

$$\begin{aligned}\Delta_2 = {} & \frac{1}{5!}\{6R_{kihj;pq} + 8R_{kihu}R_{pjqu}\}(m)x^{khpq}\partial_i\partial_j \\ & - \frac{1}{3\cdot 5!}\{54\varrho_{jk;hp} - 18R_{kuhj;pu} + 46\varrho_{ku}R_{hjpu} \\ & + 32R_{jukv}R_{hupv}\}(m)x^{khp}\partial_j,\end{aligned} \tag{5.10}$$

$$\begin{aligned}\Delta_3 = {} & \frac{8}{6!}\{R_{kihj;pqr} + 6R_{kihu}R_{pjqu;r}\}(m)x^{khpqr}\partial_i\partial_j \\ & + \frac{4}{6!}\{-8R_{kuhv}R_{juqv;p} - 8R_{juhv}R_{puqv;k} + 6R_{kuhv}R_{pjqv;u} \\ & - R_{kuhv}R_{puqv;j} - 22R_{kjhu}\varrho_{pu;q} + R_{kjhu}\varrho_{pq;u} - 16\varrho_{ku}R_{puqj;h} \\ & + 2R_{kuhj;puq} + 2R_{kuhj;pqu} - 6\varrho_{kj;hpq} + \varrho_{kh;jpq} - \varrho_{kh;pjq} \\ & - \varrho_{kh;pqj}\}(m)x^{khpq}\partial_j.\end{aligned} \tag{5.11}$$

Further, Δ_4 *satisfies*

$$\begin{aligned}\Delta_4\frac{|x|^2}{2} = {} & -\frac{1}{3\cdot 7!}\{90\varrho_{kh;pqrs} + 144R_{kuhv}R_{puqv;rs} \\ & + 135R_{kuhv,r}R_{puqv;s} + 32R_{kuhv}R_{pvqw}R_{rwsu}\}(m)x^{khpqrs}.\end{aligned} \tag{5.12}$$

P r o o f. Due to (3.1), we obtain

$$\begin{aligned}g^{ij} = {} & \delta_{ij} + \frac{1}{3}R_{kihj}(m)x^{kh} + \frac{1}{3!}R_{kihj;p}(m)x^{khp} \\ & + \frac{2}{5!}(3R_{kihj;pq} + 4R_{kihu}R_{pjqu})(m)x^{khpq} \\ & + \frac{8}{6!}\{R_{kihj;pqr} + 3\nabla_r(R_{kihu}R_{pjqu})\}(m)x^{khpqr} \\ & + \frac{5}{7!}\{2R_{kihj;pqrs} + 10\nabla^2_{rs}(R_{kihu}R_{pjqu}) - 3R_{kihu;r}R_{pjqu;s} \\ & + \frac{32}{3}R_{kihu}R_{pjqv}R_{rusv}\}(m)x^{khpqrs} + O(|x|^7).\end{aligned} \tag{5.13}$$

Hence, the formulae of Lemma 5.3 follow from substitution of relations (3.2) and (5.13) into (5.1). □

The purpose of this section is to prove

Proposition 5.4. *Let (M, g) be an n-dimensional C^∞ Riemannian manifold with $n \geqslant 2$. Then it holds that, for a function f of class C^{10} near m,*

(5.14)

$$
\begin{aligned}
E_m f(X(T_\varepsilon)) &= f(m) + \frac{\varepsilon^2}{2n}\Delta f(m) + \frac{\varepsilon^4}{4!\,n(n+2)}\Big(3\Delta^2 f + \frac{2\tau}{n}\Delta f\Big)(m) \\
&+ \frac{\varepsilon^6}{6!\,n(n+2)(n+4)}\Big\{15\Delta^3 f + \frac{30}{n}\tau\Delta^2 f + \frac{30}{n+2}\langle\nabla\Delta f, \nabla\tau\rangle \\
&+ \Big(\frac{24}{n}\Delta\tau + \frac{20}{n^2}\tau^2 - \frac{4}{n}|\varrho|^2 + \frac{4}{n}|R|^2\Big)\Delta f\Big\}(m) \\
&+ \frac{\varepsilon^8}{8!\,n(n+2)(n+4)(n+6)}\Big[105\Delta^4 f + \frac{420}{n}\tau\Delta^3 f + \frac{840}{n+2}\langle\nabla\Delta^2 f, \nabla\tau\rangle \\
&+ \frac{140(5n+12)}{n^2(n+2)}\tau^2\Delta^2 f + \frac{56(2n^2+13n+24)}{n(n+2)(n+4)}(6\Delta\tau - |\varrho|^2 + |R|^2)\Delta^2 f \\
&+ \frac{56}{n+4}\{9\langle\nabla^2\Delta f, \nabla^2\tau\rangle + 3\langle\nabla^2\Delta f, \Delta\varrho\rangle + 2\langle\nabla^2\Delta f, \dot R\rangle + 2R_{ikj\ell}\varrho_{ij}\nabla^2_{k\ell}\Delta f \\
&- 4\varrho_{ij}\varrho_{jk}\nabla^2_{ki}\Delta f\} + \frac{140}{n+2}\Big\{6\langle\nabla\Delta f, \nabla\Delta\tau\rangle + \frac{1}{3(n+2)}\varrho_{ij}\nabla_i\tau\nabla_j\Delta f \\
&+ \langle\nabla\Delta f, \nabla(|R|^2 - |\varrho|^2)\rangle + \frac{2(5n+6)}{n(n+2)}\tau\langle\nabla\Delta f, \nabla\tau\rangle\Big\} \\
&+ 8!\,n(n+2)(n+4)(n+6)U_6(0)\Delta f\Big](m) + O(\varepsilon^{10}),
\end{aligned}
$$

$$
\begin{aligned}
(5.15)\qquad E_m T_\varepsilon &= \frac{\varepsilon^2}{2n} + \frac{2\varepsilon^4}{4!\,n^2(n+2)}\tau(m) \\
&+ \frac{4\varepsilon^6}{6!\,n^2(n+2)(n+4)}\Big(6\Delta\tau + \frac{5}{n}\tau^2 - |\varrho|^2 + |R|^2\Big)(m) + \varepsilon^8 U_6(0) + O(\varepsilon^{10}),
\end{aligned}
$$

where $U_6(0)$ is given by

$$
\begin{aligned}
(5.16)\quad U_6(0) &= \frac{1}{8!\,n^2(n+2)(n+4)(n+6)}\Big\{\frac{280(5n+12)}{3n^2(n+2)}\tau^3 \\
&+ \frac{112(5n+16)(n+3)}{3n(n+2)(n+4)}\tau(6\Delta\tau - |\varrho|^2 + |R|^2) + \frac{8(19n+20)}{3(n+4)}\check\varrho \\
&- \frac{4(37n+120)}{3(n+4)}(2\langle\varrho, \dot R\rangle + 3\langle\Delta\varrho, \varrho\rangle) - \frac{16(11n+30)}{3(n+4)}\langle\varrho\otimes\varrho, \overline{R}\rangle \\
&+ \frac{48(2n+15)}{n+4}\langle\nabla^2\tau, \varrho\rangle + 270\Delta^2\tau + \frac{15(3n+62)}{n+2}|\nabla\tau|^2 - 30|\nabla\varrho|^2 \\
&- 60\alpha(\varrho) + 180\langle\Delta R, R\rangle + 135|\nabla R|^2 + \frac{220}{3}\check R + \frac{400}{3}\check{\bar R}\Big\}(m).
\end{aligned}
$$

Corollary 5.5. *Let (M,g) be a super-Einstein space. Then it holds that*

(5.17)

$$E_{m,4}f(m)=\frac{1}{8!\,n(n+2)(n+4)(n+6)}\Big[105\Delta^4f+\frac{420}{n}\tau\Delta^3f+\frac{28}{n(n+2)}\Big\{\frac{21n+46}{n}\tau^2+2(2n+7)|R|^2\Big\}\Delta^2f+\frac{1}{n}\Big\{\frac{16(51n+116)}{3n^2(n+2)}\tau^3+\frac{8(21n+56)}{n(n+2)}\tau|R|^2-\frac{5}{3}(3|\nabla R|^2-20\check{R}+16\check{\check{R}})\Big\}\Delta f\Big](m).$$

P r o o f of Proposition 5.4. We first prove (5.15). We note that, from (5.2), (5.4) and (5.7), if $r+\nu$ is odd, then

$$U_\nu^{i_1i_2\dots i_r}(0)=0. \tag{5.18}$$

The formula (5.3) with $r=0$, $k=4$, using (5.18), implies

$$E_mT_\varepsilon=\varepsilon^2U_0(0)+\varepsilon^4U_2(0)+\varepsilon^6U_4(0)+\varepsilon^8U_6(0)+O(\varepsilon^{10}). \tag{5.19}$$

The first three $U_{2\mu}(0)$ $(0\leqslant\mu\leqslant2)$ are obtained in [7]. These are also obtained in the course of our computation of $U_6(0)$.

We compute $U_6(0)$. Note first that $U_0^{i_1i_2\dots i_r}(x)$ $(0\leqslant r\leqslant4)$ are computed in [13]. It follows from Lemmas 5.1–5.3 that

$$U_0(x)=\frac{1-|x|^2}{2n},\qquad U_0(0)=\frac{1}{2n}, \tag{5.20}$$

$$\begin{aligned}U_2(x)&=G_0\Delta_0U_0(x)\\&=\frac{1}{6n(n+4)}\Big\{\varrho_{kh}(m)x^{kh}(1-|x|^2)\\&\qquad+\tau(m)\frac{1-|x|^2}{n}-\tau(m)\frac{1-|x|^4}{2(n+2)}\Big\},\\U_2(0)&=\frac{\tau(m)}{12n^2(n+2)},\end{aligned} \tag{5.21}$$

$$U_3(x)=G_0\Delta_1U_0(x)=\frac{1}{8n(n+6)}\Big\{\varrho_{kh;p}(m)x^{khp}(1-|x|^2)+2\nabla_p\tau(m)x^p\Big(\frac{1-|x|^2}{n+2}-\frac{1-|x|^4}{2(n+4)}\Big)\Big\}, \tag{5.22}$$

(5.23)

$$
\begin{aligned}
U_4(x) &= G_0(\Delta_2 U_0 + \Delta_0 U_2)(x) \\
&= \frac{4}{3\cdot 5!\, n(n+4)}\Big[10\Big(\varrho_{uv}R_{kuhv} - 2\varrho_{ku}\varrho_{hu} + \frac{\tau}{n}\varrho_{kh}\Big)(m)(U_0^{kh}(x) - U_0^{khpp}(x)) \\
&\quad + \frac{20}{3n(n+2)}\tau(m)\varrho_{kh}(m)U_0^{khpp}(x) \\
&\quad + \Big\{10\varrho_{kh}\varrho_{pq} + (n+4)(9\varrho_{kh;pq} + 2R_{kuhv}R_{puqv})\Big\}(m)U_0^{khpq}(x)\Big],
\end{aligned}
$$

and

$$
U_4(0) = \frac{4}{6!\, n^2(n+2)(n+4)}\left(6\Delta\tau + \frac{5}{n}\tau^2 - |\varrho|^2 + |R|^2\right)(m). \tag{5.24}
$$

Finally, substituting (5.20)–(5.23) into the formula

$$
U_6(0) = G_0(\Delta_4 U_0 + \Delta_2 U_2 + \Delta_1 U_3 + \Delta_0 U_4)(0)
$$

and using Lemmas 5.1–5.3, we obtain (5.16). Hence substituting (5.20)–(5.21), (5.24) and (5.16) into (5.19), the formula (5.15) follows.

Next we prove (5.14). Dynkin's formula [5] is the following:

$$
E_m f(X(T_\varepsilon)) = f(m) + E_m \int_0^{T_\varepsilon} \Delta f(X(t))\, dt. \tag{5.25}
$$

Expanding Δf at m [see (3.5)] and using (5.3), (5.25) is reduced to

$$
\begin{aligned}
E_m f(X(T_\varepsilon)) = f(m) &+ \varepsilon^2 U_0(0)\Delta f(m) + \varepsilon^4\{U_2(0)\Delta f(m) + B_2\Delta f(m)\} \\
&+ \varepsilon^6\{U_4(0)\Delta f(m) + B_4\Delta f(m)\} \\
&+ \varepsilon^8\{U_6(0)\Delta f(m) + B_6\Delta f(m)\} + O(\varepsilon^{10}),
\end{aligned}
$$

where

$$
B_j\Delta f(m) = \sum_{r=1}^{j} \frac{1}{r!} U_{j-r}^{i_1 i_2 \dots i_r}(0)(\nabla^r_{i_1 i_2 \dots i_r}\Delta f)(m).
$$

On the other hand, the terms $B_j\Delta f(m)$, $j = 2, 4, 6$ are computed in [13]. Hence we obtain (5.14). □

6. Proof of Theorem 1 (continued)

First we prepare some curvature properties of the 2^*-stein space.

Lemma 6.1. *Let (M,g) be an n-dimensional 2^*-stein space. Then it holds that*

$$\nabla_i R_{abcd}\nabla_j R_{abcd} = \nabla_p R_{iabc}\nabla_p R_{jabc} = -\frac{2\tau}{n^2}|R|^2 g_{ij} + \check{R}_{ij} + 4\check{\check{R}}_{ij}. \tag{6.1}$$

P r o o f. Since (M,g) is 2^*-steinian, we have $\nabla^2_{k\ell}(\bar{R}\circ\bar{R})_{ijk\ell} = 0$. Then we obtain

$$\nabla_i R_{abcd}\nabla_j R_{abcd} + \nabla_p R_{iabc}\nabla_p R_{jabc} = -\frac{4\tau}{n^2}|R|^2 g_{ij} + 2\check{R}_{ij} + 8\check{\check{R}}_{ij}. \tag{6.2}$$

Hence (6.1) follows from (3.19) and (6.2). □

Lemma 6.2. *Let (M,g) be an n-dimensional 2^*-stein space. Then it holds that*

$$\varphi_i = \nabla_j\left\{(\check{R}_{ij} - 2\check{\check{R}}_{ij}) - \frac{1}{6}(\check{R} - 2\check{\check{R}})g_{ij}\right\} = 0. \tag{6.3}$$

P r o o f. After calculations, we obtain

$$\nabla_j(\nabla_p R_{iabc}\nabla_p R_{jabc}) = 6\nabla_j\check{\check{R}}_{ij} - \frac{1}{12}\nabla_i(\check{R} + 16\check{\check{R}} - 3|\nabla R|^2), \tag{6.4}$$

$$\nabla_j(\nabla_i R_{abcd}\nabla_j R_{abcd}) = 8\nabla_j\check{\check{R}}_{ij} - \frac{1}{6}\nabla_i(2\check{R} + 16\check{\check{R}} - 3|\nabla R|^2). \tag{6.5}$$

Applying ∇_j to (6.1) and using (6.4)–(6.5), we have

$$\nabla_j\check{R}_{ij} = \frac{1}{6}\nabla_i\check{R}, \tag{6.6}$$

$$\nabla_j\check{\check{R}}_{ij} = \frac{1}{24}\nabla_i(3\check{R} + 16\check{\check{R}} - 3|\nabla R|^2). \tag{6.7}$$

Taking account of (6.6)–(6.7) and (3.20), we obtain (6.3). □

Now we are ready to prove the rest of Theorem 1. In the following proof, we assume that (M,g) is a super-Einstein space due to Theorem A (3).

P r o o f of Theorem 1(b). *Sufficiency.* We first note that, by (4.3) and (5.17),

$$(L_{m,4} - E_{m,4})f(m) = \frac{1}{8!\,n(n+2)(n+4)(n+6)}\Big[\overline{\mathscr{R}}f - \frac{5}{3}\{(3\nabla_i R_{abcd}\nabla_j R_{abcd} - 20\check{R}_{ij} + 16\check{\check{R}}_{ij})\nabla^2_{ij}f - \lambda\Delta f\} + \Big(82\varphi_i - \frac{5n}{18}\nabla_i\lambda\Big)\nabla_i f\Big](m). \tag{6.8}$$

Hence the proof of the sufficiency of the condition $(L1)_4$ is verified in a similar way to that of the sufficiency of Theorem 1(c), because of (6.3). Since each of the conditions $(L2)_4$–$(L4)_4$ implies $(ii)_4$, the sufficiency of each of them is clear from Theorem B (2).

Necessity. Suppose that (M, g) is a 2^*-stein space and satisfies (2.2)–(2.3). The condition $(L1)_4$ is first shown by (6.3). This with Theorem B (2) implies $(ii)_4$. Then it follows from (5.14) and (5.17) that

(6.9)

$$\begin{aligned}
L_m(\varepsilon, f) &= E_m f(X(T_\varepsilon)) + O(\varepsilon^{10}) = f(m) + \frac{\varepsilon^2}{2n}\Delta f(m) \\
&+ \frac{\varepsilon^4}{4!\, n(n+2)}\Big(3\Delta^2 f + \frac{2\tau}{n}\Delta f\Big)(m) \\
&+ \frac{\varepsilon^6}{6!\, n(n+2)(n+4)}\Big\{15\Delta^3 f + \frac{30}{n}\tau\Delta^2 f + \Big(\frac{16}{n^2}\tau^2 + \frac{4}{n}|R|^2\Big)\Delta f\Big\}(m) \\
&+ \frac{\varepsilon^8}{8!\, n(n+2)(n+4)(n+6)}\Big[105\Delta^4 f + \frac{420}{n}\tau\Delta^3 f + \frac{28}{n(n+2)}\Big\{\frac{21n+46}{n}\tau^2 \\
&+ 2(2n+7)|R|^2\Big\}\Delta^2 f + \frac{1}{n}\Big\{\frac{16(51n+116)}{3n^2(n+2)}\tau^3 + \frac{8(21n+56)}{n(n+2)}\tau|R|^2 \\
&- \frac{5}{3}(3|\nabla R|^2 - 20\check{R} + 16\check{\bar{R}})\Big\}\Delta f\Big](m) + O(\varepsilon^{10}).
\end{aligned}$$

Now the necessity of $(L4)_4$ is clear from (6.9). Further (5.15)–(5.16) and (6.9) imply $(L2)_4$. On the other hand, due to (3.6) we have

(6.10)

$$\begin{aligned}
\Phi_m(\varepsilon) &= \frac{\varepsilon^2}{2n} + \frac{2\varepsilon^4}{4!\, n^2(n+2)}\tau(m) \\
&+ \frac{4\varepsilon^6}{6!\, n^2(n+2)(n+4)}\Big(6\Delta\tau + \frac{20}{3n}\tau^2 - \frac{8}{3}|\varrho|^2 + |R|^2\Big)(m) \\
&+ \frac{\varepsilon^8}{8!\, n^2(n+2)(n+4)(n+6)}\Big\{\frac{560(5n+12)}{3n^2(n+2)}\tau^3 \\
&+ \frac{56(5n+12)}{3n(n+2)}\tau(18\Delta\tau - 8|\varrho|^2 + 3|R|^2) \\
&- \frac{128}{3}\check{\varrho} - 96\big(2\langle\varrho, \dot{R}\rangle + 3\,\langle\Delta\varrho, \varrho\rangle\big) + 128\,\langle\varrho\otimes\varrho, \bar{R}\rangle - 324\,\langle\nabla^2\tau, \varrho\rangle + 270\Delta^2\tau \\
&- 270|\nabla\tau|^2 - 135|\nabla\varrho|^2 - 270\alpha(\varrho) + 180\,\langle\Delta R, R\rangle + 135|\nabla R|^2 + \frac{220}{3}\check{R} \\
&+ \frac{400}{3}\check{\bar{R}}\Big\}(m) + O(\varepsilon^{10}).
\end{aligned}$$

Under the assumption of the Einsteinity, it follows from (5.15)–(5.16) and (6.10) that

$$E_m T_\varepsilon = \Phi_m(\varepsilon) + O(\varepsilon^{10}). \tag{6.11}$$

Hence the necessity of $(\mathrm{L3})_4$ is immediate from $(\mathrm{L2})_4$ and (6.11). □

Proof of Theorem 1(a). *Sufficiency.* By (3.16) and (6.8), we have

(6.12)
$$\begin{aligned}(M_{m,4} - E_{m,4})f(m) = \frac{1}{8!\, n(n+2)(n+4)(n+6)} \Big[& -\mathscr{R} f \\ & + 5\{(3\nabla_i R_{abcd}\nabla_j R_{abcd} - 20\check{R}_{ij} + 16\check{\bar{R}}_{ij})\nabla^2_{ij} f - \lambda\Delta f\} \\ & + (82\varphi_i + \frac{35n}{18}\nabla_i\lambda)\nabla_i f\Big](m).\end{aligned}$$

Hence we can prove all the rest in the same way as in the proof of the sufficiency of Theorem 1(b).

Necessity. The proof of the necessity is similar to that of the necessity in Theorem 1(b) and will be omitted. □

7. Proof of Theorem 2

For the proof of Theorem 2, we need the following curvature properties of the super-Einstein space.

Lemma 7.1. *Let (M,g) be an n-dimensional super-Einstein space. Then it holds that*

$$\check{R}_{ij} - 2\check{\bar{R}}_{ij} = \frac{1}{n}(\check{R} - 2\check{\bar{R}})g_{ij}, \quad \text{for } n \leqslant 6, \tag{7.1}$$

$$\check{R} - 2\check{\bar{R}} = -\frac{1}{4}\left\{\left(1 - \frac{12}{n} + \frac{40}{n^2}\right)\tau^3 + 3\left(1 - \frac{8}{n}\right)\tau|R|^2\right\}, \quad \text{for } n \leqslant 5. \tag{7.2}$$

Proof. Following [16], we define the tensor $(E^{(p)}_{ij})$ by

$$E^{(p)}_{ij} = g_{aj}\delta^{aj_1\ldots j_{2p}}_{ii_1\ldots i_{2p}} R_{i_1i_2j_1j_2}\cdots R_{i_{2p-1}i_{2p}j_{2p-1}j_{2p}},$$

for any natural number p, where

$$\delta^{aj_1\ldots j_{2p}}_{ii_1\ldots i_{2p}} = \det(\delta_{i_r j_s}) \quad (i_0 \equiv i, j_0 \equiv a).$$

Then for $p = 3$, we obtain;

$$(7.3) \quad E^{(3)}_{ij} = G_{(3)}g_{ij} - 48\Big\{(\tau^2 - 4|\varrho|^2 + |R|^2)\varrho_{ij} - 4\tau\varrho_{ip}\varrho_{jp} - 4\tau\varrho_{pq}R_{ipjq} + 8\varrho_{pq}\varrho_{ip}\varrho_{jq} + 8\varrho_{ip}\varrho_{kl}R_{jkpl} + 8\varrho_{jp}\varrho_{kl}R_{ikpl} + 8\varrho_{pq}\varrho_{qr}R_{ipjr} + 2\tau\dot{R}_{ij} - 4\varrho_{ip}\dot{R}_{jp} - 4\varrho_{jp}\dot{R}_{ip} - 4R_{ipjq}\dot{R}_{pq} - 4\varrho_{pq}R_{ipkl}R_{jqkl} - 8\varrho_{pq}R_{ikpl}R_{jkql} + 8\varrho_{pq}R_{prqs}R_{irjs} + 4\check{R}_{ij} - 8\check{\bar{R}}_{ij}\Big\},$$

where $G_{(3)}$ denotes the integrand of the Gauss-Bonnet formula, i.e.,

$$G_{(3)} \equiv E^{(3)}_{kk} = 8\{\tau^3 - 12\tau|\varrho|^2 + 3\tau|R|^2 + 16\check{\varrho} + 24\langle\varrho\otimes\varrho, \bar{R}\rangle - 24\langle\varrho, \dot{R}\rangle + 4\check{R} - 8\check{\bar{R}}\}.$$

Now note that, by definition, $E^{(3)}_{ij} = 0$ hold for $n \leqslant 6$. This with the super-Einsteinity and (7.3) implies

$$(7.4) \quad \check{R}_{ij} - 2\check{\bar{R}}_{ij} = \frac{1}{6}(\check{R} - 2\check{\bar{R}})g_{ij} + \frac{n-6}{24n}\Big\{\Big(1 - \frac{12}{n} + \frac{40}{n^2}\Big)\tau^3 + 3\Big(1 - \frac{8}{n}\Big)\tau|R|^2\Big\}g_{ij}, \quad \text{for } n \leqslant 6.$$

Hence by (7.4), we obtain (7.1)–(7.2). □

Lemma 7.2. *Let (M, g) be an n-dimensional 2^*-stein space with $3 \leqslant n \leqslant 6$. Then the following conditions are mutually equivalent, except for the case $n = 6$ in (3):*

(1) (M, g) satisfies (2.2) and (2.3);

(2) (M, g) satisfies (2.4) and (2.3);

(3) $(n \leqslant 5)$ (M, g) satisfies (2.4) and $|\nabla R|^2 =$ constant.

Proof. The equivalence of (1) and (2) follows from (2.2), (6.1) and (7.1). The equivalence of (2) and (3) follows from (2.3), (3.20) and (7.2). □

Proof of Theorem 2. The assertions (2)–(3) follow immediately from Lemma 7.2. We prove the assertion (1). But we only show the sufficiency of the assertion in the case $n = 4$, because the other assertions are clear. Assume one of the conditions $(\mathrm{i})_4$–$(\mathrm{iv})_4$, $(\mathrm{M1})_4$–$(\mathrm{M4})_4$ and $(\mathrm{L1})_4$–$(\mathrm{L4})_4$. Then by Theorem 1 and Lemma 7.2, it follows that $|\nabla R|^2$ is constant. Consequently we can trace the arguments in [21: pp. 218–220], to obtain that the eigenvalues of $W \in C^\infty(\mathrm{End}\,\Lambda^2 M)$ are constants, where W is the Weyl curvature tensor of (M, g). Hence by an unpublished result of A. Derdziński (reported in [21: Proposition 5] and see [3] for the proof), (M, g) is locally symmetric. The required result follows as in [2]. □

Proof of Theorem 3. Suppose that the condition $(\mathrm{MI})_k$ holds and choose a bi-harmonic function f near m. Due to a generalization of Dynkin's formula [1]:

$$E_m f(X(T_\varepsilon)) = f(m) + E_m T_\varepsilon \Delta f(X(T_\varepsilon)) - E_m \int_0^{T_\varepsilon} t\Delta^2 f(X(t))\, dt,$$

we have

$$\begin{aligned} E_m f(X(T_\varepsilon)) &= f(m) + E_m T_\varepsilon \Delta f(X(T_\varepsilon)) \\ &= f(m) + (E_m T_\varepsilon)(E_m \Delta f(X(T_\varepsilon))) + O(\varepsilon^{2k+2}) \\ &= f(m) + (E_m T_\varepsilon)\Delta f(m) + O(\varepsilon^{2k+2}), \end{aligned}$$

by applying (2.5) and Dynkin's formula. Hence the condition $(\mathrm{S2})_k$ follows.

Conversely, suppose that the condition $(\mathrm{S2})_k$ holds and choose a harmonic function h near m. We consider the boundary value problem

$$\begin{aligned} \Delta u_\varepsilon(x) &= h(x), \quad x \in B_m(\varepsilon), \\ u_\varepsilon(\xi) &= h(\xi), \quad \xi \in \partial B_m(\varepsilon). \end{aligned} \tag{8.1}$$

The solution u_ε of (8.1) is bi-harmonic in $B_m(\varepsilon)$. By a generalization of Dynkin's formula [1] again, we have

$$E_m u_\varepsilon(X(T_r)) = u_\varepsilon(m) + E_m T_r h(X(T_r)) \tag{8.2}$$

for all $r \in (0, \varepsilon)$. But, from the condition $(\mathrm{S2})_k$ and (8.1), we have

$$|E_m u_\varepsilon(X(T_r)) - \{u_\varepsilon(m) + (E_m T_r)h(m)\}| \leqslant K r^{2k+2} |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))} \tag{8.3}$$

for all $r \in (0, \varepsilon)$, where

$$|u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))} = \sum_{j=0}^{2k+2} \sum_{i_1, i_2, \ldots, i_j} \sup_{p \in B_m(\varepsilon)} |\partial_{i_1}\partial_{i_2}\ldots\partial_{i_j} u_\varepsilon(p)|.$$

(8.2)–(8.3) imply

$$|E_m T_r h(X(T_r)) - (E_m T_r)h(m)| \leqslant K r^{2k+2} |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))}$$

for all $r \in (0, \varepsilon)$. Letting $r \uparrow \varepsilon$, we have

$$|E_m T_\varepsilon h(X(T_\varepsilon)) - (E_m T_\varepsilon)h(m)| \leqslant K \varepsilon^{2k+2} |u_\varepsilon|_{C^{2k+2}(B_m(\varepsilon))}. \tag{8.4}$$

In the case of $k = +\infty$, we have

$$E_m T_r h(X(T_r)) = (E_m T_r) h(m), \quad r \in (0, \varepsilon)$$

first and then

$$E_m T_\varepsilon h(X(T_\varepsilon)) = (E_m T_\varepsilon) h(m)$$

in place of (8.4). These facts show that the independence formula (2.5) holds for a harmonic function h near m. Hence due to [13], $(\mathrm{MI})_k$ holds. □

Proof of Theorem 4.

Proof of the assertion for $(\mathrm{S2})_3$ and $(\mathrm{S2})_4$. This is a direct consequence of Theorem 3 and the following

Theorem C ([13]). *Let (M, g) be an n-dimensional connected C^∞ Riemannian manifold with $n \geqslant 2$. Then the following assertions hold.*

(1) *The condition $(\mathrm{MI})_3$ is necessary and sufficient in order that (M, g) be of constant scalar curvature.*

(2) *The condition $(\mathrm{MI})_4$ is necessary and sufficient in order that (M, g) be a quasi-super-Einstein space.*

□

Remark. In [15], M. Liao also proved the sufficiency of the assertion (1) in Theorem C by a different method from [13].

Proof of the assertion for $(\mathrm{S4})_3$ and $(\mathrm{S4})_4$. Suppose first that $(\mathrm{S4})_3$ holds. Then by (5.14) for each $m \in M$, we have

$$E_{m,3} f(m) = p_3(\Delta) f(m) \tag{8.5}$$

for all functions f of class C^8 near m. For the normal coordinate $(x^1, x^2, \dots, x^n)$ at m, choosing functions f so that $\Delta f = x^i$, $i = 1, 2, \dots, n$ in (8.5), we obtain that the scalar curvature τ is constant.

Suppose next that $(\mathrm{S4})_4$ holds. Then by (5.14) for each $m \in M$, we have

$$E_{m,4} f(m) = p_4(\Delta) f(m) \tag{8.6}$$

for all functions f of class C^{10} near m. Similarly, choosing functions f so that $\Delta f = x^i$, $i = 1, 2, \dots, n$ in (8.6), we obtain $|R|^2 - |\varrho|^2 =$ constant. Further choosing functions f so that $\Delta f = x^i x^j$, $i, j = 1, 2, \dots, n$ in (8.6), we obtain (2.1). Hence (M, g) is a quasi-super-Einstein space.

The necessity of each of $(\mathrm{S4})_3$ and $(\mathrm{S4})_4$ is clear from (5.14). □

The assertions (1)–(2) are proved.

Proof of Corollary. Notice that the relations $|R|^2 - |\varrho|^2 =$ constant and (2.1) are reduced to

$$|R|^2 = \text{ constant and } \dot{R}_{ij} = \frac{|R|^2}{n} g_{ij}$$

respectively, provided (M, g) is an Einstein space. Then the assertions of Corollary are clear from those of Theorem 4 (2). □

Proof of the assertion for $(S3)_3$ and $(S3)_4$. Suppose first that $(S3)_3$ holds. Then by (5.14) and (6.10) for each $m \in M$, we have

$$9\langle \nabla\Delta f, \nabla\tau\rangle(m) + 2(n+2)\left(|\varrho|^2 - \frac{\tau^2}{n}\right)(m)\Delta f(m) = 0 \tag{8.7}$$

for all bi-harmonic functions f near m. But due to [4], we can take a harmonic coordinate system $(U; x^1, x^2, \ldots, x^n)$. Choosing functions f so that $\Delta f = x^i$, $i = 1$, 2, …, n in (8.7), we obtain that τ is constant, and that $(|\varrho|^2 - \frac{\tau^2}{n})(m)\Delta f(m) = 0$. Thus (M, g) is an Einstein space.

Suppose next that $(S3)_4$ holds. Since (M, g) is Einsteinian, by (6.3) the condition $(S2)_4$ holds. Hence by Corollary, (M, g) is a super-Einstein space.

The necessity of each of $(S3)_3$ and $(S3)_4$ is clear. □

The assertion (3) is proved. □

Remark. There are quasi-super-Einstein spaces which are not Einsteinian. Indeed due to [14], the following spaces are in that category;

$$S^p(k) \times H^p(-k), \quad S^3(k) \times \mathbf{R}^p \quad \text{and } H^3(-k) \times \mathbf{R}^p \quad (p \geqslant 2),$$

where $S^n(k)$, $H^n(-k)$ and $\mathbf{R}^n$ denote n-dimensional spaces of constant sectional curvature $k > 0$, $-k < 0$ and 0, respectively.

Remark. Let M be a 4-dimensional compact orientable C^∞ manifold. Let $\mathscr{M}$ be the set of all Riemannian metrics g on M such that $\operatorname{vol} M = 1$. We define the mapping $I\colon \mathscr{M} \to \mathbf{R}$ by

$$I(g) = \int_M (|R|^2 - |\varrho|^2)\,\mathrm{d}M.$$

We then obtain that a metric $g \in \mathscr{M}$ is a critical point of I, if and only if (M, g) satisfies

$$\begin{aligned}\dot{R}_{ij} = \frac{|R|^2 - |\varrho|^2}{4} g_{ij} - \varrho^{pq} R_{ipjq} + 2\varrho_{ip}\varrho^p_j - \frac{3}{2}\Delta\varrho_{ij} \\ + \frac{1}{4}(\Delta\tau)g_{ij} + \frac{1}{2}\nabla^2_{ij}\tau.\end{aligned}$$

In particular, if (M, g_0) is a quasi-super-Einstein space, then $g_0 \in \mathscr{M}$ is a critical point of I.

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