Roman Frič; Darrell C. Kent

Regular $\mathcal{L}$-spaces

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 3, 475–486

Persistent URL: http://dml.cz/dmlcz/128345

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

REGULAR $\mathcal{L}$-SPACES

R. FRIČ, Košice and D.C. KENT, Pullman

(Received January 4, 1991)

In [FKE] the notion of regularity for various continuous structures has been discussed in connection with the extension of continuous maps. In the present paper we show that the definition of regularity for $\mathcal{L}$-spaces introduced in [FKE] meshes nicely with the γ-modification of Butzmann and Beattie. We investigate the relationships between regularity and some other properties of $\mathcal{L}$-spaces. Finally, we describe some categorical aspects of $\mathcal{L}$-regularity.

Recall that an *$\mathcal{L}$-space* $\mathbf{X}$ is a pair (X, L) where $X \neq \emptyset$ is a set and the *$\mathcal{L}$-structure* $\mathsf{L} \subset X^N \times X$ satisfies the basic two axioms of sequential convergence

($\mathcal{L}_1$) $(S, x) \in \mathsf{L}$ whenever $x \in X$ and S is the constant sequence $S(n) = x$, $n \in N$;

($\mathcal{L}_2$) If $(S, s) \in \mathsf{L}$, then $(S \circ s, x) \in \mathsf{L}$ for each subsequence $S \circ s$, where $s \in MON$ is a strictly monotone map of N into N.

According to [KNE], an $\mathcal{L}_0$-space, i.e. an $\mathcal{L}$-space with unique limits, is said to be *separated.* An $\mathcal{L}$-space satisfying the Urysohn axiom of convergence is called an *$\mathcal{L}^*$-space* and a separated $\mathcal{L}^*$-space is called an *$\mathcal{L}_0^*$-space.*

If there is a (filter) convergence structure q on X such that $\mathsf{L} = \mathcal{L}(q)$ (i.e. L is associated with q), then (X, L) is said to be an *$\mathcal{L}^f$-space.* We have $\mathcal{F} \cap \dot{x} \xrightarrow{q} x$ whenever $\mathcal{F} \xrightarrow{q} x$. Therefore, if $(S, x) \in \mathsf{L}$ and the sequence $T \in X^N$ generates the same filter of sections as the sequence $S \wedge \langle x \rangle$, then $(T, x) \in \mathsf{L}$. In fact, $\mathcal{L}^f$-spaces are characterized by this property.

We use the standard notation for sequential and filter convergence spaces (cf. [FKE], [BBU]), where some undefined notions can be found). Recall (cf. [BBU]) that if $\mathbf{X} = (X, \mathsf{L})$ is an $\mathcal{L}$-space, then the *γ-modification* $\gamma(\mathsf{L})$ of L is defined as follows: a filter $\mathcal{F}$ $\gamma(\mathsf{L})$-converges to x whenever there is a *$\gamma(\mathsf{L})$-basic* filler $\mathcal{G}$ ($\mathcal{G}$ has a countable base and if $S \in X^N$ and the filter of sections $\mathcal{F}(S)$ is finer than $\mathcal{G}$, then $(S, x) \in \mathsf{L}$) such that $\mathcal{F} \supset \mathcal{G}$; also $\gamma\mathbf{X} = (X, \gamma(\mathsf{L}))$ is said to be the γ-modification of $\mathbf{X}$. Finally, if $\mathcal{S} = \langle S_n \rangle$ is a double sequence (i.e. a sequence of sequences $S_n \in X^N, n \in N$), then by $\mathcal{F}(\mathcal{S})$ we denote the filter generated by sets $\{S_m(k);\ m \geqslant n, k \in N\}, n \in N$.

Definition 0.1. ([FKE]). Let (X, L) be an $\mathcal{L}$-space. Let $S \in X^N$ be a sequence and let $x \in X$ be a point. We say that S and x are *linked* if there exists a double sequence $\mathcal{S} = \langle S_n \rangle \in (X^N)^N$ such that for each $k \in N$ the sequence S_k L-converges to $S(k)$ and if $T \in X^N$ is a sequence such that $\mathcal{F}(T) \supset \mathcal{F}(\mathcal{S})$, then T L-converges to x; in this case we say that L *links* S and x.

Definition 0.2. ([FKE]). Let $\mathbf{X} = (X, \mathsf{L})$ be an $\mathcal{L}$-space. If a sequence $S \in X^N$ L-converges to a point $x \in X$ whenever S and x are linked, then $\mathbf{X}$ is said to be *regular*.

The obvious proof of the next proposition is left out.

Proposition 0.3. (i) *Each subspace of a regular $\mathcal{L}$-space is regular.*
(ii) *An $\mathcal{L}$-product is regular iff each of its factors is regular.*
(iii) *An $\mathcal{L}$-sum of regular $\mathcal{L}$-spaces is regular.*

Remark 0.4. Since each $\mathcal{L}$-space is an $\mathcal{L}$-quotient of a disjoint $\mathcal{L}$-sum of convergent sequences, it follows readily that an $\mathcal{L}$-quotient of a regular $\mathcal{L}$-space need not be regular.

1. Regularity in $\mathcal{L}^f$-spaces

Unless stated explicitly otherwise, in this section we deal with $\mathcal{L}^f$-spaces. We investigate the relationship between regularity and some other properties of $\mathcal{L}^f$-spaces. In particular, we prove that an $\mathcal{L}^f$-space $\mathbf{X}$ is regular iff its γ-modification $\gamma\mathbf{X}$ is a regular (filter) convergence space.

Lemma 1.1. *Let (X, L) be an $\mathcal{L}^f$-space and let $\gamma(\mathsf{L})$ be the γ-modification of L. Let $S \in X^N$ be a sequence and let $x \in X$ be a point. Then:*

(i) *Let $\mathcal{G}$ be a $\gamma(\mathsf{L})$-basic filter converging to x. If $\mathcal{F}(S) \supset cl_{\gamma(\mathsf{L})}\mathcal{G}$, then S and x are linked;*

(ii) *If S and X are linked, then there is a $\gamma(\mathsf{L})$-basic filter $\mathcal{G}$ converging to x such that $\mathcal{F}(S) \supset cl_{\gamma(\mathsf{L})}\mathcal{G}$.*

Proof. (i) Let $\{G_n;\ n \in N\}$ be a monotone base of $\mathcal{G}$. Then $\{\mathrm{cl}_{\gamma(\mathsf{L})} G_n;\ n \in N\}$ is a base of $\mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{G}$. Let $\mathcal{F}(S) \supset \mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{G}$. Put $G_0 = X, G_\infty = \bigcap_{n=0}^{\infty} G_n$. Then, for each $k \in N$, we have $S(n) \in \mathrm{cl}_{\gamma(\mathsf{L})}G_k$ for all but finitely many $n \in N$. Given $S(k)$, let $n(k)$ be the largest $n \in \{0, 1, \cdots, \infty\}$ such that $S(k) \in \mathrm{cl}_{\gamma(\mathsf{L})}G_n$ and let S_k be a sequence in $G_{n(k)}$ such that $S_k(n) \xrightarrow{\mathsf{L}} S(k)$. Let $\mathcal{S} = \langle S_k \rangle$ be the resulting double sequence. We have $\mathcal{F}(\mathcal{S}) \supset \mathcal{G}$. Therefore, if $T \in X^N$ is a sequence such that $\mathcal{F}(T) \supset \mathcal{F}(\mathcal{S})$, then $T(n) \xrightarrow{\mathsf{L}} x$. Thus $\mathcal{S}$ links S and x.

(ii) Let $\mathcal{S} = \langle S_n \rangle \in (X^N)^N$ be a double sequence linking S and x. Since $\mathcal{F}(\mathcal{S})$ has a countable base and $T(n) \xrightarrow{\mathsf{L}} x$ whenever $\mathcal{F}(T) \supset \mathcal{F}(\mathcal{S})$, we have $\mathcal{F}(\mathcal{S}) \xrightarrow{\gamma(\mathsf{L})} x$. Clearly, $\mathcal{F}(S) \supset \mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{F}(\mathcal{S})$. □

Theorem 1.2. *Let (X, L) be an $\mathcal{L}^f$-space and let $\gamma(\mathsf{L})$ be the γ-modification of L. Then (X, L) is regular iff $(X, \gamma(\mathsf{L}))$ is regular.*

Proof. 1. Let (X, L) be regular. Let $\mathcal{F} \xrightarrow{\gamma(\mathsf{L})} x$. Then there is a $\gamma(\mathsf{L})$-basic filter $\mathcal{G}$ converging to x such that $\mathcal{F} \supset \mathcal{G}$. Thus $\mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{F} \supset \mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{G}$. But $\mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{G}$ is a $\gamma(\mathsf{L})$-basic filter converging to x. Indeed, by (i) in Lemma 1.1, if $S \in X^N$ and $\mathcal{F}(S) \supset \mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{G}$, then S and x are linked and hence $S(n) \xrightarrow{\mathsf{L}} x$. Since $\mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{G}$ has a countable base, we have $\mathrm{cl}_{\gamma(\mathsf{L})}\mathcal{F} \xrightarrow{\gamma(\mathsf{L})} x$. Thus $(X, \gamma(\mathsf{L}))$ is regular.

2. Let $(X, \gamma(\mathsf{L}))$ be regular. Then from (ii) in Lemma 1.1 it follows easily that (X, L) is regular. □

Corollary 1.3. *Let (X, q) be a pretopological space and let $\mathcal{L}(q)$ be the associated $\mathcal{L}^*$-structure. If (X, q) is regular, then $(X, \mathcal{L}(q))$ is regular.*

Proof. Let $\gamma(q) = \gamma(\mathcal{L}(q))$ be the γ-modification of $\mathcal{L}(q)$. Then $q \leqslant \gamma(q)$ and $\mathcal{L}(q) = \mathcal{L}(\gamma(q))$. Clearly, $(X, \gamma(q))$ is regular and hence, by Theorem 1.2, $(X, \mathcal{L}(q))$ is regular. □

Observe that if (X, q) is a regular convergence space, then the associated $\mathcal{L}^f$-space need not be regular. Indeed, let (X, L) be an $\mathcal{L}^f$-space which fails to be regular. Let $\varphi(\mathsf{L})$ be the finest convergence structure for X such that $\mathcal{F}(S) \xrightarrow{\varphi(\mathsf{L})} x$ whenever $S(n) \xrightarrow{\mathsf{L}} x$ (cf. [BBH]). Then $(X, \varphi(\mathsf{L}))$ is regular and L is associated with $\varphi(\mathsf{L})$ (i.e. $\mathcal{L}(\varphi(\mathsf{L})) = \mathsf{L}$).

Proposition 1.4. *Let (X, L) be an $\mathcal{L}^*$-space. Then the following are equivalent:*

(i) *(X, L) is regular;*

(ii) *Let $S \in X^N, x \in X$ and let $(S, x) \notin \mathsf{L}$. Then there exists $s \in MON$ such that whenever $\mathcal{S} = \langle S_n \rangle \in (X^N)^N$ is a double sequence such that for each $k \in N$ we have $(S_k, S(s(k))) \in \mathsf{L}$, then for each $f \in N^N$ there exists $g \in N^N$, $g \geqslant f$, such that $(S_g, x) \notin \mathsf{L}$, where $S_g = \langle S_n(g(n)) \rangle$.*

P r o o f. (i) implies (ii). Let $S \in X^N, x \in X$ and let $(S,x) \notin \mathsf{L}$. Since $\mathsf{L} = \mathsf{L}^*$, there exists $s \in MON$ such that $(S \circ s, x) \notin \mathsf{L}$. Suppose that, on the contrary, there exists a double sequence $\mathcal{S} = \langle S_n \rangle \in (X^N)^N$ such that for each $k \in N$ we have $(S_k, S(s(k))) \in \mathsf{L}$ and for some $f \in N^N$ we have $(S_g, x) \in \mathsf{L}$ whenever $g \in N^N$, $g \geqslant f$. Then the double sequence $\langle\langle S_k(n + f(k))\rangle\rangle$ links $S \circ s$ and x. If (X, L) is regular, then $(S \circ s, x) \in \mathsf{L}$, a contradiction.

(ii) implies (i). Let $S \in X^N, x \in X$ and let $\mathcal{S} = \langle S_n \rangle \in (X^N)^N$ be a double sequence which links S and x. Suppose that, on the contrary, $(S,x) \notin \mathsf{L}$. Since for each $s \in MON$ the double sequence $\langle S_{s(n)} \rangle$ links $S \circ s$ and x, this would contradict (ii). □

Recall (cf. [FRI]) that a topological space (X,q) is said to be *c-regular* if a sequence $S \in X^N$ $\mathsf{L}(q)$-converges to $x \in X$ whenever $\mathcal{F}(S) \supset \mathrm{cl}_q \mathcal{N}_x$, where $\mathcal{N}_x$ is the neighborhood filter of x.

Proposition 1.5. *Let (X,q) be a c-regular topological space and let L be the associated $\mathcal{L}^*$-structure. Then the following holds:*

(iii) Let $S \in X^N, x \in X$ and let $(S,x) \notin \mathsf{L}$. Then there exists $s \in MON$ such that whenever $\mathcal{S} = \langle S_n \rangle \in (X^N)^N$ is a double sequence such that for each $k \in N$ we have $(S_k, S(s(k))) \in \mathsf{L}$, then there exists $f \in N^N$ such that for each $g \in N^N$, $g \geqslant f$, we have $(S_g, x) \notin \mathsf{L}$.

P r o o f. Let $S \in X^N, x \in X$ and let $(S,x) \notin \mathsf{L}$. Then there exists a closed neighborhood U of x and $s \in MON$ such that for all $k \in N$ we have $S(s(k)) \notin U$. Let $\mathcal{S} = \langle S_n \rangle \in (X^N)^N$ be a double sequence such that for each $k \in N$ we have $(S_k, S(s(k))) \in \mathsf{L}$. Let V be an open neighborhood of x such that $U \supset \mathrm{cl}_q V$. Then for each $k \in N$ there exists $f(k) \in N$ such that $S_k(n) \notin V$ for all $n \geqslant f(k)$. Thus $(S_g, x) \notin \mathsf{L}$ whenever $g \in N^N$ and $g \geqslant f$. □

Corollary 1.6. *Let (X,q) be a c-regular topological space and let (X,L) be the associated $\mathcal{L}^*$−space. Then (X,L) is regular.*

Corollary 1.7. *Let (X,L) be a sequentially regular $\mathcal{L}_0^*$-space. Then (X,L) is regular.*

The next two examples show that the converse of both corollaries are false.

Example 1.8. Let (X,q) be the regular topological space on which each continuous function is constant, constructed by *J.* Novák in [NOR]. It is a sequential space having unique sequential limits. The associated $\mathcal{L}_0^*$-space fails to be sequentially regular, but, by Corollary 1.6, it is regular.

Example 1.9. Let (X, q) be the sequentially regular Fréchet space which fails to be regular, constructed in [FNO]. By Corollary 1.7, the associated $\mathcal{L}_0^*$-space is regular.

Remark 1.10. The topological sum of the associated $\mathcal{L}_0^*$-spaces from the previous two examples shows that there is a regular $\mathcal{L}_0^*$-space whose topological modification is neither regular nor sequentially regular.

2. Categories of $\mathcal{L}$-spaces

For the readers convenience and a later reference, in this section we survey some categorical $\mathcal{L}$-folklore. Additional information can be found, e.g., in [FKO], [HER], [BBH], [KNE].

Our reference category will be the topological category $\mathcal{L}$ of $\mathcal{L}$-spaces and $\mathcal{L}$-continuous maps. Recall that in $\mathcal{L}$ monomophisms are exactly one-to-one continuous maps and epimorphisms are exactly continuous onto maps. Let (X, L) be an $\mathcal{L}$-space. Consider the following additional axioms of convergence:

$(\mathcal{AL})$ If $S \in X^N$ and $(\langle S(n+1)\rangle, x) \in \mathsf{L}$, then $(S, x) \in \mathsf{L}$;

$(\mathcal{LL})$ If $(S, x) \in \mathsf{L}$ and $T \in X^N$ is defined by $T(2n-1) = S(n)$ and $T(2n) = x$, $n \in N$ (i.e. $T = S \wedge \langle x\rangle$), then $(T, x) \in \mathsf{L}$;

$(\mathcal{ML})$ If $(S, x), (T, x) \in \mathsf{L}$ and $U \in X^N$ is defined by $U(2n-1) = S(n)$ and $U(2n) = T(n), n \in N$ (i.e. $U = S \wedge T$), then $(U, x) \in \mathsf{L}$.

In addition to $\mathcal{L}^f$ and $\mathcal{L}^*$, consider the following (full and isomorphisms-closed) subcategories of $\mathcal{L}$:

$\mathcal{L}^a$ of all $\mathcal{L}$-spaces satisfying $(\mathcal{AL})$;

$\mathcal{L}^l$ of all $\mathcal{L}$-spaces satisfying $(\mathcal{LL})$;

$\mathcal{L}^m$ of all $\mathcal{L}$-space satisfying $(\mathcal{ML})$;

and, further,

$$\mathcal{L}^t = \mathcal{L}^a \cap \mathcal{L}^l \text{ of all spaces satisfying } (\mathcal{AL}) \text{ and } (\mathcal{LL});$$

similar "intersection" notational convention will be applied to other categories as well, e.g., $\mathcal{L}_0^t = \mathcal{L}^t \cap \mathcal{L}_0$. Denote $\mathbf{T} = \{a, l, m, t, f, *\}$. For $c \in \mathbf{T}, \mathcal{L}^c$ will be the corresponding subcategory of $\mathcal{L}$.

Good properties of various subcategories of $\mathcal{L}$ follow from the fact that $\mathcal{L}$-axioms behave extremely well with respect to intersections of $\mathcal{L}$-structures. The obvious proofs of the next few propositions are omitted.

Lemma 2.1. *Let X and A be nonempty sets and for each $a \in A$ let L_a be an $\mathcal{L}$-structure for X. Put $\mathsf{L}_A = \bigcap_{a \in A} \mathsf{L}_a = \{(S, x) \in X^N \times X\,;\ (S, x) \in \mathsf{L}_a \text{ for all } a \in A\}$. Then*

(i) *L_A is an $\mathcal{L}$-structure for X;*

(ii) *If each $\mathsf{L}_a, a \in A$, satisfies an axiom $(\cdot) \in \{(\mathcal{L}_0), (\mathcal{AL}), (\mathcal{LL}), (\mathcal{ML}), (\mathcal{FL}), (\mathcal{L}_3)\}$, then so does L_A.*

Let X and A be nonempty sets and for each $a \in A$ let (Y_a, M_a) be an $\mathcal{L}$-space and f_a a map of X into Y_a. Then the initial $\mathcal{L}$-structure for X given by $\{f_a\,;\ a \in A\}$, i.e. the coarsest $\mathcal{L}$-structure L for X such that each f_a is $\mathcal{L}$-continuous, can be described as follows:

for $a \in A$, put $\mathsf{L}_a = \{(S, x) \in X^N \times X\,;\ (f_a \circ S, f_a(x)) \in \mathsf{M}_a\}$; then L_a is an $\mathcal{L}$-structure and $\mathsf{L} = \mathsf{L}_A = \bigcap_{a \in A} \mathsf{L}_a$.

Lemma 2.2. *If each M_a, $a \in A$, satisfies an axiom $(\cdot) \in \{(\mathcal{AL}), (\mathcal{LL}), (\mathcal{ML}), (\mathcal{FL}), (\mathcal{L}_3)\}$, then so does the initial $\mathcal{L}$-structure L_A given by the family $\{f_a\,;\ a \in A\}$. If for each $(S, x) \in \mathsf{L}_A$ and for each $y \in X$, $y \neq x$, there exists $a \in A$ such that $(S, y) \notin \mathsf{L}_a$, then L_A is separated.*

Corollary 2.3. *Let $c \in \mathbf{T}$. Then*

(i) *$\mathcal{L}^c$ is closed under formation of products and subspaces (i.e. subobjects) in $\mathcal{L}$;*

(ii) *$\mathcal{L}^c$ is epireflective in $\mathcal{L}$.*

Let (X, L) be an $\mathcal{L}$-space. Let $c \in \mathbf{T}$. Since $X^N \times X$ is an $\mathcal{L}^*$-structure and hence $\mathcal{L}^c$-structure for X, we can form L_c, the intersection of all $\mathcal{L}^c$-structures for X coarser than L.

Lemma 2.4. *Let f be a continuous map of an $\mathcal{L}$-space (X, L) into an $\mathcal{L}$-space (Y, M). Then f is a continuous map of (X, L_c) into (Y, M_c). Each identity of an $\mathcal{L}$-space (X, L) to the $\mathcal{L}^c$-space (X, L_c) is $\mathcal{L}^c$-universal.*

Corollary 2.5. *Let $c \in \mathbf{T}$. Then:*

(i) *$\mathcal{L}^c$ is a bireflective subcategory of $\mathcal{L}$;*

(ii) *$\mathcal{L}^c$ is a topological category.*

Proof. (i) is a trivial consequence of Lemma 2.4 and (ii) follows from (i) and Theorem 2.2.12 in [PRE]. □

Definition 2.6. Let $c \in \mathbf{T}$. Let (X, L) be an $\mathcal{L}$-space. Then L_c is said to be the $\mathcal{L}^c$-*modification* of L, resp. (X, L_c) is said to be the $\mathcal{L}^c$-*modification* of (X, L).

Remark 2.7. It is easy to see that if (X, L) is an $\mathcal{L}^a$-space and f is a map of X into a set Y, then the final $\mathcal{L}$-structure M_f for Y need not satisfy axiom $(\mathcal{AL})$. A final structure in $\mathcal{L}^a$ is the $\mathcal{L}^a$-modification of the corresponding final structure formed in $\mathcal{L}$. Similarly for other axioms. Observe that extremal epimorphisms in $\mathcal{L}^c, c \in \mathbf{T}$, are exactly quotients.

Now, let us turn to the category $\mathcal{L}_0$ of all separated $\mathcal{L}$-spaces and its subcategories. Since the quotient of a separated $\mathcal{L}$-space need not be separated, $\mathcal{L}_0$ fails to be a topological category. Recall that in $\mathcal{L}_0$ epimorphisms are exactly continuous maps with topologically dense range, monomorphisms are exactly one-to-one continuous maps, $\mathcal{L}_0$ is well- and co-well-powered and (epi, extremal mono)-factorizable and (extremal epi, mono)-factorizable. Thus (cf. [PRE]), a subcategory of $\mathcal{L}_0$ is epireflective if it is closed under formation of products and closed subspaces (i.e. subobjects) in $\mathcal{L}$.

Corollary 2.8. *Let $c \in \mathbf{T}$. Then:*

(i) $\mathcal{L}_0^c = \mathcal{L}^c \cap \mathcal{L}_0$ *is closed under formation of products and subspaces (i.e. subobjects) in $\mathcal{L}$;*

(ii) $\mathcal{L}_0^c$ *is epireflective in $\mathcal{L}_0$.*

Lemma 2.9. *Let $c \in \mathbf{T}$. Then:*

(i) *Let (X, L) be a separated $\mathcal{L}$-space. Then (X, L_c) is separated;*

(ii) *Let (X, L) be a separated $\mathcal{L}$-space. Then the identity map of (X, L) into (X, L_c) is $\mathcal{L}_0^c$-universal;*

(iii) *$\mathcal{L}_0^c$ is a bireflective subcategory of $\mathcal{L}_0$.*

Proof. (i) follows form Lemma 2.1. The straightforward proofs of the other assertions are omitted. □

In this section we investigate regularization, i.e. a process of replacing an $\mathcal{L}$-structure with a coarser one satisfying a certain regularity condition. By a regularity condition we mean that if a sequence S and a point x are linked with a double sequence $\mathcal{S} = \langle S_k \rangle$ of a specified type, then S converges to x. In order to avoid pathological situations, it is natural to require that a convergent sequence is always linked with its limit. Consequently, for a given type of linkage it is natural to work in a suitable subcategory of $\mathcal{L}$-spaces. For the regularity defined in [FKE] and adopted in the present paper the suitable subcategory is $\mathcal{L}^f$. In $\mathcal{L}^f$ the following holds: if S converges to x, then the double sequence $\langle S_k \rangle$, where S_k is the constant sequence generated by $S(k)$, links S and x.

Lemma 3.1. (i) *Let $\mathbf{Y} = (Y, \mathsf{M})$ be a regular $\mathcal{L}^f$-space. Let $X \neq \emptyset$ be a set, let g be a map of X into Y, and let L_g be the initial $\mathcal{L}^f$-structure for X generated by g. Then $\mathbf{X} = (X, \mathsf{L}_g)$ is a regular space.*

(ii) *Let $\{\mathsf{L}_a ;\ a \in A\}$ be a set of regular $\mathcal{L}^f$-structures for the same set $X \neq \emptyset$. Then $\mathsf{L}_A = \bigcap_{a \in A} \mathsf{L}_a$ is a regular $\mathcal{L}^f$-structure.*

Proof. (i) Let $S \in X^N$ and $x \in X$. Let $\mathcal{S} = \langle S_k \rangle \in (X^N)^N$ be a double sequence linking S and x in (X, L_g). Clearly, $g \circ \mathcal{S} = \langle g \circ S_k \rangle \in (Y^N)^N$ links $g \circ S$ and $g(x)$. Thus $(g \circ S, g(x)) \in \mathsf{M}$ and hence $(S, x) \in \mathsf{L}_g$.

(ii) is trivial. □

Since for each set $X \neq \emptyset$, $X^N \times X$ is an $\mathcal{L}^*$-structure on X, it follows from Lemma 3.1 that for each $\mathcal{L}^f$-structure L on X there is a regular $\mathcal{L}^f$-structure L_r (resp. $\mathcal{L}^*$-structure L_{r*}) on X such that $\mathsf{L} \subset \mathsf{L}_r$ (resp. $\mathsf{L} \subset \mathsf{L}_{r*}$) which is the finest of all such $\mathcal{L}^f$-structures (resp. $\mathcal{L}^*$-structures).

Definition 3.2. Let $\mathbf{X} = (X, \mathsf{L})$ be an $\mathcal{L}^f$-space (resp. $\mathcal{L}^*$-space). Then L_r (resp. L_{r*}) is said to be the *regular modification* of L in $\mathcal{L}^f$ (resp. in $\mathcal{L}^*$) and $r\mathbf{X} = (X, \mathsf{L}_r)$ (resp. $r^*\mathbf{X} = (X, \mathsf{L}_{r*})$) is said to be the *regular modification* of $\mathbf{X}$ in $\mathcal{L}^f$ (resp. in $\mathcal{L}^*$).

Problem 3.3. Does there exist an $\mathcal{L}^*$-space $\mathbf{X} = (X, \mathsf{L})$ such that $\mathsf{L}_r \neq \mathsf{L}_{r*}$?

Denote by $\mathcal{L}^r$ the (full and isomorphism-closed) subcategory of $\mathcal{L}^f$ whose objects consist of all regular spaces.

Proposition 3.4. *Let $\mathbf{X} = (X, \mathsf{L})$ be an $\mathcal{L}^f$-space and let $r\mathbf{X} = (X, \mathsf{L}_r)$ be its regular modification in $\mathcal{L}^f$. Then*

(i) *If g is a continuous map of* $\mathbf{X}$ *into a regular* $\mathcal{L}^f$*-space* $\mathbf{Y}$, *then g is a continuous map of* $r\mathbf{X}$ *into* $\mathbf{Y}$;

(ii) *The identity map of* $\mathbf{X}$ *into* $r\mathbf{X}$ *is* $\mathcal{L}^r$*-universal.*

Proof. (i) Let L_g be the initial $\mathcal{L}^f$-structure on X generated by g. By Lemma 3.1, (X, L_g) is regular. Since $\mathsf{L} \subset \mathsf{L}_g$, we have $\mathsf{L}_r \subset \mathsf{L}_g$ and the assertion follows.

(ii) is a straightforward consequence of (i). □

Remark 3.5. An analogous result holds for $\mathcal{L}^*$-spaces and regular modifications in $\mathcal{L}^*$.

Corollary 3.6. (i) *$\mathcal{L}^r$ is a bireflective subcategory of $\mathcal{L}^f$ and $\mathcal{L}^r \cap \mathcal{L}^*$ is a bireflective subcategory of $\mathcal{L}^*$.*

(ii) *$\mathcal{L}_0^r = \mathcal{L}^r \cap \mathcal{L}_0$ is an epireflective subcategory of $\mathcal{L}^f$ and $\mathcal{L}^r \cap \mathcal{L}_0^*$ is an epireflective subcategory of $\mathcal{L}^*$.*

Next we describe how the epireflection of $\mathcal{L}^f$ into $\mathcal{L}_0^r$ is split into an extremal epireflection and a bireflection. The objects of the category in the middle will be defined analogously to the R-Hausdorff spaces in [KRI].

Definition 3.7. Let $\mathbf{X}$ be an $\mathcal{L}^f$-space and let $r\mathbf{X}$ be its regular modification. If $r\mathbf{X}$ is separated, then $\mathbf{X}$ is said to be *r-separated.* Denote $\mathcal{L}_s^r$ the (full and isomorphism-closed) subcategory of $\mathcal{L}^f$ consisting of r-separated spaces.

Observation 3.8. *Since $\mathsf{L} \subset \mathsf{L}_r$, it follows that every r-separated space is separated.*

Lemma 3.9. *Let $\mathbf{X} = (X, \mathsf{L})$ be an $\mathcal{L}^f$-space and let $r\mathbf{X} = (X, \mathsf{L}_r)$ be its regular modification. Assume that whever $x, y \in X, x \neq y$, then there is a continuous map of $\mathbf{X}$ into an r-separated space $\mathbf{Y} = (Y, \mathsf{M})$ such that $f(x) \neq f(y)$. Then $(S, x) \in \mathsf{L}_r$ implies $(S, y) \notin \mathsf{L}_r$.*

Proof. The assertion follows from (ii) of Proposition 3.4. □

Theorem 3.10. *Let $\mathbf{X}$ be an $\mathcal{L}^f$-space. Then the following are equivalent:*

(i) *$\mathbf{X}$ is r-separated;*

(ii) *Whenever $x, y \in X, x \neq y$, then there is a continuous map f of $\mathbf{X}$ into an r-separated $\mathcal{L}^f$-space $\mathbf{Y}$ such that $f(x) \neq f(y)$.*

Proof. (i) $\Rightarrow$ (ii) is trivial. The converse follows from Lemma 3.9. □

Recall that extremal epimorphisms in $\mathcal{L}^f$ are exactly quotients.

Corollary 3.11. (i) *$\mathcal{L}^r_s$ is closed under formation of products and weak subobjects in $\mathcal{L}$ (and hence in $\mathcal{L}^f$).*

(ii) *$\mathcal{L}^r_s$ is extremal epireflective in $\mathcal{L}^f$.*

Proof. (i) follows from Theorem 3.10. (ii) follows from (i) and Theorem 2.2.4 in [PRE]. □

Remark 3.12. (ii) can be proved directly as follows. Let $\mathbf{X} = (X, \mathsf{L})$ be an $\mathcal{L}^f$-space. Define an equivalence relation on X by $x \sim y$ iff for each continuous map f of $\mathbf{X}$ into each r-separated space $\mathbf{Y}$ we have $f(x) = f(y)$. Let g_x be the quotient map of X onto $X/\sim$ and let $\mathsf{L}^\sim$ be the final $\mathcal{L}^f$-structure on $X/\sim$. The rest is clear.

The next two propositions are straightforward consequences of the previous results.

Proposition 3.13. (i) *Let $\mathbf{X} = (X, \mathsf{L})$ be an r-separated space and let $r\mathbf{X} = (X, \mathsf{L}_r)$ be its regular modification. Then the identity map of $\mathbf{X}$ into $r\mathbf{X}$ is $\mathcal{L}^r_0$-universal.*

(ii) *$\mathcal{L}^r_s$ is bireflective in $\mathcal{L}^r_0$.*

(iii) *An $\mathcal{L}^f$-space is r-separated iff it is a weak subobject (in $\mathcal{L}$) of a regular $\mathcal{L}^f_0$-space.*

Proposition 3.14. *Let $\mathbf{X} = (X, \mathsf{L})$ be an $\mathcal{L}^f$-space. Let $\sim$ be the equivalence relation described in Remark 3.12, let g_x be the corresponding quotient map of X into $X^\sim = X/^\sim$, and let L^r be the induced quotient $\mathcal{L}^f$-structure on $X^\sim$. Let $r\mathbf{X}^\sim = (X^\sim, \mathsf{L}^\sim_r)$ be the regular modification of $\mathbf{X}^\sim = (X^\sim, \mathsf{L}^\sim)$. Then $id_{X^\sim} \circ g_x$ is the epireflection of $\mathbf{X}$ into $\mathcal{L}^r_0$.*

Remark 3.15. Analogous relationships are valid for $\mathcal{L}^*$-spaces, r-separated $\mathcal{L}^*$-spaces and regular $\mathcal{L}^*_0$-spaces.

Using a construction analogous to that of the "regularity series" for filter convergence space (see [KRI]), we next define inductively a chain of $\mathcal{L}^f$-structures which terminate in the r-modification of an $\mathcal{L}^f$-structure L on X.

Let (X, L) be an $\mathcal{L}^f$-space. Define $r\mathsf{L} \subset X^N \times X$ as follows: $(S, x) \in r\mathsf{L}$ if S and x are linked. It is easy to see that $r\mathsf{L}$ is an $\mathcal{L}^f$-structure coarser than L. Inductively, define $\mathsf{L}_0 = \mathsf{L}$, $\mathsf{L}_{\alpha+1} = r\mathsf{L}_\alpha$ and $\mathsf{L}_\alpha = \bigcup_{\beta<\alpha} \mathsf{L}_\beta$ if α is a limit ordinal. Clearly, each L_α is an $\mathcal{L}^f$-structure. Let (X, L_r) be the regular modification of (X, L). Since $\mathsf{L}_r \supset \mathsf{L}_\alpha$

for each ordinal number α, there exists the least ordinal number $\alpha(\mathsf{L}) = \alpha$ such that $\mathsf{L}_r = \mathsf{L}_\alpha$. For $\gamma = \gamma(\mathsf{L})$ define a mapping γ_1 of X into the power set of all filters on X as follows: $\mathcal{F} \in \gamma_1(x)$ if there is a filter $\mathcal{G}$ which γ-converges to x and $\mathcal{F} \supset \mathrm{cl}_\gamma \mathcal{G}$. Clearly, γ_1 is a filter convergence structure on X.

Proposition 3.16. $\mathsf{L}_1 = \mathcal{L}(\gamma_1)$.

Proof. The assertion follows by Lemma 1.1. □

Problem 3.17. Let (X, L) be an $\mathcal{L}^f$-space. Is it true that $\gamma(\mathsf{L}_1) = \gamma_1$?

Problem 3.18. Characterize the smallest ordinal number α such that $\alpha(\mathsf{L}) \leqslant \alpha$ for each $\mathcal{L}^f$-space (X, L).

Remark 3.19. As pointed out in [FKE], the notion of linkage between a sequence S and a point via a double sequence $\mathcal{S}$ can be modified by imposing on $\mathcal{S}$ various diagonal conditions (cf. [FVO], [NOC], [FZA]). Observe that in different subcategories of $\mathcal{L}$-spaces different conditions can be used to define a suitable notion of $\mathcal{L}$-regularity and $\mathcal{L}$-regular modification.

References

[BBU] *Beattie, R. and Butzmann, H.-P.*: Sequentially determined convergence spaces, Czechoslovak Math. J. *37* (1987), 231–247.

[BBH] *Beattie, R., Butzmann, H.-P. and Herrlich, H.*: Filter convergence via sequential convergence., Comment. Math. Univ. Carolin. *27* (1986), 69–81.

[FRI] *Frič, R.*: Regularity and extension of mappings in sequential spaces., Comment. Math. Univ. Carolin. *15* (1974), 161–171

[FKE] *Frič, R. and Kent, D.C.*: Regularity and extension of maps., Math. Slovaca, to appear.

[FKO] *Frič, R. and Koutník, V.*: Sequential structures. Convergence Structures and Applications to Analysis, Abh. Akad. Wiss. DDR, Abt. Math.-Naturwiss.-Technik N4, 1979, pp. 37–56.

[FNO] *Frič, R. and Novák, J.*: A tiny peculiar Fréchet space., Czechoslovak Math. J. *34* (1984), 22–27.

[FVO] *Frič, R. and Vojtáš, P.*: Diagonal conditions in sequential convergence. Convergence Structures 1984, Proc. Conf. on Convergence, Bechyně, 1984, Akademie-Verlag, Mathematical Research/Mathematische Forschung Bd. 24, Berlin, 1985, pp. 77–94.

[FZA] *Frič, R. and Zanolin, F.*: Strict completions of $\mathcal{L}_0^*$-groups., Czechoslovak Math. J., to appear.

[HER] *Herrlich, H.*: Sequential structures: Categorical aspects. Convergence Structures 1984, Proc. Conf. on Convergence, Bechyně, 1984, Akademie-Verlag, Mathematical Research/Mathematische Forschung Bd. 24, Berlin, 1985, pp. 165–176.

[KRI] *Kent, D.C. and Richardson, G.D.*: The regularity series of a convergence space., Bull. Austral. Math. Soc. *13* (1975), 21–44.

[KNE] *Kneis, G.*: Completion functors for categories of convergence spaces, I, Acceptability functors., Math. Nachr. *129* (1986), 283-312.

[NOR] *Novák, J.*: Regular space on which every continuous function is constant, Časapis. Pěst. Mat. Fys. *73* (1948), 56–68. (In Czech.)

[NOC] *Novák, J.*: Convergence of double sequences. Convergence structures 1984 (Proc. Conf. on Convergence, Bechyně, 1984), Akademie-Verlag, Mathematical Research/Mathematische Forschung Bd. 24, Berlin.

[PRE] *Preuss, G.*: Theory of topological structures. D., Reidel Publ. Co., Dordrecht, pp. 1987.

Authors' addresses: Matematický ústav SAV, Grešákova 6, 040 01 Košice, Czechoslovakia; Department of Mathematics, Washington State University, Pullman, WA 99164-3113, U.S.A.