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T_2-FRAMES AND ALMOST COMPACT FRAMES

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In the theory of frames (or "pointless topologies"), several authors have tried to find a suitable form of separation axioms. Our purpose is to describe a T_2-axiom in the form usual in the case of regular frames. T_2-frames coincide for topological spaces with Hausdorff spaces but they are described independently on points. We also investigate almost compact frames and H-closed extensions of T_2-frames (see [3], 6.1., h, for spaces).

All unexplained facts concerning frames can be found in Johnstone [10] or in [16]. Recall that a frame is a complete lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge s \colon s \in S\}$$

holds for all $a \in L$, $S \subseteq L$.

The known facts (see [10] or [16]) indicate the importance of the opposite category *Loc* of locales to the category *Frm* of frames. We will work in the category *Frm*. If *Top* is the category of topological spaces, then the set $O(T)$ of all open sets of $T \in$ *Top* is a frame. Frames isomorphic to some $O(T)$ are called spatial (or a topology).

In [16], frames in which primes (i.e. $\wedge$-irreducible and $\neq 1$ elements) are dual atoms (called T_1-frames), are investigated. The category of all T_1-frames is the smallest epireflective subcategory in *Frm* containing all T_1-spaces. Every T_1-frame is a homomorphic image of a T_1-space and every spatial T_1-frame is a T_1-space. We will now investigate similar problems for T_2-frames.

The category of T_2-frames is an epireflective subcategory in *Frm* containing all Hausdorff spaces but there exists a T_2-frame which is not a homomorphic image of a Hausdorff topology. Spatial T_2-frames are Hausdorff spaces. There exists a compact T_2-frame which is not regular. An almost compact frame which is a homomorphic image of a Hausdorff topology is a topology. A compact T_2-frame exists which is not a homomorphic image of a Hausdorff topology. An H-closed extension of a T_2-frame is an almost compact T_2-frame. H-closed extension of a frame L is a topology iff L is a topology. H-closed extension of a complete Boolean algebra, which is not dually atomic, is dually atomic and not conjunctive.

1. T_2-FRAMES

The main separation axiom for frames is regularity which is defined in the following way:

A frame L is *regular* if

$$a = \bigvee(b \in L : b \lhd a)$$

holds for all $a \in L$, where $b \lhd a$ means $b^* \vee a = 1$.

Dowker and Strauss [5], Simmons [17] (and Kerstan [12], as well) proposed to define N_2-frames as frames L satisfying the condition (N_2):

For any $a, b \in L$ with $a \vee b = 1$, $a \neq 1 \neq b$ there are elements $x, y \in L$ such that $x \wedge y = 0$, $x \nleq a$, $y \nleq b$.

Spatial N_2-frames coincide with Hausdorff spaces and every regular frame is N_2. However, it does not seem that these N_2-frames form an epireflective subcategory in Frm.

Recall that a frame L is conjunctive if for any $a, b \in L$ with $a \nleq b$ there is an element $c \in L$ such that $a \vee c = 1$, $b \vee c \neq 1$ (see Simmons [17]). If we combine the (N_2)-condition and the conjunctivity then we obtain the following condition (S_2):

For any $a, b \in L$ with $1 \neq a \nleq b$ there is an element $c \in L$ such that $c^* \nleq a$, $c \nleq b$.

This approximation of the Hausdorff axiom on frames fulfilling the condition (S_2) was investigated by P. T. Johnstone and Sun Shu-Hao in [11]. These frames, which are called Hausdorff in [11], are N_2-frames and need not be conjunctive. Hausdorff frames form an epireflective subcategory in Frm (equivalently, closed under arbitrary homomorphic images and sums), spatial Hausdorff frames coincide with Hausdorff spaces.

The problem of Hausdorff frames was discussed also by Isbell [9] which introduced strongly Hausdorff frames. In a latter paper [6], Dowker and Strauss proposed a definition equivalent to Isbell's strong Hausdorffness.

The following equivalent of Hausdorff spaces for frames is in [16]. We say that an element $a \in L$, $a \neq 1$ of a frame L is semiprime if

$$x \wedge y = 0 \Rightarrow x \leqslant a \quad \text{or} \quad y \leqslant a,$$

for any $x, y \in L$.

If we denote $D(L)$, $P(L)$ resp., $S(L)$ resp., the set of all dual atoms, points resp., semiprime elements resp., in L then $D(L) \subseteq P(L) \subseteq S(L)$ and $a \leqslant b$, $b \neq 1$, $a \in S(L) \Rightarrow b \in S(L)$. We say that L is an S-frame if $S(L) = D(L)$. It is equivalent to the fact that semiprime elements form an antichain. The category $\mathcal{S}$ of all S-frames is a monocoreflective subcategory in Frm such that $\mathcal{S} \cap Sob = Haus$, where Sob ($Haus$, resp.) is the category of all sober (Hausdorff, resp.) spaces. Every N_2-frame is an S-frame.

A. Pultr in [15] defines Hausdorffness on a frame L with the following condition: If $a, b \in L$, $a \| b$ then elements $k, l \in L$ exist such that $k \wedge l = 0$, $k, l \leqslant a \vee b$, $k \not\leqslant a$, $1 \not\leqslant b$.

We will describe our candidate for a T_2-axiom on frames.

Definition. Let L be a frame and let $a \,\Box\, b$ denote $a \leqslant b$ and $a^* \not\leqslant b$ for $a, b \in L$. Then L is called a *T_2-frame* if

$$a = \bigvee(x \in L : x \,\Box\, a)$$

holds for any $1 \neq a \in L$.

L is a T_2-frame iff for any $a, b \in L$ with $1 \neq a \not\leqslant b$ there is $l \in L$ such that $l \leqslant a$, $l^* \not\leqslant a$, $l \not\leqslant b$. It is evident that T_2-frames form a subcategory in Frm which we will denote by Frm_2. T_2-frames are exactly Hausdorff frames introduced by P. T. Johnstone and Sun Shu-Hao in [11].

If L is a frame then we will denote $\Box a = \{x \in L : x \,\Box\, a\}$ for $a \in L$ and $S_L = \{l \in L : l^* \neq 0\}$.

Proposition 1.1. *If L is a frame, $a, b, c, d, a_i, b_i \in L$ $(i \in I)$ then it holds:*

a) $\Box a \subseteq \downarrow a$, $b \in \Box a \Rightarrow \downarrow b \subseteq \Box a$; $\Box a = \downarrow a \Leftrightarrow a^* \neq 0$

b) $a \leqslant b$, $b \,\Box\, c \Rightarrow a \,\Box\, c$; $a \,\Box\, b$, $b \,\Box\, c \Rightarrow a \,\Box\, c$

c) $a_i \,\Box\, b_i$ $(i \in I) \Rightarrow \bigwedge(a_i : i \in I) \,\Box \bigwedge(b_i : i \in I)$

d) $1 \neq a \Rightarrow 0 \,\Box\, a$; 1 non $\Box a$

e) $a \leqslant b$, $a^{**} \,\Box\, b^{**} \Rightarrow a \,\Box\, b$

f) $a \,\Box\, b \Rightarrow a^* \neq 0 \Rightarrow a \,\Box\, a$, $a \,\Box\, a^{**}$

Definition. We say that a frame L is *T_2'-frame* if for any $a \in L$, $a \neq 1$ there is an ideal A in L such that $a = \bigvee A$ and $x \,\Box\, a$ holds for any $x \in A$.

It is clear that any T_2'-frame is a T_2-frame. T_2'-frames form a subcategory in Frm which we will denote by Frm_2'.

Proposition 1.2. *A T_2-frame is an S-frame and also a T_1-frame.*

Proof. If $a \in S(L)$ and $b \in L$ exists such that $1 > b > a$ then $b \nleq a$, $b \neq 1$ and therefore $l \in L$ exists such that $l \leqslant b$, $l \nleq a$, $l^* \nleq b$. These facts imply $l^* \leqslant a \leqslant b$, a contradiction. We have $S(L) = D(L)$ and $P(L) = D(L)$ evidently. $\square$

Proposition 1.3. *For each T_0-space T the following are equivalent:*

1. *T is a Hausdorff topological space.*
2. *$O(T)$ is a T_2-frame.*
3. *$O(T)$ is a T_2'-frame.*

Proof. $1 \Rightarrow 3$: If $1 \neq a \in O(T)$ then there exists $\gamma \in T \setminus a$ and for any $\alpha \in a$ there exist $u, v \in O(T)$ such that $u \wedge v = 0$, $\alpha \in u$, $\gamma \in v$. We have $l_\alpha = u \wedge a \leqslant a$ and if $l_\alpha^* \leqslant a$ then $v \leqslant u^* \leqslant l_\alpha^* \leqslant a$, which is a contradiction with $\gamma \notin a$. The ideal A generated by the set $\{l_\alpha : \alpha \in a\}$ has the properties $a = \bigvee A$ and $x \,\square\, a$ for any $x \in A$. $O(T)$ is a T_2'-frame.

$3 \Rightarrow 2$ is clear.

$2 \Rightarrow 1$: The implication follows from 1.2 and [11], Cor. 2.4. $\square$

Remark. If L is a T_2-frame then $d = \bigvee(x \in L : x \lhd d)$ for any $d \in D(L)$.

Definition. Let L be a frame such that for any $l \in L$, $l \neq 1$ there exists $d \in D(L)$, $d \geqslant l$. Then L is called ***dually atomic***.

Proposition 1.4. *Let L be a dually atomic frame. Then the following are equivalent:*

1. *L is a T_2-frame.*
2. *L is a T_2'-frame.*
3. *$d = \bigvee(x \in L : x \lhd d)$ for any $d \in D(L)$.*
4. *L is a Hausdorff frame.*

Proof. $1 \Rightarrow 3$: $d = \bigvee(x \in L : x \lhd d)$ and $x \,\square\, d$ implies $x \lhd d$.

$3 \Rightarrow 2$: If $1 \neq l \in L$, $l \leqslant d \in D(L)$ then $l = l \wedge d = l \wedge \bigvee(x \in L : x \lhd d) = \bigvee(l \wedge x : x \lhd d)$. Further, $x^* \vee d = 1$ implies $(x \wedge l)^* \vee d = 1$, $(x \wedge l)^* \nleq d$, $(x \wedge l)^* \nleq l$. We have $l = \bigvee A$, $A \subseteq \square l$, where A is the ideal generated by the set $\{l \wedge x : x \lhd d\}$.

$2 \Rightarrow 1$ and $1 \Rightarrow 4$ is clear.

$4 \Rightarrow 3$: If $d \nleq \bigvee(x \in L : x \lhd d)$ then $1 \neq d \nleq \bigvee(x \in L : x \lhd d)$ and thus $l \in L$ exists such that $l \nleq \bigvee(x \in L : x \lhd d)$, $l^* \nleq d$. Then $l \lhd d$ and $l \leqslant \bigvee(x \in L : x \lhd d)$ holds, a contradiction. $\square$

Proposition 1.5. *Any regular frame is a conjunctive T_2'-frame.*

Proof is evident.

Theorem 1.6. *The categories Frm_2 and Frm'_2 are closed with respect to homomorphic images.*

Proof. Let $f\colon K \to L$ be a surjective homomorphism of frames and $K \in Frm_2$. We shall prove that $L \in Frm_2$: If $a, b \in L$, $1 \neq a \not\leqslant b$ then we consider elements $f^0(a) = \bigvee(z \in K\colon f(z) \leqslant a)$ and $f^0(b) = \bigvee(z \in K\colon f(z) \leqslant b)$. Since $1 \neq f^0(a) \not\leqslant f^0(b)$ and $x \in K$ exists such that $x \leqslant f^0(a)$, $x \not\leqslant f^0(b)$ and $x^* \not\leqslant f^0(a)$. It means that $y = f(x) \leqslant a$ and if $y^* \leqslant a$ then $f(x^*) \leqslant f(x)^* \leqslant a$, i.e., $x^* \leqslant f^0(a)$, a contradiction. Together $y \leqslant a$, $y \not\leqslant b$, $y^* \not\leqslant a$ holds and L is a T_2-frame. Moreover, if $K \in Frm'_2$ then $x_i \leqslant f^0(a)$, $x_i \not\leqslant f^0(b)$, $x_i^* \not\leqslant f^0(a)$ for $i = 1, 2$ and $(x_1 \vee x_2)^* \not\leqslant f^0(a)$ implies $y_1 \vee y_2 \leqslant a$, $y_1 \vee y_2 \not\leqslant b$ and $(y_1 \vee y_2)^* \not\leqslant a$ for $y_i = f(x_i)$, $i = 1, 2$. It means that $L \in Frm'_2$. $\square$

Now we shall investigate sums of a set $(L_\gamma\colon \gamma \in \Gamma)$ of frames and we shall prove that T_2-frames are closed under sums. Sums of frames were studied by C. H. Dowker and D. Strauss in [7], by P. T. Johnstone in [10] and by I. Kříž [13]. We shall use methods of C. H. Dowker and D. Strauss but results are similar to results of P. T. Johnstone.

Let $(L_\gamma\colon \gamma \in \Gamma)$ be a set of frames and write B for the set-theoretical product of the L_γ. Clearly, B is a frame and the projections $\pi_\gamma\colon B \to L_\gamma$ are frame homomorphisms. Let us define $L = \{x \in B\colon \pi_\gamma(x) = 1$ for all but finitely many $\gamma \in \Gamma\}$. Clearly, L is a meet semilatice. We define $Z = \downarrow L$ to be the frame of all lower sets of L. We denote $M = \{x \in L\colon (\exists\gamma \in \Gamma)(\pi_\gamma(x) = 0)\}$ and $Q = \{W \subseteq L\colon (\exists\gamma \in \Gamma)(\pi_\beta(x_1) = \pi_\beta(x_2)$ for any $x_1, x_2 \in W$ and any $\beta \in \Gamma\backslash\{\gamma\})\}$. We shall say that an element $U \in Z$ is Σ-coherent if

(i) $M \subseteq U$,
(ii) $W \in Q$, $W \subseteq U$ implies $W \subseteq U$.

Now, we have that L, M are Σ-coherent. Let us define a map $j\colon Z \to Z$ by the prescription $C \mapsto \bigwedge\{U \geqslant C\colon U$ is Σ-coherent$\}$. Clearly, the intersection of Σ-coherent elements is again Σ-coherent. Now, we have $C \leqslant j(C) = j(j(C))$ for all $C \in Z$ and $j(C \wedge D) \leqslant j(C) \wedge j(D)$ for all $C, D \in Z$ and write $E = \{b \in L\colon b \wedge d \in j(C \wedge D)$ for all $d \in D\}$. We shall show that E is Σ-coherent. Let $b \in M$. Then $b \wedge d \in M \subseteq j(C \wedge D)$ and we have $M \subseteq E$. If $W \subseteq E$, $W \in Q$, $d \in D$ then $w \wedge d \in j(C \wedge D)$, i.e., $\bigvee w \wedge d \in j(C \wedge D)$. Now, we have $\bigvee W \in E$. If we define $F = \{b \in L\colon c \wedge b \in j(C \wedge D)$ for all $C \in E\}$ then F is again Σ-coherent. Now, we have $j(C) \wedge j(D) \leqslant E \wedge F \leqslant j(C \wedge D)$. Hence j is a nucleus on Z and Z_j together with induced operations $\bigvee$ and $\wedge$ is a frame.

Proposition 1.7. *Let $L_\gamma \to K$, $\gamma \in \Gamma$ be a system of frame homomorphisms. Then there exists a unique frame homomorphism $f\colon Z_j \to K$ such that*

$$f(j(\downarrow x)) = \bigwedge_{\gamma\in\Gamma} f_\gamma(\pi_\gamma(x))$$

holds for all $x \in L$.

Proof. We shall show that following definition of f is correct:

$$f(j(U)) = \bigvee_{x\in U} f(j(\downarrow x)) \text{ for any } U \in Z.$$

Let $U, V \in Z, j(U) = j(V)$. We define $s(U) = \{x \in L\colon f(j(\downarrow x)) \leqslant f(j(U))\}$. Then $U \subseteq s(U)$, $s(U)$ is Σ-coherent. Namely, $M \subseteq s(U)$ from the fact that $x \in M$ implies $f(j(\downarrow x)) = 0$. It is easy to verify that $W \in Q$, $W \leqslant s(U)$ implies $\bigvee W \in s(U)$. Hence $V \leqslant j(V) = j(U) \leqslant s(U)$, i.e., $f(j(V)) \leqslant f(s(U)) \leqslant f(j(U))$. The symetric argument concludes the proof. □

For each $\gamma \in \Gamma$, we now define a map $i_\gamma\colon L_\gamma \to Z_j$ by the prescription $x \mapsto j(\downarrow \tau_\gamma(x))$, where $\tau_\gamma(x) \in L$, $\pi_\gamma\tau_\gamma(x) = x$, $\pi_\beta\tau_\gamma(x) = 1$ for any $\beta \in \Gamma \setminus \{\gamma\}$.

Theorem 1.8. *Z_j is the sum of frames L_γ with injections i_γ.*

Proof. Clearly, i_γ are frame homomorphisms. Let $f_\gamma\colon L_\gamma \to L$, $\gamma \in \Gamma$ be a set of frame homomorphisms. Then there exists a unique frame homomorphism $f\colon Z_j \to K$ such that $f \circ i_\gamma = f_\gamma$. □

Lemma 1.9. *Let $X \neq L$, $X \in Z_j$ hold. Then the following propositions hold:*

(i) *If $\{x_i\}_{i\in I}$ is a chain, $x_i \in X$ then $\bigvee_{i\in I} x_i \in X$.*

(ii) *If $N(X) = \{x \in X : y \neq x \Rightarrow x \not\leqslant y$ for any $y \in X\}$ then $N(X) \neq \emptyset$ and $X = \bigcup\{\downarrow x : x \in N(X)\}$*

(iii) *If $x, y \in N(X)$ then card $\{\gamma \in \Gamma : \pi_\gamma(x) \neq 1$ or $\pi_\gamma(y) \neq 1\} \geqslant 2$.*

(iv) *If $x \in N(X)$, $x \geqslant \tau_\gamma(0)$ for some $\gamma \in \Gamma$ then $\pi_\gamma(y) \geqslant \pi_\gamma(x)$ for all $y \in N(X)$.*

Proof. (i) Let us define an ordering on I such that $i < j$ iff $x_i < x_j$ for all $i, j \in I$. Clearly, I is a chain. Since $X \neq L$, $I \neq \emptyset$ then there exists an index $i \in I$ and an index $\gamma_0 \in \Gamma$ such that $\pi_{\gamma_0}(x_i) \neq 1$.Hence there exists a set $\Gamma_i = \{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ such that $\pi_\gamma(x_i) \neq 1$ iff $\gamma \in \Gamma_i$. Let us define chains $\{z_k^l\}_{k\geqslant l}$ for any $l \geqslant i$: $\pi_\gamma(z_k^l) = \pi_\gamma(x_i)$ for $\gamma \neq \gamma_l$ and $\pi_{\gamma_1}(z_k^l) = \pi_{\gamma_1}(x_k)$. Clearly, $z_k^l \leqslant x_k$, i.e., $z_k^l \in X$ for all $k \geqslant l \geqslant i$. Obviously, we have $\bigvee_{l\leqslant k} z_k^l \in X$ since $\{z_k^l\}_{k\geqslant l} \in Q$ for any $l \geqslant i$. Let us consider a chain $\{w_l\}_{l\geqslant i}$ defined by the prescription $w_l = \bigvee_{k\geqslant l} z_k^l$. Then $\pi_{\gamma_1}(w_{l_1}) = \pi_{\gamma_1}(w_{l_2})$ holds for all $l_1,\ l_2 \geqslant i$. By induction we can see that the proposition holds.

(ii) Clearly, $\bigcup(\downarrow x : x \in N(X)) \subseteq X$. Let $x \in X$. We define $\Gamma_x = \{\gamma_l, \ldots, \gamma_n\}$ to be the set of indices such that $\gamma \in \Gamma_x$ iff $\pi_\gamma(x) \neq 1$. Now, we put $x^0 = x$, $x^k = \bigvee\{y \geqslant x^{k-1} : y \in X, \pi_\beta(y) = \pi_\beta(x^{k-1})$ for $\beta \neq \gamma_k\}$ for $1 \leqslant k \leqslant n$. Clearly, $x^k \in X$ for any $1 \leqslant k \leqslant n$ and $x^n \in N(X)$. Namely, if there exists $z \in X$ such that $z > x^n$ then there exists $\gamma_i \in \Gamma_x$ such that $\pi_{\gamma_i}(z) > \pi_{\gamma_i}(x^n) \geqslant \pi_{\gamma_i}(k^{i-1})$ because $x^n \geqslant x^{n-1} \geqslant \ldots \geqslant x^1 \geqslant x^0$. Now, we can define an element $y \in x$ such that $\pi_\beta(y) = \pi_\beta(x^{i-1})$, $\pi_{\gamma_i}(y) = \pi_{\gamma_i}(z)$. Then $y \leqslant x^i$, i.e., $\pi_{\gamma_i}(z) = \pi_{\gamma_i}(y) \leqslant \pi_{\gamma_i}(x^i)$, a contradiction.

(iii) Let $x, y \in N(X)$, card $\{\gamma : \pi_\gamma(x) \neq 1$ or $\pi_\gamma(y) \neq 1\} = 1$. Then $x, y \in L$ and $\pi_\gamma(x) = \pi_\gamma(y) = 1$ for $\gamma \in \Gamma \setminus \{\gamma_0\}$, $\pi_{\gamma_0}(x) \neq 1$, $\pi_{\gamma_0}(y) \neq 1$ and $\pi_{\gamma_0}(x) \| \pi_{\gamma_0}(y)$. It implies $x \vee y \in X$, $x \vee y > x$, a contradiction.

(iv) Let define $z \in L$ by the prescription $\pi_\beta(z) = \pi_\beta(y)$ if $\beta \neq \gamma$ and $\pi_\beta(z) = \pi_\gamma(x)$ otherwise. Then $z \in X$, i.e., $z \vee y \in X$. Now, we have $z \vee y \leqslant y$, i.e., $\pi_\gamma(x) \leqslant \pi_\gamma(y)$. □

Now, we give an explicit description of the sets $S(Z_j), P(Z_j)$ and $D(Z_j)$.

Proposition 1.10. (i) *Let* $X \in S(Z_j)$. *Then* $x \in N(X)$ *implies* $x = \bigwedge_{i=1}^{n} \tau_{\gamma_i}(x_{\gamma_i})$, *where* $x_{\gamma_i} \in S(L_{\gamma_i})$.

(ii) $X \in P(Z_j) \Leftrightarrow X = \bigcup_{\gamma\in\Gamma} (\downarrow \tau_\gamma(x_\gamma) : x_\gamma \in P(L_\gamma))$.

(iii) $X \in D(Z_j) \Leftrightarrow X = \bigcup_{\gamma\in\Gamma} (\downarrow \tau_\gamma(x_\gamma : x_\gamma \in D(L_\gamma))$.

Proof. (i) Let $x \in N(X)$, $x_\gamma = \pi_\gamma(x) \neq 1$ for some $\gamma \in \Gamma$. If $x_\gamma \notin S(L_\gamma)$ then there exist elements $u_\gamma, v_\gamma \in L_\gamma$ such that $u_\gamma \wedge v_\gamma = 0$, $u_\gamma, v_\gamma \not\leqslant x_\gamma$. Let us define elements $u, v \in L$ by the prescription $\pi_\beta(u) = \pi_\beta(v) = \pi_\beta(x)$, for $\beta \neq \gamma$, $\pi_\gamma(u) = u_\gamma$, $\pi_\gamma(v) = v_\gamma$. Clearly, $u, v \in X$. Now, we have $j(\downarrow u) \wedge j(\downarrow v) = M$, $j(\downarrow u) \not\leqslant X$, $j(\downarrow v) \not\leqslant X$, a contradiction.

(ii) Let $X \in P(Z_j)$, $x \in N(X)$. As in (i) we can verify that $x_\gamma = \pi_\gamma(x) \neq 1$ implies $x_\gamma \in P(L_\gamma)$. Let $\gamma_1, \gamma_2 \in \Gamma$ such that $\pi_{\gamma_1}(x) \in P(L_{\gamma_1})$, $\pi_{\gamma_2}(x) \in P(L_{\gamma_2})$, $\gamma_1 \neq \gamma_2$. Then we can define elements $y, z \in L$ by the prescription $\pi_\beta(y) = \pi_\beta(x)$ for $\beta \neq \gamma_2$, $\pi_\beta(z) = \pi_\beta(x)$ for $\beta \neq \gamma_1$ and $\pi_{\gamma_2}(y) = \pi_{\gamma_1}(z) = 1$. Clearly, $y, z \in L$. Now, we have $j(\downarrow y) \wedge j(\downarrow z) \leqslant X$, $j(\downarrow y) \not\leqslant X$, $j(\downarrow z) \not\leqslant X$, a contradiction. From 1.9, (iv) the proposition follows.

Conversely, let $X = \bigcup_{\gamma\in\Gamma} (\downarrow \tau_\gamma(x_\gamma) : x_\gamma \in P(L_\gamma))$ and $I, J \in Z_j$, $I \wedge J \leqslant X$. Let $y \notin X$, $y \in I$. Then for any $\gamma \in \Gamma$ we have $y_\gamma = \pi_\gamma(y) \not\leqslant x_\gamma$. If $z \in J$ then

$j(\downarrow y) \wedge j(\downarrow z) \leqslant X$, i.e., there exists $\gamma_0 \in \Gamma$ such that $y_{\gamma_0} \wedge x_{\gamma_0} \leqslant x_{\gamma_0}$. Now, we have $z_{\gamma_0} \leqslant x_{\gamma_0}$, i.e., $z \leqslant \tau_{\gamma_0}(x_{\gamma_0})$. Clearly, $J \subseteq X$.

(iii) Let $x \in N(X)$, $X \in D(Z_j)$. If $x_\gamma = \pi_\gamma(x) \neq 1$, $x_\gamma \notin D(L_\gamma)$ for some $\gamma \in \Gamma$ then there exists an element $u_\gamma \in L_\gamma$ such that $u_\gamma \neq 1$, $u_\gamma > x_\gamma$. Let us define an element $u \in L$ such that $\pi_\beta(u) = \pi_\beta(x)$ for $\beta \neq \gamma$ and $\pi_\gamma(u) = u_\gamma$. Clearly, $X \leqslant X \vee j(\downarrow u)$, $X \neq X \vee j(\downarrow u)$, $X \vee j(\downarrow u) \neq L$, a contradiction.

Conversely, let $X = \bigcup\limits_{\gamma \in \Gamma} (\downarrow \tau_\gamma(x_\gamma) \colon x_\gamma \in D(L_\gamma))$ and let $y \notin X$. Then for any $\gamma \in \Gamma$ we have $y_\gamma = \pi_\gamma(y) \nleq x_\gamma$, i.e. $x_\gamma \vee y_\gamma = 1$. We define $\Gamma_y \subseteq \Gamma$ such that $\pi_\gamma(y) \neq 1 \Leftrightarrow \gamma \in \Gamma_y$. Clearly, Γ_y is finite, $\Gamma_y = \{\gamma_1, \dots, \gamma_n\}$. Then we have $j(\downarrow y) \vee \bigvee\limits_{i=1}^{n} j(\downarrow \tau_{\gamma_i}(x_i)) = 1$. Now, $I \nsupseteq X$ implies $I \vee X = L$. □

The preceding proposition has the following consequences:

Corollary 1.11. (i) *A sum of T_1-frames is a T_1-frame.*
(ii) *A sum of S-frames is an S-frame.*

Proof. (i) It follows immediately from 1.10, (ii) and (ii).
(ii) It follows from 1.10, (i) and (iii) and 1.9, (iv). □

Theorem 1.12. *A sum of T_2-frames is a T_2-frame.*

Proof. Let $X \in Z_j$, $X \neq L$. Then $X = \bigcup\{\downarrow x \colon x \in N(X))\}$. For any $x \in N(X)$ let us define $R(\downarrow x) = \prod\limits_{\gamma \in \Gamma} K_\gamma$, where $K_\gamma = L_\gamma$ if $\pi_\gamma(x) = 1$ and $K_\gamma = \Box\pi_\gamma(x)$ otherwise. Then $\bigvee R(\downarrow x) = \downarrow x$, $R(\downarrow x) \subseteq \Box \downarrow x$. Let us show that $R(\downarrow x) \subseteq \Box x$. Let $y \in R(\downarrow x)$, $y \notin \Box x$. Then there exists $\gamma_0 \in \Gamma$ such that $\pi_{\gamma_0}(y) \neq 1$ and we can define an element $l \in L$ such that $\pi_{\gamma_0}(l) = \pi_{\gamma_0}(y)^*$, $\pi_\gamma(l) = 1$ for all $\gamma \in \Gamma \setminus \{\gamma_0\}$. Since $y \notin \Box X$ then $l \in X$, i.e., there exists an element $z \in N(X)$ such that $z \geqslant l$. Now, we have $\pi_\gamma(z) = 1$ for all $\gamma \in \Gamma \setminus \{\gamma_0\}$, i.e., $\pi_{\gamma_0}(x) \| \pi_{\gamma_0}(z)$, because $\pi_{\gamma_0}(l) \nleq \pi_{\gamma_0}(x)$ and $x \nleq z$. Let us define an element $\bar{z} \in L$ such that $\pi_{\gamma_0}(\bar{z}) = \pi_{\gamma_0}(z)$, $\pi_\gamma(\bar{z}) = \pi_\gamma(x)$ for all $\gamma \in \Gamma \setminus \{\gamma_0\}$. Clearly, $\bar{z} \in X$, i.e., $\bar{z} \vee x \in X$. But $\bar{z} \vee x > x$, a contradiction. □

An alternative proof of this Theorem may be given in the same way as in [11], Th. 2.9.

We do not know whether T_2'-frames are closed under sums.

Corollary 1.13. *T_2-frames form a monocoreflective subcategory in frm.*

Proof follows from 1.12, 1.6 and [8], 37.4.

Let us recall that for a frame L we put $N = \{(x, y) \in L \times L \colon x \wedge y = 0\}$.

Proposition 1.14. *Let L be a Hausdorff frame, $P \in P(L+L)$, $P \supseteq N$. Then $P = \downarrow (m,1) \vee \downarrow (1,m)$ for some $m \in D(L)$.*

Proof. Clearly, $P = \downarrow (m,1) \vee \downarrow (1,n)$ for some $m,n \in D(L)$. Let $m \neq n$. Then there exists an element $l \in L$ such that $l^* \nleq n$, $l \nleq m$. Then $(l,l^*) \in N$, i.e., $j(\downarrow (l,l^*)) \leqslant P$. Now, we have $P \geqslant \downarrow (m,1) \vee \downarrow (1,n) \vee \downarrow (l,l^*) \geqslant \downarrow (1,l^*) \vee \downarrow (1,n) = \downarrow (1,1)$, a contradiction. □

An alternative proof of 1.14 follows [11], Proposition 1.8 and Definition 2.2.

2. Almost compact frames

A Hausdorff topological space which is not regular has no T_2-compactification because a compact T_2-space is normal. It is natural to ask if some T_2-extension of the Hausdorff space exists with some properties of compactification. Of course, a compact Hausdorff space is closed in any T_2-extension. Therefore, it is necessary to restrict on non compact Hausdorff spaces and it is known (see cf. [3], 6.1.h, pp. 238-241) that suitable spaces for this situation are exactly almost compact spaces. These spaces are H-closed and T_2-extensions with some properties of compactifications are Katětov H-closed extensions. We shall consider these problems for frames.

Definition. We say that a frame L is *almost compact* if the following condition is fulfilled:

If $\bigvee(x_i : i \in I) = 1$ then a finite subset $K \subseteq I$ exists such that $(\bigvee(x_i : i \in K))^{**} = 1$ where $x_i \in L$ for $i \in I$.

Remark. Recall that L is a compact frame if the facts $X \subseteq L$, $\bigvee X = 1$ imply the existence of a finite set $K \subseteq L$ with properties $K \subseteq X$, $\bigvee K = 1$. Every compact conjunctive frame is spatial ([9], 2.11). Stone-Čech compactification for frames is investigated by Banaschewski and Mulvey in [2].

Proposition 2.1. *1. A compact frame is almost compact.*

2. A frame L is not almost compact iff an ideal Q in L exists such that $Q \subseteq S_L$ and $\bigvee Q = 1$.

Proof. 1. follows from Definition immediately.

2. If L is not almost compact then a set $\{x_i : i \in I\}$ exists such that $\bigvee(x_i : i \in I) = 1$ and thus $[\bigvee(x_i : i \in K)]^{**} \neq 1$ for any finite subset $K \subseteq I$. If we put $Q = < \{x_i : i \in I\} >$ then Q is an ideal in L such that $Q \subseteq S_L$ and $\bigvee Q = 1$.

If Q is an ideal in L, $Q \subseteq S_L$, $\bigvee Q = 1$ and L is almost compact then $q \in Q$ exists such that $q^{**} = 1$, a contradiction. □

Proposition 2.2. *For a Hausdorff frame L the following are equivalent:*

1. *L is almost compact and regular.*
2. *L is compact and conjunctive.*

Proof. $1 \Rightarrow 2$: Clearly, any regular frame is conjunctive. If $\bigvee(x_i : i \in I) = 1$ then $\bigvee(y_{j_i} : y_{j_i} \lhd x_i, j_i \in J_i, i \in I) = 1$ where $x_i = \bigvee(y_{j_i} : y_{j_i} \lhd x_i, j_i \in J_i)$ for any $i \in I$, $J_i \bigcap J_j \neq \emptyset \Rightarrow i = j$. These facts imply that a finite set $K \subseteq \bigcup(J_i : i \in I)$ exists such that $1 = [\bigvee(y_k : k \in K)]^{**}$. Clearly, $[\bigvee(y_k : k \in K)]^{**} \lhd \bigvee(x_i : k \in J_i \text{ for } k \in K)$, i.e., $\bigvee(x_i : k \in J_i \text{ for } k \in K) = 1$.

$2 \Rightarrow 1$: If L is compact and conjunctive then L is spatial, i.e., L is a Hausdorff topology. Hence L is regular and almost compact (see 2.1). □

Proposition 2.3. *Let L be a frame, L_r be a Boolean algebra of regular elements in L ($L_r = \{a^{**} : a \in L\}$). Then $K(L) = \{(u,v) : u \in L,\ v \in L_r,\ u \leqslant v\}$ is a frame with the following properties:*

1. *L is dually atomic iff $K(L)$ is dually atomic.*
2. *L is compact iff $K(L)$ is compact.*
3. *$K(L)$ is a T_2-frame iff L is a T_2-frame and L fulfils the condition:*
   $d \in D(L) \Rightarrow d^* = 0$.
4. *$K(L)$ is not conjunctive.*
5. *L is almost compact iff $K(L)$ is almost compact.*

Proof. 1. $d \in D(L) \Leftrightarrow (d,1) \in D(K(L))$.

2. Let $\bigvee((x_i, y_i) : i \in I) = (1,1)$. Then $\bigvee(x_i : i \in I) = 1$, i.e., there exists a finite subset $K \subseteq I$ such that $\bigvee((x_i, y_i) = (1,1)$. Conversely, if $\bigvee(x_i : i \in I) = 1$, $x_j \in L$ then $\bigvee((x_i, 1) : i \in I) = (1,1)$, i.e., there exists a finite subset $K \subseteq I$ such that $\bigvee((x_i, 1) : i \in K) = (1,1)$, i.e., $\bigvee(x_i : i \in K) = 1$.

3. $\Rightarrow$: Clearly, L is a homomorphic image of $K(L)$. Let $d \in D(L)$, $d^* \neq 0$, i.e., $d = d^{**}$. Then $(1,1) \neq (d,1) \nleq (d,d)$. If there exist $x, y \in K(L)$ such that $x \wedge y = 0$, $x = (x_1, x_2) \nleq (d,1)$, $y = (y_1, y_2) \nleq (d,d)$ then $y \leqslant (d,1)$, i.e., $y_2 \nleq d$, $y_1 \leqslant d$, $x_2 \nleq d$. Now, $x_2 \wedge y_2 = 0$, i.e., $x_2 \leqslant d$, a contradiction.

$\Leftarrow$: Let L be a Hausdorff frame. If $(u_1, v_1), (u_2, v_2) \in K(L)$, $(1,1) \neq (u_1, v_1) \nleq (u_2, v_2)$ then we have the following cases:

a) If $1 \neq u_1 \nleq u_2$ then $l \in L$ exists such that $l \nleq u_2$, $l^* \nleq u_1$. Hence $(l, l^{**}) \in K(L)$, $(l^*, l^*) \in K(L)$ and $(l, l^{**}) \nleq (u_2, v_2)$, $(l^*, l^*) \nleq (u_1, v_1)$.

b) If $1 \neq u_1 \leqslant u_2$, $1 \neq v_1 \nleq v_2$ then $l \in L$ exists such that $l \nleq v_2$, $l^* \nleq v_1$. Hence $(0, l^{**}) \in K(L)$, $(l^*, l^*) \in K(L)$ and $(0, l^{**}) \nleq (u_2, v_2)$, $(l^*, l^*) \nleq (u_1, v_1)$.

c) If $1 \neq u_1 \leqslant u_2$, $1 = v_1 \nleq v_2$ then $u_1 \leqslant u_2 \leqslant v_2 \neq 1$ holds. If $v_2 \nleq u_1$ then $1 \neq v_2 \nleq u_1$ and $(0, v_2^*) \in K(L)$, $(v_2, v_2) \in K(L)$. Now, we have $(0, v_2^*) \nleq$

$(u_2, v_2), (v_2, v_2) \not\leq (u_1, v_1)$. If $v_2 = u_1$ then $u_1 = u_2 = v_2 \neq 1$. Clearly, $v_2^* \neq 0$, i.e., $v_2 \in D(L) = S(L)$ and thus we can find $x \in L$ such that $x \not\leq v_2, x^* \not\leq v_2$. Therefore we have $(0, x^*) \in K(L)$, $(x^{**}, x^{**}) \in K(L)$ and $(0, x^*) \not\leq (u_2, v_2), (x^*, x^*) \not\leq (u_1, v_1)$. Finally, $K(L)$ is a Hausdorff frame.

The same way we can verify the property 3.

4. We have $(0,0)$, $(0,1) \in K(L)$, $(0,1) \not\leq (0,0)$ and $(0,1) \vee (u,v) = (1,1)$ for $(u,v) \in K(L)$ implies $u = 1 = v$, i.e., $(u,v) = (1,1)$. Finally, $K(L)$ is not conjunctive.

5. Let L be almost compact, $\bigvee((x_i, y_i) : i \in I) = (1,1)$, $(x_i, y_i \in K(L)$ for $i \in I)$. Then $\bigvee(x_i : i \in I) = 1$, i.e., there exists a finite set $K \subseteq I$ such that $[\bigvee(x_i : i \in K)]^{**} = 1$. Now, we have $[\bigvee(x_i, y_i) : i \in K]^{**} \geqslant [\bigvee(x_i : i \in K), (\bigvee(x_i : i \in K))^{**}]^{**} = (1,1)$.

Conversely, let $K(L)$ be almost compact, $\bigvee(x_i : i \in I) = 1$, $x_i \in L$ for $i \in I$. Then $\bigvee((x_i, x_i^{**}) : i \in I) = (1,1)$, i.e., there exists a finite set $K \subseteq I$ such that $[\bigvee(x_i, x_i^{**}) : i \in K)]^{**} = (1,1)$, i.e., $(\bigvee_{i \in K} x_i^{**})^* = \bigwedge_{i \in K} x_i^* = 0$. Now, we have $[\bigvee(x_i : i \in K)]^{**} = 1$.

□

R e m a r k . Proposition 2.3 is motivated by the paper of Murchiston and Stanley ([14], example 2).

Proposition 2.4. *There exists a compact Hausdorff frame which is not regular.*

P r o o f . Let L be the closed interval [0,1] with usual topology. Clearly, $O(I)$ is a compact Hausdorff frame and $d \in D(O(I))$ implies $d^* = 0$. Now, we have from 2.3 that $K(O(I))$ is a compact Hausdorff frame, which is not regular. □

Corollary 2.5. *There exists a compact Hausdorff frame which is not spatial.*

Definition. Let $f : K \to L$ be a surjective homomorphism of frames, $f^0(0) = \bigvee(x \in K : f(x) = 0)$ and let $f(a) = f(b) \Rightarrow a \vee f^0(0) = b \vee f^0(0)$ hold for any $a, b \in K$. Then f is called a *closed homomorphism.*

R e m a r k s . 1. If $f : K \to L$ is a closed homomorphism of frames then an isomorphism $i : \uparrow f^0(0) \to L$ exists such that $f(k) = i(k \vee f^0(0))$ for any $k \in K$.

2. The composition of closed homomorphisms is a closed homomorphism.

Definition. A homomorphism $f : K \to L$ of frames is called *dense* (*codense*, resp.) if $f(k) = 0 \Rightarrow k = 0$ ($f(k) = 1 \Rightarrow k = 1$, resp.) holds for any $k \in K$.

Proposition 2.6. *Let $f : K \to L$ be a surjective homomorphism of T_2'-frames and $R = \uparrow f^0(0)$. If L is almost compact then R is almost compact and $f(D(R)) = D(L)$.*

P r o o f. R is a T_2'-frame because R is a homomorphic image of K.

Now, let $\bar{f}\colon R \to L$ be a map such that $\bar{f} = f|_R$. Evidently, $\bar{f}$ is a dense homomorphism and we shall prove that $\bar{f}$ is also codense.If $\bar{f}(r) = 1$, $r \neq 1$ then $r = \bigvee A$, where $A \subseteq \Box r$ is an ideal, $A \subseteq S_R$. Because L is almost compact there exists $a \in A$ such that $[\bar{f}(a)]^{**} = 1$, $a \leqslant r$, $a^* \nleq r$. Namely, $a^* \neq 0$ holds. We can prove that $\bar{f}(x)^*$ for any $x \in R$ and therefore $\bar{f}(a^*) = [\bar{f}(a)]^{**} \wedge \bar{f}(a^*) = [\bar{f}(a)]^{**} \wedge \bar{f}(a)^* = 0$, which is in a contradiction with density of $\bar{f}$.

Immediately, we get that R is almost compact because $\bigvee(x_i\colon i \in I) = 1$ implies $\bigvee(\bar{f}(x_i)\colon i \in I) = 1$, i.e., a finite set $H \subseteq I$ exists such that $[\bigvee(\bar{f}(x_i)\colon i \in H)]^{**} = 1$. Hence $\bar{f}((\bigvee(x_i\colon i \in H))^{**}) = 1$ and from codensity of $\bar{f}$ we get $[\bigvee(x_i\colon i \in H)]^{**} = 1$.

If $m \in D(R)$ and $f(m) \notin D(L)$ then $\bar{f}(m) \neq 1$ holds and an element $y \in L$, $y \in 1$ exists such that $\bar{f}(m) < y$. Since $y = \bar{f}(x)$ for suitable $x \in R$ we have $x \nleq m$ and also $y = \bar{f}(x) \vee \bar{f}(m) = \bar{f}(x \vee m) = \bar{f}(1) = 1$, a contradiction. $\square$

Proposition 2.7. *Let T be a T_1-space, $f\colon O(T) \to L$ be a codense surjective homomorphism of frames. Then L is spatial.*

P r o o f. Clearly, $f(D(O(T))) = D(L)$ and thus L is dually atomic. Let us show that is conjunctive. Now, let $a \nleq b$, $a, b \in L$. Then there exist elements $c, d \in O(T)$ such that $c \nleq d$, $f(c) = a$, $f(d) = b$. Since $O(T)$ is spatial and $P(O(T)) = D(O(T))$ we have an element $m \in (D(O(T))$ such that $m \vee c = 1$, $m \geqslant d$. Evidently, $n = f(m) \in D(L)$, $n \vee a = 1$, $1 \neq n \geqslant b$.

Finally, L is conjunctive and dually atomic, i.e., L is spatial. $\square$

Proposition 2.8. *An almost compact frame which is a homomorphic image of a Hausdorff topology is a topology.*

P r o o f. Let T be a T_2-space, L a frame and $f\colon O(T) \to L$ be a surjective homomorphism. Clearly, L is a T_2'-frame and $R =\uparrow f^0(0)$ is a Hausdorff topology. If we put $\bar{f} = f|_R$ then we have from 2.6 that $\bar{f}$ is codense. Now, by 2.7 L is a topology. $\square$

The preceding results (2.5 and 2.8) establish the following.

Corollary 2.9. *A compact Hausdorff frame exists which is not a homomorphic image of a Hausdorff topology.*

Corollary 2.10. *Frames which are homomorphic images of Hausdorff topologies are either Hausdorff topologies or T_2'-frames which are not almost compact.*

Proposition 2.11. *Let L be a frame. Then a subframe $T(L) = \{(u,v) \in L \times L : u \leqslant v\}$ has following properties:*

1. *L is compact iff $T(L)$ is compact*
2. *L is almost compact iff $T(L)$ is almost compact.*
3. *$T(L)$ is not conjunctive.*
4. *$T(L)$ is a T_2-frame iff L is a T_2-frame and $D(L) = \emptyset$.*

Proof. The proof is similar to the proof of 2.3. □

Proposition 2.12. *If L is an almost compact frame then $S(L) \neq \emptyset$.*

Proof. Let F be a maximal filter in L. We put $a = \bigvee\{x^* : x \in F\}$. Clearly, $a \neq 1$. If $x, y \in L$, $x \wedge y = 0$, $x \nleq a$ then $x^* \notin F$, i.e., $x^{**} \in F$ and it implies $x^* \leqslant a$ and $y \leqslant a$. We have $a \in S(L)$. □

Corollary 2.13. *If L is an almost compact Hausdorff frame then $D(L) \neq \emptyset$.*

3. H-Closed extensions

Finally, we investigate some properties of the Katětov H-closed extension for frames. The construction of the H-closed extension for a given Hausdorff topological space is described for example in [4].

Definition. Let L be a frame. We say that a set $F \subseteq L$, $F \neq \emptyset$ is an *α-filter* if

(i) $0 \notin F$,
(ii) $a, b \in F \Rightarrow \wedge b \in F$,
(iii) $a \in F$, $b \geqslant a \Rightarrow b \in F$,
(iv) $\bigvee(a^* : \in F) = 1$ hold.

A maximal α-filter is called a *β-filter*. Evidently, $\bigwedge F = 0$ for any α-filter F.

Proposition 3.1. *A frame L is not almost compact iff there exists a β-filter in L.*

Proof. If L is not almost compact then 2.1.2 implies that an ideal 0 in L exists such that $Q \subseteq S_L$ and $\bigvee Q = 1$. If F is a filter in L generated by $\{a^* : a \in Q\}$ then $1 = \bigvee(a^{**} : a \in Q)$ and F is an α-filter in L.

If L is almost compact and F is a β-filter in L then $\bigvee(a^* : a \in F) = 1$ and there exists a finite set $K \subseteq F$ such that $1 = [\bigvee(a^* : a \in K)]^{**} = [\bigwedge(a : a \in K)]^*$. It means that $0 = \bigwedge K \in F$, a contradiction. □

Lemma 3.2. *If L is a frame and F_1, F_2 are β-filters in L, $F_2 \neq F_1$ then for all $x \in F_2 \setminus F_1$ there exists $y \in F_1$ such that $x \wedge y = 0$.*

Proof. Let $x \in F_2 \setminus F_1$. We put $U = \{b \in L : b \wedge x \geqslant a \wedge x$ for a suitable element $a \in F_1\}$. The set U fulfils (ii), (iii) and (iv) from the Definition of α-filters and $x \in U$ holds. These facts imply that $U \supseteq F_1$, $U \neq F_1$ and thus $0 \in U$ because U is not a β-filter. It means that $x \wedge y = 0$ for a suitable element $y \in F_1$. □

Definition. Let L be a frame and $\{F_j : j \in J\}$ be the set of all β-filters in L. The frame $L_\beta \subseteq L \times 2^J$, generated by $\{(l, \emptyset) : l \in L\} \cup \{(a, \{j\}) : a \in F_j\}$ with operations $(a_1, I_1) \wedge (a_2, I_2) = (a_1 \wedge a_2, I_1 \cap I_2)$, $(a_1, I_1) \vee (a_2, I_2) = (a_1 \vee a_2, I_1 \cup I_2)$ is *an H-closed extension* of L.

Let us remark that $x \in L_\beta$ iff $x = (a, I)$ where $a \in \bigcap(F_j : j \in I)$ and $I \subseteq J$.

Let $a \in L$. Then we shall denote $I_a = \{j \in J : a \in F_j\}$.

Lemma 3.3. *If L is a frame then $(l, I)^* = (l^*, I_{l^*})$ holds in L_β.*

Proof. We have $(l, I) \wedge (l^*, I_{l^*}) = (0, \emptyset)$ because $0 = l \wedge l^* \in F_i$ for $i \in I \cap I_{l^*}$. If $(l, I)^* = (k, K)$ then $k \leqslant l^*$ and now we have $K \subseteq I_k \subseteq I_{l^*}$. □

Remark. Let L be a frame, $g_L : L_\beta \to L$ be a map such that $g_L((a, I)) = a$ for any $(a, I) \in L_\beta$. Then g_L is a dense surjective homomorphism of frames.

Proposition 3.4. *An H-closed extension of a T_2-frame is a T_2-frame.*

Proof. Let L be a Hausdorff frame, $(1, J) \neq (a_1, I_1) \not\leqslant (a_2, I_2)$. Then we have the following cases:

(i) If $1 \neq a_1 \not\leqslant b_1$ then there exists an element $l \in L$ such that $l^* \not\leqslant a_1$, $l \not\leqslant a_2$. Clearly, $(l^*, \emptyset) \wedge (l, \emptyset) = (0, \emptyset)$, $(l^*, \emptyset) \not\leqslant (a_1, I_1)$, $(l, \emptyset) \not\leqslant (a_2, I_2)$.

(ii) If $1 = a_1 \not\leqslant a_2$ then there exists $j_0 \in J \setminus J_1$. Now, there exists an element $s \in F_{j_0}$ such that $s^* \not\leqslant a_2$. Clearly, $(s, \{j_0\}) \wedge (s^*, \emptyset) = (0, \emptyset)$, $(s, \{j_0\}) \not\leqslant (a_1, I_1)$, $(s^*, \emptyset) \not\leqslant (a_2, I_2)$.

(iii) Let $a_1 \leqslant a_2$, $J \neq J_1 \not\leqslant I_2$. Then there exist $j_1 \in I_1 \setminus I_2$, $j_2 \notin I_1$, i.e., there exist elements $x \in F_{j_1}$, $y \in F_{j_2}$ such that $x \wedge y = 0$. Now, we have $(x, \{j_1\}) \wedge (y, \{j_2\}) = (0, \emptyset)$, $(x, \{j_1\}) \not\leqslant (a_2, I_2)$, $(y, \{j_2\}) \not\leqslant (a_1, I_1)$.

(iv) Let $a_1 \leqslant a_2$, $J = I_1 \not\leqslant I_2$. Then $a_1 \neq 1$ and there exists an index $j_0 \in J \setminus I_2$. Now, there exists an element $s \in F_{j_0}$ such that $s^* \not\leqslant a_1$. Clearly, $(s, \{j_0\}) \wedge (s^*, \emptyset) = (0, \emptyset)$, $(s, \{j_0\}) \not\leqslant (a_2, I_2)$, $(s^*, \emptyset) \not\leqslant (a_1, I_1)$.

If L is a T_2-frame then by the same arguments as in the first part we can verify that L_β is a T_2-frame. □

Theorem 3.5. *If L is a frame then L_β is an almost compact frame.*

Proof. If L is almost compact then $J = \emptyset$ and thus $L = L_\beta$ is almost compact. Let L be not almost compact. Then $J \neq \emptyset$ and let us suppose that L_β is not almost compact. Then an ideal Q in L_β exists such that $Q \subseteq S_{L_\beta}$, $\bigvee Q = (1, J)$. Namely, $\bigvee(a\colon (a, I) \in Q = 1$, i.e., $\bar{Q} = \{a \in L\colon (a, I) \in Q\}$ is an ideal in L such that $\bar{Q} \subseteq S_L$, $\bigvee \bar{Q} = 1$. If we consider an α-filter F generated by $\{a^*\colon a \in \bar{Q}\}$ then a β-filter $F_{j_0} \supseteq F$ exists. There exists an element $a \in F_{j_0}$ such that $(a, I) \in \bar{Q}$, $j_0 \in I$. But $a^* \wedge a = 0$, which is in a contradiction with the fact that F_{j_0} is a β-filter. □

Remark. It is well known (see [3] or [4] that if L is a spatial T_2-frame then L_β is a spatial almost compact T_2-frame.

Definition. A frame L is called *T_2-closed*, if L is a T_2-frame, and any surjective homomorphism $f\colon K \to L$ is closed, where K is a T_2-frame.

Proposition 3.6. *An T_2-closed frame L is almost compact.*

Proof. If L is not almost compact then $g_L\colon L_\beta \to L$ is a dense surjective homomorphism, i.e., g_L is an isomorphism and L_β is almost compact, a contradiction. □

Lemma 3.7. *If L is a frame then it holds:*

1. *If $l \in L$, $l^* = 0$ then $l \in F$ for any β-filter F in L.*
2. *L is Boolean algebra iff $l^* \neq 0$ holds for any $1 \neq l \in L$.*

Proof. 1. If $l \notin F$ then $\uparrow (l \wedge F) \supsetneqq F$ holds and it means that there exists $f_0 \in F$ such that $l \wedge f_0 = 0$, i.e., $0 \neq f_0 \leqslant l^*$, a contradiction.

2. We have $(l \vee l^*)^* = 0$ and thus $l \vee l^* = 1$. Hence l^* is a complement of l and L is a Boolean algebra. □

Proposition 3.8. *Let L be a frame which is not almost compact. Then it holds:*

1. *If d is a dual atom in L then (d, J) is a dual atom in L_β.*
2. *If (a, I) is a dual atom in L_β then $(a, I) = (1, J \setminus \{j\})$ for a suitable element $j \in J$ or $(a, I) = (d, J)$ where $d \in D(L)$.*

Proof. 1. Let us suppose that a β-filter F_j exists such that $d \notin F_j$. Namely, $d \ngeqslant x$ for any $x \in F_j$, i.e., $d \vee x = 1$. Since $x^* = x^* \wedge (d \vee x) \leqslant d$ holds for any $x \in F_j$ we have $1 = \bigvee(x^*\colon x \in F_j) \leqslant d$, a contradiction.

2. If $(a, I) \in D(L_\beta)$ and $a = 1$ then $j \in J \setminus I$ exists such that $(a, I) = (1, J \setminus \{j\})$. In the case $a \neq 1$ it holds $a = d \in D(L)$ and $(a, I) = (d, J)$.

Now we give an explicit description of the sets $S(L_\beta)$ and $P(L_\beta)$. □

Proposition 3.9. *Let L be a frame which is not almost compact. Then the following propositions hold:*

1. $(a, I) \in S(L_\beta)$ *iff* $a = 1$, $I = J \setminus \{j\}$ *for some* $j \in J$ *or* $a \in S(L)$, $I = J$.
2. $(a, I) \in P(L_\beta)$ *iff* $a = 1$, $I = J \setminus \{j\}$ *for some* $j \in J$ *or* $a \in P(L)$, $I = J$.

Proof. 1. $\Rightarrow$: Let $(a, I) \in S(L_\beta)$. Then $a \in S(L) \cup \{1\}$. Namely, if $a \neq 1$, $a \notin S(L)$ then there exist elements $x, y \in L$ such that $x \wedge y = 0$, $x \not\leqslant a$, $y \not\leqslant a$. Clearly, $(x, \emptyset) \wedge (y, \emptyset) = (0, \emptyset)$, $(x, \emptyset) \not\leqslant (a, I)$, $y, \emptyset) \not\leqslant (a, I)$, a contradiction.

(i) Let $a = 1$. If $I \neq J \setminus \{j\}$ for any $j \in J$ then there exist $j_1, j_2 \in J$, $j_1 \neq j_2$ such that $j_1 \notin U$, $j_2 \notin I$. Clearly, $(x, \{j_1\} \wedge (y, \{j_2\}) = (0, \emptyset)$, $(x, \{j_1\}) \not\leqslant (a, I)$, $(y, \{j_2\}) \not\leqslant (a, I)$ for suitable elements $x \in F_{j_1}$, $y \in F_{j_2}$ such that $x \wedge y = 0$.

(ii) Let $a \neq 1$, $a \in S(L)$. If $I \in J$ then there exists $j \in J \setminus I$. Now, there exists an element $x \in F_j$ such that $x^* \not\leqslant a$. Clearly, $(x, \{j\}) \wedge (x^*, \emptyset) = (0, \emptyset)$, $(x, \{j\}) \not\leqslant (a, I)$, $(x^*, \emptyset) \not\leqslant (a, I)$, a contradiction.

$\Leftarrow$: Clearly, $(1, J \setminus \{j\}) \in D(L_\beta) \subseteq S(L_\beta)$. Consider (a, J) for some $a \in S(L)$. If $(x_1, I_1) \wedge (x_2, I_2) = (0, \emptyset)$ then $x_1 \wedge x_2 = 0$, i.e., $x_1 \leqslant a$ or $x_2 \leqslant a$. Now, we have $I_1 \subseteq I_{x_1} \subseteq J$ or $I_2 \subseteq I_{x_2} \subseteq J$, i.e., $(a, j) \in S(L_\beta)$.

2. The proof for $P(L_\beta)$ is similar. □

Corollary 3.10. (i) *An H-closed extension of a T_1-frame is a T_1-frame.*

(ii) *An H-closed extension of an S-frame is an S-frame.*

(iii) *L is dually atomic implies L_β is dually atomic.*

Proof follows from 3.8 and 3.9.

Proposition 3.11. *Let L be a frame. Then the following conditions are equivalent:*

1. *L is spatial.*
2. *L_β is spatial.*

Proof. If L is almost compact we are ready. Let L be not almost compact.

$1 \Rightarrow 2$: Let $(1, J) \neq (a, I) \in L_\beta$. Then we have the following cases:

(i) If $a = 1$ then we have $(a, I) = \bigwedge\{(1, J \setminus \{j\}) : j \notin I\}$.

(ii) If $a \neq 1$ then $(a, I) = \bigwedge\{(p, I_p) : p \geqslant a,\ p \in P(L)\} \wedge \bigwedge\{(1, J \setminus \{j\}) : j \notin U\}$.

$2 \Rightarrow 1$: Let $1 \neq a \in L$. Then we have $(a, \emptyset) = \bigwedge\{(p, I_p) \in P(L_\beta) : (p, I_p) \geqslant (a, \emptyset) \wedge \bigwedge\{(1, J \setminus \{j\}) : j \in J\}$. □

Proposition 3.12. *Let L be a complete Boolean algebra. Then L_β is dually atomic.*

Proof. Let $\{(x_i, J) : x_i \neq 1 \text{ for } i \in I\}$ be a chain in L_β, $\bigvee_{i \in I} (x_i, J) = (1, J)$. Since L_β is almost compact there exists a finite set $K \subseteq I$ such that $(1, J) =$

$[\bigvee\{(x_i, J): i \in K\}]^{**} = (x_k, J)^{**} = (x_k^{**}, J) = (x_k, J)$ for a suitable $k \in K$, a contradiction. The rest follows from 3.8. □

Corollary 3.13. *Let L be a complete Boolean algebra which is not dually atomic. Then $K(L)_\beta$ is an almost compact T_2-frame which is not dually atomic.*

Proof. If $a \in L$, $\uparrow a \cap D(L) = \emptyset$ then $(a, 1) \in K(L)$, $\uparrow (a, 1) \cap D(K(L)) =$ (see 2.3) and $((a, 1), J) \in K(L)_\beta$. The rest follows from 3.8. □

Corollary 3.14. *Let L be a complete Boolean algebra which is not dually atomic. Then L_β is not conjunctive.*

Proof follows immediately from 3.11 and from the fact that L is not a topology.

Corollary 3.15. *Homomorphic images of Hausdorff topologies form a subcategory in Frm which is not closed under H-extensions.*

Proof. Let L be a complete Boolean algebra which is not dually atomic. Then from 2.8 we have that L_β is not a homomorphic image of a Hausdorff topology. □

We do not know whether our class of T_2-frames is the monocoreflective hull of Hausdorff spatial frames. This problem for regular frames was solved negatively by I. Kříž.

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