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REPRESENTATIONS OF RIESZ SPACES AS SPACES OF MEASURES I

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INTRODUCTION

Representation theory plays an important role in the theory of Riesz spaces, and a number of papers on this subject have appeared, e. g. [N], [Y], [MO], [O], [Vu], [P], [JK1, 2], [Ve1-3], [Fr1], [B], [BN], [VL], [H1, 2], [MW], [Ko], [We], [Wn], [L], [BR], [Fl] in the case of abstract Riesz spaces, and [KK1, 2], [Ka1, 2], [BK], [D], [M], [Sc1], [Wo], [G], [FP1, 2] in the case of Banach lattices. But in almost all of them, Riesz spaces are represented as spaces of continuous functions. It seems that representations as spaces of measures have been treated only in the case of abstract (L)-spaces (see Kakutani [Ka1, 2]) or in the case the Riesz space in question is a space of (abstract) measures itself (see Constantinescu [C]). Of course the possibility of representing Riesz spaces with separating order continuous dual as spaces of measures follows—by applying the Radon-Nikodym Theorem—already from Fremlin's Theorem [Fr1], which states that such spaces are isomorphic to spaces of locally integrable functions. In the present paper we give concrete representations for these spaces which reflect many properties of the duality. Applying our results, we can introduce measure theoretic notions such as atomical and atomfree elements, Hellinger integrability, or measurability in Riesz spaces (see [Fi2, 5, 6]).

We will show (Corollary 2.17) that a Riesz space E can be represented as an order dense Riesz subspace of a band $\mathscr{M}$ of measures iff it is separated by the set E^π of its order continuous linear forms; E is an ideal of $\mathscr{M}$ iff E is Dedekind complete; $E = \mathscr{M}$ iff E is hypercomplete (Hypercomplete Riesz spaces are investigated in [Fi3, 4]). This representation can even be given in the following concrete form: Applying the Ogasawara-Maeda Theorem for the extended order continuous dual E^ϱ of E, we show that if E^π separates E, if e is a weak unit of E^ϱ and if $R \subset E^\pi$ is a set of components of e with $e = \sup R$ (such pairs (e, R) always exist), then there is a locally compact hyperstonian space Y (unique up to a canonical homeomorphism) with the following properties: There exist an injective Riesz homomorphism

$$v\colon E \to \mathscr{M}(Y) := \{\mu\colon \mu \text{ is a normal Radon measure on } Y\}$$

and a Riesz isomorphism

$$\hat{u}\colon E^{\varrho} \to C_{\infty}(Y) := \{f \in \overline{\mathbf{R}}^{Y} : f \text{ is continuous}, \{y \in Y : |f(y)| \neq \infty\} \text{ is dense in } Y\}$$

such that vE is order dense in $\mathscr{M}(Y)$, $\hat{u}e = 1_Y$, $\operatorname{supp} \hat{u}g$ is compact for all $g \in R$, $Y = \bigcup_{g\in R} \operatorname{supp} \hat{u}g$, and $\xi(x) = \int \hat{u}\xi \, \mathrm{d}(vx)$ for all $(x,\xi) \in E \times E^{\varrho}$ with $\xi(x) \in \mathbf{R}$ (Theorem 2.16, Corollary 2.20). In the special case where E is a space of (abstract) measures, this result was proved by Constantinescu [C; 2.3.6, 2.3.8]. As an immediate consequence (Corollary 2.18) of our theorem we obtain the well-known result that each AL-space is Riesz isomorphic and isometric to $\mathscr{M}(Y)$ for some compact hyperstonian space Y, from which one can easily deduce Kakutani's Theorem on AL-spaces.

Some more properties of these representations are discussed in the last part of section 2. In particular we prove the existence of a greatest representation (Proposition 2.22) and investigate the relations of the representations of a Riesz space to the representations of its subspaces.

In the appendix we show that Fremlin's Theorem is an easy consequence of our theorem.

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1. Notations and terminology

Let E be a Riesz space. We set

$$E_{+} := \{x \in E : x \geqslant 0\}.$$

If $x \in E_{+}$, then $y \in E$ is called a component of x iff $\inf(y, x - y) = 0$.

For $A \subset E$,

$$A^{\perp} := \{y \in E\colon \inf(|x|, |y|) = 0 \text{ for all } x \in A\}.$$

A Riesz subspace F of E is called order dense in E iff for every $x \in E_{+}$, $x = \sup\{y \in F : 0 \leqslant y \leqslant x\}$. If E is Archimedean, an equivalent formulation is: For each $x \in E_{+}$, $x \neq 0$, there is an $y \in F$ with $0 < y \leqslant x$ [AB; 1.14].

E is called laterally complete iff each disjoint family in E_+ has a supremum. If E is laterally complete and Archimedean, then E possesses weak units [AB; 23.2]. E is called universally complete iff E is Dedekind complete and laterally complete. If F is a universally complete Riesz space such that E is an order dense Riesz subspace of F, then F is called a universal completion of E.

We put

$$E^+ := \{\xi \in \mathbf{R}^E : \xi \text{ is an order bounded linear form on } E\},$$
$$E^\pi := \{\xi \in E^+ : \xi \text{ is order continuous}\}.$$

We call E a π-space iff E^π separates E. Each π-space is Archimedean [Z; 88.2]. If F is an ideal or an order dense Riesz subspace of E, then we have $\xi|_F \in F^\pi$ for each $\xi \in E^\pi$, which implies that F is a π-space if E is one.

Let E be Archimedean. We denote by E^ϱ the extended order continuous dual of E [LM]; hence if we put $\Phi := \{F : F \text{ is an order dense ideal of } E\}$, then $E^\varrho = \bigcup_{F \in \Phi} F^\pi$ is the inductive limit of the spaces F^π with respect to the injective maps $F^\pi \to G^\pi$, $\xi \mapsto \xi|_G$ $(F, G \in \Phi,\ G \subset F)$, where we identify $\xi \in F^\pi$ and $\eta \in G^\pi$ if $\xi|_{F \cap G} = \eta|_{F \cap G}$ $(F, G \in \Phi)$. For each $\xi \in E^\varrho$ there exists a greatest order dense ideal $E[\xi]$ of E on which ξ is defined as an order continuous linear form [LM; 1.3]. E^ϱ is a universally complete Riesz space [LM; 1.5]. If E is a π-space, then F^π is an order dense ideal of E^ϱ whenever $F \in \Phi$ [LM; 3.1], and E^ϱ is the universal completion of E^π.

We say that $R \subset (E^\pi)_+$ satisfies the hc-condition iff for each upward-directed family $(x_\iota)_{\iota \in I}$ from E_+ with $\sup_{\iota \in I} g(x_\iota) < \infty$ for every $g \in R$, there exists $\sup\{x_\iota : \iota \in I\}$ in E. If e is a weak unit of E^ϱ and $R \subset (E^\pi)_+$ is a set of components of e satisfying the hc-condition, then we call (e, R) an hc-pair of E. E is called hypercomplete iff there exists an hc-pair of E.

For a set X and a subset A of X, let 1_A^X or simply 1_A denote the characteristic function of A (on X). Given $f, g \in \overline{\mathbf{R}}^X$, we put $\{f = g\} := \{x \in X : f(x) = g(x)\}$, and we define analogously the sets $\{f < g\}$, $\{f = \alpha\}$, ... $(\alpha \in \overline{\mathbf{R}})$.

If $\mathfrak{R}$ is a δ-ring of subsets of X, then we set

$$\mathscr{M}(\mathfrak{R}) := \{\mu \in \mathbf{R}^{\mathfrak{R}} : \mu \text{ is } \sigma\text{-additive}\},$$
$$\mathscr{M}_b(\mathfrak{R}) := \{\mu \in \mathscr{M}(\mathfrak{R}) : \mu \text{ is bounded}\},$$
$$\mathscr{M}_c(\mathfrak{R}) := \{\mu \in \mathscr{M}(\mathfrak{R}) : X \backslash A \text{ is } \mu\text{-null for some } A \in \mathfrak{R}\}.$$

If $\mathscr{M}$ is a band of $\mathscr{M}(\mathfrak{R})$, then we put $\mathscr{M}_b := \mathscr{M} \cap \mathscr{M}_b(\mathfrak{R})$ and $\mathscr{M}_c := \mathscr{M} \cap \mathscr{M}_c(\mathfrak{R})$.

Let Y be a Hausdorff space. We set

$$\begin{aligned}
C(Y) &:= \{f \in \mathbf{R}^Y : f \text{ is continuous}\},\\
C_b(Y) &:= \{f \in C(Y) \colon f \text{ is bounded}\},\\
C_c(Y) &:= \{f \in C(Y) \colon \operatorname{supp} f \text{ is compact}\},\\
C_\infty(Y) &:= \{f \in \overline{\mathbf{R}}^Y : f \text{ is continuous, } \{|f| = \infty\} \text{ is nowhere dense}\},\\
\mathfrak{B}_c(Y) &:= \{B \colon B \text{ is a relatively compact Borel set of } Y\}.
\end{aligned}$$

$\mu \in \mathscr{M}(\mathfrak{B}_c(Y))$ is called normal iff μ is interior regular with respect to approximation by open sets.

$$\begin{aligned}
\mathscr{M}(Y) &:= \{\mu \in \mathscr{M}(\mathfrak{B}_c(Y)) \colon \mu \text{ is a normal Radon measure}\},\\
\mathscr{M}_b(Y) &:= (\mathscr{M}(Y))_b, \quad \mathscr{M}_c(Y) := (\mathscr{M}(Y))_c,\\
C_i(Y) &:= \{f \in C(Y) \colon f \in \mathscr{L}^1(\mu) \text{ for every } \mu \in \mathscr{M}(Y)\}.
\end{aligned}$$

If Y is locally compact, then the map

$$\mathscr{M}(Y) \to C_c(Y)^\pi, \quad \mu \mapsto \ell_\mu$$

is a Riesz isomorphism, where ℓ_μ denotes the map $C_c(Y) \to \mathbf{R}$, $f \mapsto \int f \, d\mu$.

Now let Y be a locally compact Stonian (= extremally disconnected) space. Then $C_\infty(Y)$ is a universally complete Riesz space [LZ; 47.4], and for $\mu, \nu \in \mathscr{M}(Y)$ we have that $\operatorname{supp} \mu$ and $\operatorname{supp} \nu$ are open-closed and

$$\begin{aligned}
\inf(|\mu|, |\nu|) = 0 &\iff \operatorname{supp} \mu \cap \operatorname{supp} \nu = \emptyset,\\
\mu \ll \nu &\iff \operatorname{supp} \mu \subset \operatorname{supp} \nu.
\end{aligned}$$

A locally compact Stonian space Y is called hyperstonian iff $\bigcup_{\mu \in \mathscr{M}(Y)} \operatorname{supp} \mu$ is dense in Y, or equivalently, iff $C_c(Y)^\pi$ separates $C_c(Y)$.

If Y is a completely regular Hausdorff space, then βY stands for the Stone-Čech compactification of Y.

E always denotes an Archimedean Riesz space and e a weak unit of E^ϱ. (Since E^ϱ is laterally complete and Archimedean, E^ϱ has weak units. $\xi \in (E^\varrho)_+$ is a weak unit of E^ϱ iff there is an order dense ideal F of E such that ξ is defined on F and is a weak unit of F^π, or equivalently, is a strictly positive element of F^π.)

R_e stands for the set of all components of e in E^π.

We begin with the basic definition which will turn out to be appropriate for our purposes.

Definition 2.1. An e-measure representation (shortly: e-mr) of E is a triple $(Y, \hat{u}, v)$ satisfying

(α) Y is a locally compact hyperstonian space;

(β) $\hat{u}\colon E^\varrho \to C_\infty(Y)$ is a Riesz isomorphism with $\hat{u}e = 1_Y$;

(γ) $v\colon E \to \mathscr{M}(Y)$ is an injective Riesz homomorphism such that vE is order dense in $\mathscr{M}(Y)$;

(δ) for all $(x,\xi) \in E \times E^\varrho$ we have

$$x \in E[\xi] \iff \hat{u}\xi \in \mathscr{L}^1(vx) \Longrightarrow \xi(x) = \int \hat{u}\xi \, \mathrm{d}(vx).$$

Some words should be said about this definition. Of course, (γ) is just the embedding of E we are aiming for. (δ) expresses the fact that the canonical duality of E and E^ϱ is related in a natural way to the canonical duality of $\mathscr{M}(Y)$ and $C_\infty(Y)$. The condition "$\hat{u}e = 1_Y$" means that our fixed weak unit e corresponds to the natural weak unit 1_Y of $C_\infty(Y)$. The property "$\hat{u}$ is a Riesz isomorphism" can easily be derived from (α), (γ), (δ), but we prefer to include it in the definition.

If E possesses an e-mr, then by (γ) E is necessarily a π-space (since $\mathscr{M}(Y)$ is one). *From now on we assume E to be a π-space.*

If Y and $\hat{u}$ satisfy conditions (α) and (β), we write for each component ξ of e in E^ϱ:

$$U_\xi := \operatorname{supp} \hat{u}\xi = \{\hat{u}\xi = 1\}.$$

Before proving our main results about existence and uniqueness of the objects just defined we want to state some basic properties.

Proposition 2.2. *If $(Y, \hat{u}, v)$ is an e-mr of E and if $A \subset Y$ is open-compact, then there exists $g \in R_e$ with $\hat{u}g = 1_A$.*

Proof. By 2.1(β) there exists $g \in E^\varrho$ with $\hat{u}g = 1_A$. Again by 2.1(β), g is a component of e. Since $1_A \in \mathscr{L}^1(vx)$ for all $x \in E$, it follows from 2.1(δ) that $E = E[g]$. Hence $g \in E^\pi$, and so $g \in R_e$. □

Corollary 2.3. *If $(Y, \hat{u}, v)$ is an e-mr of E, then there exists $R \subset E^\pi$ such that g is a component of e and U_g is compact whenever $g \in R$ and such that* $Y = \bigcup_{g \in R} U_g$.

Proof. Set $\mathfrak{K} := \{K \subset Y : K \text{ is open-compact}\}$. By 2.2, $R := \{\hat{u}^{-1} 1_K : K \in \mathfrak{K}\}$ meets the requirements. □

If $(Y, \hat{u}, v)$ is an e-mr of E and $R \subset E^\pi$ satisfies all the conditions described in the preceding corollary, then we call R associated to $(Y, \hat{u}, v)$. The e-mr's to which a given R is associated are important for the sake of compactness arguments (note that the U_g's are open).

We want to give an example which might help the reader to get a better intuition for the investigations made in this paper.

Example 2.4. Let X be a set and $\mathfrak{R}$ a δ-ring of subsets of X. Let E be a band of $\mathscr{M}(\mathfrak{R})$ (e.g. $E = \mathscr{M}(\mathfrak{R})$), and set

$$\dot{1}_A : E \to \mathbf{R}, \quad \mu \mapsto \mu(A) \qquad \text{for every } A \in \mathfrak{R},$$
$$\dot{1}_X : E_b \to \mathbf{R}, \quad \mu \mapsto \int 1_X \mathrm{d}\mu.$$

Then $\dot{1}_X$ is a (very natural!) weak unit of E^ϱ, and $R := \{\dot{1}_A : A \in \mathfrak{R}\}$ is a set of components of $\dot{1}_X$ in E^π with $\dot{1}_X = \sup\{\dot{1}_A : A \in \mathfrak{R}\}$. Moreover, $(\dot{1}_X, R)$ is an hc-pair of E.

In this special case, Constantinescu proved the existence and uniqueness of a $\dot{1}_X$-mr of E to which R is associated [C; 2.3.6,2.3.8].

Proposition 2.5. *If $(Y, \hat{u}, v)$ is an e-mr of E and if R, S are subsets of R_e such that U_g is compact for every $g \in R$, then the following are equivalent.*

(a) $\bigcup_{g \in R} U_g \subset \bigcup_{h \in S} U_h$.

(b) *For each $g \in R$ there exist $h_1, \ldots, h_n \in S$ with $g \leqslant \sup\{h_k : 1 \leqslant k \leqslant n\}$.*

This shows that if $R, S \subset E^\pi$ are both associated to $(Y, \hat{u}, v)$, then they are closely related.

Corollary 2.6. *If $(Y, \hat{u}, v)$ is an e-mr of E, if (e, R) is an hc-pair of E such that R is associated to $(Y, \hat{u}, v)$, and if $S \subset E^\pi$ is associated to $(Y, \hat{u}, v)$, then (e, S) is an hc-pair of E too.*

Proposition 2.7. *If $(Y, \hat{u}, v)$ is an e-mr of E, then the following are equivalent.*

(a) *Y is paracompact.*

(b) *There exists* $R \subset E^\pi$ *associated to* $(Y, \hat{u}, v)$ *such that* $\inf(g, h) = 0$ *whenever* $g, h \in R$, $g \neq h$.

Proof. (a)$\Rightarrow$(b): We can write Y as the disjoint union of a family $(A_\iota)_{\iota \in I}$ of open-closed, σ-compact subsets of Y. Since Y is Stonian, each A_ι is the disjoint union of a countable family $(A_{\iota\lambda})_{\lambda \in L_\iota}$ of open-compact subsets of Y. Put $R := \{\hat{u}^{-1} 1_{A_{\iota\lambda}} : (\iota, \lambda) \in I \times L_\iota\}$.

(b)$\Rightarrow$(a) is obvious. $\square$

The following propositions show how the embedding properties of E in $\mathscr{M}(Y)$ improve under additional completeness requirements on E.

Proposition 2.8. *If* $(Y, \hat{u}, v)$ *is an e-mr of* E, *then we have*

(a) vE *is an ideal of* $\mathscr{M}(Y)$ $\iff$ E *is Dedekind complete.*

(b) v *is surjective (and hence a Riesz isomorphism)* $\iff$ *there is an hc-pair* (e, R) *of* E *such that* R *is associated to* $(Y, \hat{u}, v)$.

(c) *If* v *is surjective, then* E *is hypercomplete and* $\hat{u}(E^\pi) = C_i(Y)$.

Proof. (a). "$\Rightarrow$" follows from 2.1(γ) and the Dedekind completeness of $\mathscr{M}(Y)$, while "$\Leftarrow$" follows from [AB; 2.2].

(b). "$\Rightarrow$" By 2.3 there exists $R \subset E^\pi$ such that R is associated to $(Y, \hat{u}, v)$. Then $(\hat{u}e, \{\hat{u}g : g \in R\})$ is an hc-pair of $\mathscr{M}(Y)$, and since v is a Riesz isomorphism, it follows from 2.1(δ) that (e, R) is an hc-pair of E.

"$\Leftarrow$" Let $\mu \in \mathscr{M}(Y)_+$ and put $F := \{z \in E : 0 \leqslant vz \leqslant \mu\}$. For each $g \in R$ we have $\sup_{z \in F} g(z) = \sup_{z \in F} \int \hat{u}g \, \mathrm{d}(vz) \leqslant \int \hat{u}g \, \mathrm{d}\mu < \infty$, and hence there exists $x := \sup F$. We get $vx = \mu$.

(c) follows from (b) and [C; 1.6.1]. $\square$

Proposition 2.9. *If* $(Y, \hat{u}, v)$ *is an e-mr of* E, *then we have*

(a) $v(E[e])$ *is an order dense Riesz subspace of* $\mathscr{M}_b(Y)$.

(b) $v(E[e])$ *is an ideal of* $\mathscr{M}_b(Y)$ $\iff$ $E[e]$ *is Dedekind complete.*

(c) $v(E[e]) = \mathscr{M}_b(Y)$ $\iff$ $(e, \{e\})$ *is an hc-pair of* $E[e]$.

Proof. (a). For each $x \in E[e]$ we have $1_Y = \hat{u}e \in \mathscr{L}^1(vx)$ (2.1(δ)), and so $vx \in \mathscr{M}_b(Y)$, which implies that $v(E[e]) \subset \mathscr{M}_b(Y)$. The remaining statements are obvious.

(b). "$\Rightarrow$" follows from 2.1(γ), while "$\Leftarrow$" follows from 2.1(γ), (a) and [AB; 2.2].

(c). "$\Rightarrow$" follows from 2.1(δ).

"$\Leftarrow$" Let $\mu \in \mathscr{M}_b(Y)_+$ and set $F := \{x \in E[e]_+ : vx \leqslant \mu\}$. (a) implies that $\mu = \sup vF$. We have $\sup_{x \in F} e(x) = \sup_{x \in F} \int 1_Y \mathrm{d}(vx) = \int 1_Y \mathrm{d}\mu < \infty$, and hence there exists z, the supremum of F in $E[e]$. Thus $z = \sup F$ (in E), and we conclude by 2.1(δ) that $\mu = vz \in v(E[e])$. $\square$

The proof that E possesses an e-mr relies essentially on the Ogasawara-Maeda Theorem which we need in the following version.

Theorem 2.10. *There exist a compact hyperstonian space Z and a Riesz isomorphism $u\colon E^{\varrho} \to C_\infty(Z)$ with $ue = 1_Z$. If Z' is another compact Stonian space such that E^{ϱ} and $C_\infty(Z')$ are Riesz isomorphic, then Z and Z' are canonically homeomorphic.*

Proof. All but the fact that Z is hyperstonian follows from [B; Thm. 6 and Rem. (4), (7)]. Now $E^{\varrho\varrho}$ separates E^{ϱ} by [LM; 2.2], and thus Z is hyperstonian by [Fil; Thm. 3]. $\square$

The next proposition, which is easy to prove, serves as well as the following ones for the proof of Theorem 2.16.

Proposition 2.11. *Take Z, u as in 2.10, take $\xi \in (E^{\varrho})_+$, let T be a set of components of e, and put $U := \bigcup_{\tau \in T} \operatorname{supp} u\tau$. Then the following hold.*

(a) *ξ is a component of e $\iff$ $u\xi$ is the characteristic function of some open-compact subset of Z.*
(b) *U is open; we have $\overline{U} = Z \iff e = \sup T$.*

Proposition 2.12. *There exists a subset R of R_e of pairwise disjoint elements such that $e = \sup R$.*

Proof. Take Z, u as in 2.10 and set $\Phi := \{\mathfrak{U}\colon \mathfrak{U}$ is a disjoint family of open-compact subsets of Z such that $u^{-1}1_U \in E^{\pi}$ for every $U \in \mathfrak{U}\}$. By Zorn's Lemma, Φ has a maximal element $\mathfrak{W}$ (with respect to inclusion). Since $u(E^{\pi})$ is an order dense ideal of $C_\infty(Z)$ [LM; 3.1] and since Z is Stonian, we conclude that $\overline{\bigcup_{W \in \mathfrak{W}} W} = Z$. By 2.11, $R := \{u^{-1}1_W : W \in \mathfrak{W}\}$ meets the requirements. $\square$

Proposition 2.13. *Take $R \subset R_e$ satisfying $e = \sup R$, take Z, u as in 2.10, and put $Y := \bigcup_{g \in R} \operatorname{supp} ug$. Then we have*

(a) *Y is a locally compact hyperstonian space with $Z = \beta Y$.*
(b) *$\hat{u}\colon E^{\varrho} \to C_\infty(Y)$, $\xi \mapsto u\xi|_Y$ is a Riesz isomorphism such that $C_b(Y) \subset \hat{u}(E[e]^{\pi})$.*
(c) *$U_\xi = \{\hat{u}\xi = 1\}$ is open-closed in Y for all components ξ of e, U_g is open-compact in Y for all $g \in R$, and $Y = \bigcup_{g \in R} U_g$.*

P r o o f. Y is a dense open subset of Z, by 2.11. Hence (a) is well-known, (b) is a consequence of (a), and (c) follows from 2.11. □

Example 2.14. We have in general $C_b(Y) \neq \hat{u}(E[e]^\pi)$:

Take a locally compact hyperstonian space X and set $E := \mathscr{M}_c(X)$. By [C; 1.6.1], $E^\pi = C(X)$, and hence $E^\varrho = C_\infty(X)$. Put $e := 1_X$ and $R := \{1_A : A \subset X,\ A \text{ open-compact}\}$. Then $Z = \beta X$, and for every $f \in C_\infty(X)$, uf is the continuous extension of f to βX. We have $Y = X$ and therefore $\hat{u}$ =id. Then $E[e]^\pi = E^\pi = C(Y)$, but $C(Y) \neq C_b(Y)$ in general (e.g. $\mathbf{N}$ with the discrete topology).

Proposition 2.15. *If $(Y, \hat{u}, v)$ is an e-mr of E, then $\hat{u}^{-1}(C_c(Y)) \subset E^\pi$.*

P r o o f. Let $\xi \in \hat{u}^{-1}(C_c(Y))_+$. Set $f := \hat{u}\xi$, $U := \operatorname{supp} f$ and $\alpha := \sup f(Y)$. Then $\hat{u}^{-1}1_U \in E^\pi$ by 2.2, and from $0 \leqslant \xi \leqslant \alpha\hat{u}^{-1}1_U$ we conclude that $\xi \in E^\pi$. □

We are now in a position to prove our main result, i.e. the existence of an e-mr of E. The following theorem says even more: One can find a representation space Y which is (via $\hat{u}$) the union of the supports of a given set of components of e, a useful property for compactness arguments. Moreover, in general there exist many different e-mr's (corresponding to different R's) so that in concrete cases the user can choose the best one for his purposes.

Theorem 2.16. *Take $R \subset R_e$ with $e = \sup R$ (such a set R always exists, by 2.12). Then there exists an e-mr $(Y, \hat{u}, v)$ of E to which R is associated.*

In particular, if Z is a compact hyperstonian space and $u: E^\varrho \to C_\infty(Z)$ is a Riesz isomorphism with $ue = 1_Z$, one can choose

$$Y := \bigcup_{g \in R} \operatorname{supp} ug \qquad \text{and} \qquad \hat{u}: E^\varrho \to C_\infty(Y), \quad \xi \mapsto u\xi|_Y.$$

It follows that Y can be chosen compact if e belongs to E^π.

P r o o f. We take Z, u, Y, $\hat{u}$ as indicated in the assertion of the theorem (see 2.10). By 2.13, conditions (α) and (β) of 2.1 are satisfied.

Let $x \in E$. Since any compact subset of Y is covered by finitely many U_g $(g \in R)$, we see that $\hat{u}^{-1}f \in E^\pi$ for all $f \in C_c(Y)$. Thus

$$\ell_x : C_c(Y) \to \mathbf{R}, \quad f \mapsto (\hat{u}^{-1}f)(x)$$

is well-defined and hence an element of $C_c(Y)^\pi$. We conclude the existence of a unique $\mu_x \in \mathscr{M}(Y)$ such that $\ell_x(f) = \int f \, d\mu_x$ for all $f \in C_c(Y)$. We set

$$v: E \to \mathscr{M}(Y), \quad x \mapsto \mu_x.$$

Obviously v is linear.

To show that v is a Riesz homomorphism, let $x, z \in E_+$ with $\inf(x, z) = 0$. Take an open-compact subset A of Y. Since E^π is Dedekind complete and $\inf(px, pz) = p(\inf(x, z)) = 0$ (where $p\colon E \to E^{\pi\pi}$ denotes the canonical embedding), and since $\hat{u}^{-1}1_A \in E^\pi$ by 2.2, we can find $f_1, f_2 \in C_\infty(Y)$ satisfying

$$\inf(f_1, f_2) = 0, \qquad f_1 + f_2 = 1_A, \qquad \int f_1 \mathrm{d}\mu_x = 0 = \int f_2 \mathrm{d}\mu_z$$

[Z; 90.6]. Hence f_1, f_2 must be characteristic functions of open-compact sets. Since A is arbitrary, we conclude that $\operatorname{supp}\mu_x \cap \operatorname{supp}\mu_z = \emptyset$, and thus that $\inf(\mu_x, \mu_z) = 0$.

Let $x \in E$ with $vx = 0$. Since $\operatorname{supp} v(x^+) \cap \operatorname{supp} v(x^-) = \emptyset$, it is easily checked that $x^+ = 0$ and $x^- = 0$. Hence v is injective.

To show that 2.1(δ) is satisfied, take $(x, \xi) \in E \times E^\varrho$. We may assume that $x \geqslant 0$ and $\xi \geqslant 0$. Put $\mathscr{H} := \{\eta \in \hat{u}^{-1}(C_c(Y))\colon 0 \leqslant \eta \leqslant \xi\}$. Then $\xi = \sup \mathscr{H}$, $\hat{u}\xi = \sup \hat{u}(\mathscr{H})$, and the claimed implications follow from the equation

$$\xi(x) = (\sup \mathscr{H})(x) = \sup_{\eta \in \mathscr{H}} \eta(x) = \sup_{\eta \in \mathscr{H}} \int \hat{u}\eta \, \mathrm{d}(vx) = \int \hat{u}\xi \, \mathrm{d}(vx),$$

which holds provided that $x \in E[\xi]$ or $\hat{u}\xi \in \mathscr{L}^1(vx)$.

It remains to show that vE is order dense in $\mathscr{M}(Y)$. Applying the property just proved, we easily see that $\bigcup_{x \in E} \operatorname{supp} vx$ is dense in Y. Hence if $\mu \in \mathscr{M}(Y)$, $\mu > 0$ is given, there exists $x \in E_+$ such that $\nu := \inf(vx, \mu) > 0$. Since $0 < \nu \leqslant vx$, the map

$$\ell\colon E^\pi \to \mathbf{R}, \quad \xi \mapsto \int \hat{u}\xi \, \mathrm{d}\nu$$

is well-defined. We have $\ell \in E^{\pi\pi}$ and $0 < \ell \leqslant px$ (where $p\colon E \to E^{\pi\pi}$ denotes again the canonical embedding). By [AB; 3.11], there exists a $z \in E$ with $0 < pz \leqslant \ell$. It follows that $0 < vz \leqslant \nu \leqslant \mu$. □

Remark. It is always possible to find an e-mr $(Y, \hat{u}, v)$ of E such that Y is paracompact, as 2.12, 2.16 and 2.7 show. But if E is hypercomplete, it can happen that for no hc-pair (e, R) the representation space Y, constructed relative to R, is paracompact, as [Fi3; 2.7] together with 2.7 and 2.6 shows.

We want to formulate an easy consequence of what we have proved in order to clarify our result; in particular 2.17(c) shows the significance of the hypercomplete spaces: They can always be considered as bands of spaces of measures.

Corollary 2.17. *For a Riesz space F we have*

(a) *F is a π-space $\iff$ there exist a δ-ring $\mathfrak{R}$ and a band $\mathscr{M}$ of $\mathscr{M}(\mathfrak{R})$ such that F is an order dense Riesz subspace of $\mathscr{M}$.*

(b) *F is a Dedekind complete π-space $\iff$ there exist a δ-ring $\mathfrak{R}$ and a band $\mathscr{M}$ of $\mathscr{M}(\mathfrak{R})$ such that F is an order dense ideal of $\mathscr{M}$.*

(c) *F is hypercomplete $\iff$ there exist a δ-ring $\mathfrak{R}$ and a band $\mathscr{M}$ of $\mathscr{M}(\mathfrak{R})$ such that $F = \mathscr{M}$.*

Proof. (a). "$\Rightarrow$" Apply 2.16. "$\Leftarrow$" Note that $\mathscr{M}$ is a π-space.
(b). "$\Rightarrow$" Use 2.16 and 2.8(a). "$\Leftarrow$" follows from [AB; 2.2].
(c). "$\Rightarrow$" Use 2.16 and 2.8(b). "$\Leftarrow$" follows from [Fi3; 2.5]. □

Theorem 2.16 implies the following well-known result [Sc2; 9.2].

Corollary 2.18. *If E is an AL-space, then there is a compact hyperstonian space Y such that E is Riesz isomorphic and isometric to $\mathscr{M}(Y)$.*

Proof. By [Fi3; 2.12] the map $e\colon E \to \mathbf{R}$, $x \mapsto \|x^+\| - \|x^-\|$ is a weak unit of E^π, and $(e, \{e\})$ is an hc-pair of E. Applying 2.16 (with $R := \{e\}$) and 2.8(b) we obtain the first part of the assertion, and in addition we have

$$\|x\| = e(|x|) = \int 1_Y \mathrm{d}|vx| = \|vx\| \qquad \text{for all } x \in E.$$

□

The next problem to be tackled is that of uniqueness of the measure representation. For this purpose we prove

Proposition 2.19. *Suppose that for $k = 1, 2$, $(Y_k, \hat{u}_k, v_k)$ is an e-mr of E such that $R_k \subset E^\pi$ is associated to $(Y_k, \hat{u}_k, v_k)$. If $R_1 \subset R_2$, then there exists a map φ: $Y_1 \to Y_2$ with the following properties:*

(i) *φ is injective, continuous, open, and φY_1 is dense in Y_2;*
(ii) *$\varphi(\operatorname{supp} \hat{u}_1 g) = \operatorname{supp} \hat{u}_2 g$ for all $g \in R_1$;*
(iii) *$\hat{u}_1 \xi = \hat{u}_2 \xi \circ \varphi$ for all $\xi \in E^\varrho$;*
(iv) *φ is $v_1 x$-proper and $\varphi(v_1 x) = v_2 x$ for all $x \in E$.*

Moreover, φ is uniquely determined by (iii).

Proof. The map $u := \hat{u}_2 \circ \hat{u}_1^{-1}$ is a Riesz isomorphism with $u 1_{Y_1} = 1_{Y_2}$. Since $C_b(Y_k)$ is the ideal of $C_\infty(Y_k)$ generated by 1_{Y_k}, we get $u(C_b(Y_1)) = C_b(Y_2)$.

For $f \in C_\infty(Y_k)$, we denote by f^* the continuous extension of f to $Z_k := \beta Y_k$. Then

$$w\colon C(Z_1) \to C(Z_2), \quad f \mapsto \left(u(f|_{Y_1})\right)^*$$

is a norm preserving Riesz isomorphism. By [Se; 7.8.4] there exists a homeomorphism $\psi\colon Z_1 \to Z_2$ satisfying $wf = f \circ \psi^{-1}$ for all $f \in C(Z_1)$.

Take $g \in R_1$ and set $A_k := \operatorname{supp} \hat{u}_k g$. For each $\mu \in \mathscr{M}(Z_2)$ we obtain

$$\int 1^{Z_2}_{\psi A_1} \mathrm{d}\mu = \int 1^{Z_1}_{A_1} \circ \psi^{-1} \mathrm{d}\mu = \int w 1^{Z_1}_{A_1} \mathrm{d}\mu = \int ((u \circ \hat{u}_1)(g))^* \mathrm{d}\mu$$
$$= \int (\hat{u}_2 g)^* \mathrm{d}\mu = \int 1^{Z_2}_{A_2} \mathrm{d}\mu,$$

and hence that $\psi A_1 = A_2$. Since $R_1 \subset R_2$, it follows that $\psi Y_1 \subset Y_2$. This enables us to define

$$\varphi \colon Y_1 \to Y_2, \quad y \mapsto \psi y.$$

Then φ satisfies (i) and (ii). Since $wf = f \circ \psi^{-1}$ for all $f \in C(Z_1)$, we have $h = uh \circ \varphi$ for every $h \in C_b(Y_1)$. Thus for each $\xi \in (E^\varrho)_+$

$$\hat{u}_1 \xi = \sup_{n \in \mathbf{N}} \inf \left(\hat{u}_1 \xi, n 1_{Y_1}\right) = \sup_{n \in \mathbf{N}} \hat{u}_1(\inf(\xi, ne))$$
$$= \sup_{n \in \mathbf{N}} \left(u \circ \hat{u}_1(\inf(\xi, ne))\right) \circ \varphi = \sup_{n \in \mathbf{N}} \inf \left(\hat{u}_2 \xi, n 1_{Y_2}\right) \circ \varphi = \hat{u}_2 \xi \circ \varphi,$$

which implies (iii).

Now let $x \in E_+$. For each $g \in R_2$ we have $\hat{u}_2 g \circ \varphi = \hat{u}_1 g \in \mathscr{L}^1(v_1 x)$ by 2.1(δ), i.e. $\varphi^{-1}(\operatorname{supp} \hat{u}_2 g)$ is $v_1 x$-integrable. But each compact $K \subset Y_2$ is covered by finitely many sets $\operatorname{supp} \hat{u}_2 g$, with $g \in R_2$, and hence $\varphi^{-1} K$ is $v_1 x$-integrable. Thus φ is $v_1 x$-proper. For each $f \in C_c(Y_2)$ we have

$$\int f \, \mathrm{d}\varphi(v_1 x) = \int f \circ \varphi \, \mathrm{d}(v_1 x) = \int u^{-1} f \, \mathrm{d}(v_1 x)$$
$$= \left((\hat{u}_1^{-1} \circ u^{-1})(f)\right)(x) = (\hat{u}_2^{-1} f)(x) = \int f \, \mathrm{d}(v_2 x),$$

and hence $\varphi(v_1 x) = v_2 x$, which proves (iv).

The uniqueness of φ is easy to see. □

Corollary 2.20. *Suppose that for $k = 1, 2$, $(Y_k, \hat{u}_k, v_k)$ is an e-mr of E such that $R \subset E^\pi$ is associated to $(Y_k, \hat{u}_k, v_k)$. Then Y_1 and Y_2 are canonically homeomorphic.*

Proof. Take φ as in the preceding proposition. Then by 2.19(ii), $\varphi Y_1 = Y_2$. □

Remarks. (i) It follows that if $R \subset R_e$ satisfies $e = \sup R$, then the e-mr of E to which R is associated is uniquely determined.

(ii) If X is a set, $\mathfrak{R}$ is a δ-ring of subsets of X and $\mathscr{M}$ is a band of $\mathscr{M}(\mathfrak{R})$, then by Theorem 2.16 and (i) we get the existence and uniqueness of a representation and of a bounded representation of $(X, \mathfrak{R}, \mathscr{M})$ [C; §2]. Our notation $(Y, \hat{u}, v)$ for an e-mr has been chosen in coincidence with the notations in [C].

Of course the e-mr's belonging to different R's are different in general. But we can single out a "greatest element" among them. For this purpose we define a preorder $\preccurlyeq$ on

$$\Phi(E) := \{(Y, \hat{u}, v) \colon (Y, \hat{u}, v) \text{ is an } e\text{-mr of } E\}$$

by setting $(Y_1, \hat{u}_1, v_1) \preccurlyeq (Y_2, \hat{u}_2, v_2)$ iff there exists a mapping $\varphi\colon Y_1 \to Y_2$ satisfying

(i) φ is injective, continuous and open;

(ii) $\hat{u}_1\xi = \hat{u}_2\xi \circ \varphi$ for all $\xi \in E^{e}$;

(iii) φ is v_1x-proper and $\varphi(v_1x) = v_2x$ for all $x \in E$.

If such φ exists, then by (ii) it is uniquely determined, and φY_1 is dense in Y_2.

Proposition 2.21. *Let $(Y_k, \hat{u}_k, v_k) \in \Phi(E)$ $(k = 1, 2)$ such that $(Y_1, \hat{u}_1, v_1) \preccurlyeq (Y_2, \hat{u}_2, v_2) \preccurlyeq (Y_1, \hat{u}_1, v_1)$. Then Y_1 and Y_2 are canonically homeomorphic.*

Proof. If $\varphi\colon Y_1 \to Y_2$ and $\psi\colon Y_2 \to Y_1$ are the corresponding mappings, then from (ii) one easily deduces that $\psi \circ \varphi = \mathrm{id}$, and from this relation that ψ is surjective, hence a homeomorphism. □

Proposition 2.22. *Take $(Y_0, \hat{u}_0, v_0) \in \Phi(E)$ such that R_e is associated to it (2.16). Then $(Y, \hat{u}, v) \preccurlyeq (Y_0, \hat{u}_0, v_0)$ for each $(Y, \hat{u}, v) \in \Phi(E)$.*

Proof. Everything follows from 2.3 and 2.19. □

Hence by Remark (i) above we can talk of "the greatest element of $\Phi(E)$". The next two propositions give further informations about this element.

Proposition 2.23. *Take Z, u as in 2.16 and set*

$$\mathfrak{K}_x := \{K \subset Z \colon K \text{ is open-compact, } x \in E[u^{-1}1_K]\},$$
$$K_x := \bigcup_{K \in \mathfrak{K}_x} K \quad \text{for all } x \in E_+.$$

Then

$$\bigcup_{g \in R_e} \operatorname{supp} ug = \operatorname{int}\Bigl(\bigcap_{x \in E_+} K_x\Bigr).$$

Proof. "$\subset$" Let $g \in R_e$. For all $x \in E_+$ we have $x \in E[g]$, hence $\operatorname{supp} ug \in \mathfrak{K}_x$. It follows that $\operatorname{supp} ug \subset \bigcap_{x \in E_+} K_x$, and since $\operatorname{supp} ug$ is open, this implies the claim.

"$\supset$" Let K be an open-compact subset of $\operatorname{int}\bigl(\bigcap_{x \in E_+} K_x\bigr)$ and put $g := u^{-1}1_K$. For each $x \in E_+$ we can find $K_1, \dots, K_n \in \mathfrak{K}_x$ with $K \subset \bigcup_{i=1}^{n} K_i$; hence $x \in E[g]$. Thus $E = E[g]$ which implies $g \in E^{\pi}$ and therefore $g \in R_e$. □

Proposition 2.24. *For $(Y, \hat{u}, v) \in \Phi(E)$, the following are equivalent.*

(a) *R_e is associated to $(Y, \hat{u}, v)$.*

(b) *U_g is compact for all $g \in R_e$.*

Proof. (a)⇒(b) follows from the definition, while (b)⇒(a) is a consequence of 2.3. □

Our concluding results concern relations between measure representations of a Riesz space and its subspaces.

If F is an order dense Riesz subspace of E, then the map

$$E^{\varrho} \to F^{\varrho}, \quad \xi \mapsto \xi|_{E[\xi]\cap F}$$

is a Riesz isomorphism [LM; 2.6]. In the following two propositions, this map is denoted by ψ.

Proposition 2.25. *Let F be an order dense ideal of E, let f be a weak unit of F^{ϱ}, and let $(Y, \hat{u}, v)$ be an f-mr of F to which $R \subset F^{\pi}$ is associated. If $\psi^{-1}g \in E^{\pi}$ for every $g \in R$, then there exists a unique map $\bar{v}\colon E \to \mathscr{M}(Y)$ such that $\bar{v}|_F = v$ and $(Y, \hat{u} \circ \psi, \bar{v})$ is a $\psi^{-1}f$-mr of E; $\psi^{-1}R$ is associated to $(Y, \hat{u} \circ \psi, \bar{v})$.*

Proof. The existence is proved in complete analogy with Theorem 2.16, while the uniqueness is easy to see. □

Proposition 2.26. *Let F be an order dense Riesz subspace of E, and let $(Y, \hat{u}, v)$ be an e-mr of E. Then $(Y, \hat{u} \circ \psi^{-1}, v|_F)$ is an $e|_{E[e]\cap F}$-mr of F. If $R \subset E^{\pi}$, then R is associated to $(Y, \hat{u}, v)$ iff ψR is associated to $(Y, \hat{u} \circ \psi^{-1}, v|_F)$.*

Proposition 2.27. *Let F be a band of E, and let $(Y, \hat{u}, v)$ be an e-mr of E. We define $X := \overline{\bigcup_{x \in F} \operatorname{supp} vx}$ and identify $\mathscr{M}(X)$ with $\{\mu \in \mathscr{M}(Y)\colon \operatorname{supp}\mu \subset X\}$. For each $f \in C_\infty(X)$ put*

$$F_f := v^{-1}(\mathscr{M}(X)[f]) \qquad \text{and} \qquad \xi_f\colon F_f \to \mathbf{R}, \quad x \mapsto \int f \, \mathrm{d}(vx).$$

Then

(a) *$vF = vE \cap \mathscr{M}(X)$ is an order dense Riesz subspace of $\mathscr{M}(X)$.*

(b) *F_f is an order dense ideal of F and $v(F_f)$ is an order dense Riesz subspace of $\mathscr{M}(X)$, for every $f \in C_\infty(X)$.*

(c) *$\xi_f \in (F_f)^{\pi}$ for each $f \in C_\infty(X)$.*

(d) *The map $\varphi\colon C_\infty(X) \to F^{\varrho}$, $f \mapsto \xi_f$ is a Riesz isomorphism.*

(e) $\xi|_{E[\xi]\cap F} \in F^\varrho$ *and* $\hat{u}\xi|_X = \varphi^{-1}(\xi|_{E[\xi]\cap F})$ *for all* $\xi \in E^\varrho$*; in particular* $1_X = \varphi^{-1}(e|_{E[e]\cap F})$.

(f) $(X, \varphi^{-1}, v|_F)$ *is an* $e|_{E[e]\cap F}$*-mr of* F*, and if* $R \subset E^\pi$ *is associated to* $(Y, \hat{u}, v)$*, then* $\{\varphi(1_{U_g}|_X) : g \in R\}$ *is associated to* $(X, \varphi^{-1}, v|_F)$.

Proof. (a). "$\subset$" follows from the definition of X.

"$\supset$" Let $\mu \in vE \cap \mathscr{M}(X)$. There exists $x \in E$ with $vx = \mu$. For every $z \in F^\perp$ we have $\operatorname{supp} vz \cap X = \emptyset$ and hence $\inf(|z|, |x|) = 0$. Thus $x \in F^{\perp\perp} = F$. The order denseness is obvious.

(b) follows from (a).

(c). If $x_\iota \downarrow 0$ in F_f, then by (b) $vx_\iota \downarrow 0$ in $\mathscr{M}(X)$. This implies our claim.

(d). The only non-trivial assertion is that φ is surjective. So take $\xi \in (F^\varrho)_+$. Then $G := F[\xi] + F^\perp$ is an order dense ideal of E, and each $x \in G$ can be decomposed in a unique way as $x = x_1 + x_2$, with $x_1 \in F[\xi]$ and $x_2 \in F^\perp$. Put $\eta \colon G \to \mathbf{R}$, $x \mapsto \xi(x_1)$. Then $\eta \in G^\pi \subset E^\varrho$, and for $f := \hat{u}\eta|_X$ we obtain $\xi_f = \xi$.

(e) follows from (d), and (f) follows from (a), (d), (e). □

Appendix: Representations as spaces of integrable functions

In this appendix we show how two well-known representation theorems can easily be derived from our main results. Part (a) of the following theorem goes back to Luxemburg and Zaanen and was published by Fremlin (see [Fr1; Thm. 6] or [Fr2; p.184]). Part (b) was essentially proved by Vietsch [Vi; 5.9]; see also [Z; 120.11].

Theorem A. *Let E be a π-space.*

(a) *There exists a normal Radon measure μ on some locally compact, paracompact hyperstonian space Y such that E is Riesz isomorphic to an order dense Riesz subspace of*

$$L^1_{loc}(\mu) := \{f \in L^0(\mu) \colon f 1_K \in L^1(\mu) \text{ for every compact } K \subset Y\}.$$

Moreover, E is Dedekind complete iff E embeds as an ideal into $L^1_{loc}(\mu)$.

(b) *If E^π has a weak unit e (or equivalently, if E admits a strictly positive order continuous linear functional e), then E embeds into $L^1(\mu)$, with the same properties as in (a).*

Proof. Let e be a weak unit of E^ϱ. By 2.16, there exists an e-mr $(Y_1, \hat{u}_1, v_1)$ of E. Fix a maximal disjoint system $(x_\iota)_{\iota\in I}$ of E_+ with the property that each $v_1 x_\iota$ has compact support, and set $g_\iota := \hat{u}_1^{-1} 1_{\operatorname{supp} v_1 x_\iota}$ for each $\iota \in I$. Since $(g_\iota)_{\iota\in I}$ is a disjoint family from R_e with $e = \sup\{g_\iota : \iota \in I\}$, there exists (again by 2.16) an e-mr $(Y, \hat{u}, v)$ of E to which $\{g_\iota : \iota \in I\}$ is associated. By 2.7, Y is paracompact.

For each $\xi \in E^\varrho$ with $\inf(\xi, g_\iota) = 0$ we have $\int \hat{u}_1 \xi \, d(v_1 x_\iota) = 0$, hence $\xi(x_\iota) = 0$, and hence $\int \hat{u}\xi \, d(vx_\iota) = 0$. This implies that $\operatorname{supp} vx_\iota \subset U_{g_\iota}$ for each $\iota \in I$. From the maximality of $(x_\iota)_{\iota \in I}$ we conclude that $\operatorname{supp} vx_\iota = U_{g_\iota}$ for each $\iota \in I$. We put $\mu := \sum_{\iota \in I} vx_\iota$. Then μ belongs to $\mathscr{M}(Y)$, and in view of $\operatorname{supp} \mu = Y$ we see that $\mathscr{M}(Y)$ is the band generated by μ. By the Radon-Nikodym Theorem, $\mathscr{M}(Y)$ is Riesz isomorphic to $L^1_{loc}(\mu)$, which implies (together with 2.8(a)) assertion (a).

To prove (b), we need only observe that in this case $vE \subset \mathscr{M}_b(Y)$ (since $\int 1_Y d(vx) = \int \hat{u}e \, d(vx) = e(x) \in \mathbf{R}$ for each $x \in E$), and that $\mathscr{M}_b(Y)$ is Riesz isomorphic to $L^1(\mu)$. □

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