Czechoslovak Mathematical Journal

A. M. Fink

Bounds on the deviation of a function from its averages

Czechoslovak Mathematical Journal, Vol. 42 (1992), No. 2, 289–310

Persistent URL: http://dml.cz/dmlcz/128336

Czechoslovak Mathematical Journal, 42 (117) 1992, Praha

BOUNDS ON THE DEVIATION OF A FUNCTION FROM ITS AVERAGES

A. M. FINK, Ames

(Received August 13, 1990)

1. INTRODUCTION

Ostrowski's Inequality, Ostrowski [4] is

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant \left(\frac{1}{4} + \frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right)(b-a)\|f'\|_\infty. \tag{1}$$

This is a best possible inequality since the term in ( ) cannot be replaced by a smaller function. On geometrical grounds Mahajani [2] proved that if $\int_a^b f(x)\,\mathrm{d}x = 0$, then on $[a,b]$

$$\left|\int_a^x f(t)\,\mathrm{d}t\right| \leqslant \frac{(b-a)^2}{8}\|f'\|_\infty. \tag{2}$$

If further $f(a) = f(b) = 0$ then $\frac{1}{8}$ may be replaced by $\frac{1}{16}$. We argue that the last two are also best possible by embedding these inequalities into a family of (best possible) inequalities. In this family f' is replaced by $f^{(n)}$ and $\|\ \|_\infty$ is replaced by any of the usual p-norms $\|\ \|_p$. Moreover we also consider bounds on the quantity

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2}. \tag{3}$$

Inequalities of this type come from Iyengar [3].

Generalizations of the above three classes of inequalities appear in Milovanovič [4], [5] and Milovanovič and Pečarić [6], [7]: They use Taylor's formula with the Lagrange form of the remainder to derive formuli that express the quantity to be bounded in terms of $f^{(n)}(\xi)$. It is easy to lose the cases of equality this way. It is more appropriate to use Taylor's formula with integral remainder. We get best possible results because we know two things. We know the cases of equality in Hölder's inequality and we know the characterization of best approximations of functions by polynomials. These two facts combine in nice ways to get our best possible inequalities.

2. On Ostrowski's Inequality

We embed Ostrowski's inequality (1) into a family of inequalities by consideration of inequalities

$$\left|\frac{1}{n}\Big(f(x)+\sum_{k=1}^{n-1}F_k(x)\Big)-\frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant K(n,p,x)\|f^{(n)}\|_p, \tag{4}$$

where

$$F_k(x)=\frac{n-k}{k!}\,\frac{f^{(k-1)}(a)(x-a)^k-f^{(k-1)}(b)(x-b)^k}{b-a} \tag{5}$$

so that we are estimating a "two point expansion of f". For $n=1$ we take the sum to be zero. We reserve the notation $K(n,p,x)$ for the best possible constant. As is usual, we take $\dfrac{1}{p}+\dfrac{1}{p'}=1$ with $p'=1$ when $p=\infty$, $p'=\infty$ when $p=1$, and $\|f^{(n)}\|_p=\left(\int_a^b|f^{(n)}(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}$ with $\|f^{(n)}\|_\infty=\operatorname*{ess\,sup}_{a\leqslant t\leqslant b}|f^{(n)}(t)|$. Let $B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t$ be the beta function.

Theorem A. *Let $f^{(n-1)}(t)$ be absolutely continuous on $[a,b]$ with $f^{(n)}\in L^p(a,b)$. Then the inequality (4) holds with*

$$K(n,p,x)=\frac{1}{n!}\,\frac{[(x-a)^{np'+1}+(b-x)^{np'+1}]^{1/p'}}{b-a}\,B((n-1)p'+1,p'+1)^{1/p'}, \tag{6}$$

if $1<p\leqslant\infty$, and

$$K(n,1,x)=\frac{(n-1)^{n-1}}{n^n\ n!}\,\frac{\max\{(x-a)^n,(b-x)^n\}}{b-a}. \tag{7}$$

Moreover, for $p>1$ the inequality (4) is best possible in the strong sense that for any $x\in[a,b]$ there is an f for which equality holds at x.

Before we offer a proof, we note that $K(1,\infty,x)=\dfrac{(x-a)^2+(b-x)^2}{2(b-a)}=\left[\dfrac{1}{4}+\dfrac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)$ so that inequality (1) is best possible.

Proof of Theorem A. We start with Taylor's formula

$$f(x) = f(y) + \sum_{k=1}^{n-1} \frac{f^{(k)}(y)(x-y)^k}{k!} + \frac{1}{(n-1)!}\int_y^x (x-t)^{n-1} f^{(n)}(t)\,\mathrm{d}t$$

and integrate with respect to y. To integrate the last term we may write $\int_a^b \mathrm{d}y \int_y^x \mathrm{d}t = \int_a^x \mathrm{d}y \int_y^x \mathrm{d}t + \int_x^b \mathrm{d}y \int_y^x \mathrm{d}t = \int_a^x \mathrm{d}t \int_a^t \mathrm{d}y - \int_x^b \mathrm{d}y \int_x^y \mathrm{d}t = \int_a^x \mathrm{d}t \int_a^t \mathrm{d}y - \int_x^b \mathrm{d}t \int_t^b \mathrm{d}y$. In this way

$$\text{(8)} \quad f(x)(b-a) = \int_a^b f(y)\,\mathrm{d}y + \sum_{k=1}^{n-1} I_k + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t)(x-t)^{n-1}k(t,x)\,\mathrm{d}t,$$

where

$$I_k(x) = \int_a^b \frac{f^{(k)}(y)(x-y)^k}{k!}\,\mathrm{d}y$$

and

$$\text{(9)} \qquad k(t,x) = \begin{cases} t-a, & a \leqslant t \leqslant x \leqslant b; \\ t-b, & a \leqslant x < t \leqslant b. \end{cases}$$

The form of I_k suggests an integration by parts (see [6]). Then

$$I_k = I_{k-1} - (b-a)F_k(n-k)^{-1}, \quad 1 \leqslant k \leqslant n-1.$$

Write this as $(n-k)(I_k - I_{k-1}) = -(b-a)F_k$ and sum from 1 to $(n-1)$. Simplification leads to $\sum_{k=1}^{n} I_k = -(b-a)\sum_{k=1}^{n-1} F_k + (n-1)I_0$. Insert this identity into (8) and rearrange to get

(10)

$$\frac{1}{n}\Big(f(x) + \sum_{k=1}^{n-1} F_k(x)\Big) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y = \frac{1}{n!(b-a)}\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\,\mathrm{d}t.$$

For fixed x, we apply Hölder's inequality ($1 < p \leqslant \infty$) and

$$\text{(11)} \quad \Big|\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\,\mathrm{d}t\Big| \leqslant \|f^{(n)}\|_p \Big(\int_a^b (x-t)^{(n-1)p'}|k(t,x)|^{p'}\,\mathrm{d}t\Big)^{\frac{1}{p'}},$$

with equality when $|f^{(n)}(t)|^p = A|(x-t)^{n-1}k(t,x)|^{p'}$. For $p = 1$ or ∞, the obvious interpretation should be taken. The integral on the right hand side of (11) needs to be calculated. Write it as

$$\int_a^x (x-t)^{(n-1)p'}(t-a)^{p'}\,\mathrm{d}t + \int_x^b (t-x)^{(n-1)p'}(b-t)^{p'}\,\mathrm{d}t.$$

In the first of these let $t = sx + (1-s)a$ and in the second let $t = ub + (1-u)x$. Then

$$\left(\int_a^b |(x-t)^n k(t,x)|^{p'}\,\mathrm{d}t\right)^{\frac{1}{p'}} = [(x-a)^{np'+1}+(b-x)^{np'+1}]^{\frac{1}{p'}}\left(\int_0^1 (1-s)^{(n-1)p'}s^{p'}\,\mathrm{d}s\right)^{\frac{1}{p'}}$$

and the equation (6) follows. Equality holds in (10) when

$$f^{(n)}(t) = |(x-t)^{(n-1)}k(t,x)|^{p'-1}\operatorname{sgn}\{(x-t)^{n-1}k(t,x)\}$$

so for a fixed x we have an extremal. This holds for $1 < p \leqslant \infty$. For $p = 1$, (11) is replaced by

$$\left|\int_a^b (x-t)^{n-1}k(t,x)f^{(n)}(t)\,\mathrm{d}t\right| \leqslant \|f^{(n)}\|_1 \sup_{a\leqslant t\leqslant b} |(x-t)^{n-1}k(t,x)|.$$

It is an elementary exercise to show that the supremum is $\frac{1}{n}\left(\frac{n-1}{n}\right)^{n-1}\max\{(x-a)^n,(b-x)^n\}$. To argue that this is best possible one should take $f^{(n)}(t) = \delta(t-t_0)$ where t_0 is the point that gives the supremum. If $x > \frac{a+b}{2}$, $t_0 = \frac{1}{n}x + \frac{n-1}{n}a$ we take $f_0(t) = \frac{(t-t_0)_+^{n-1}}{(n-1)!}$ where $x_+ = \max(0,x)$. A calculation shows that the left hand side of (4) is equal to $\frac{(n-1)^{n-1}}{n!n}\cdot\frac{(x-a)^n}{b-a}$ for f_0. Since f_0 is not an admissible function we approximate $\delta(t-t_0)$ by

$$f_\varepsilon^{(n)}(t) \equiv \begin{cases} \varepsilon^{-1}, & t \in (t_0-\varepsilon, t_0) \\ 0, & \text{else} \end{cases}$$

and take $f_\varepsilon^j(a) = 0$, $n = 0, \ldots, n-1$. As $\varepsilon \downarrow 0$, $f_\varepsilon^{(j)}$ converges to $f_0^{(j)}(x)$ proving that (7) gives the best possible constant. □

Milovanič and Pečarič [6] have obtained the above result for $p = \infty$ but were not able to establish the cases of equality.

The inequality (1) when specialized to the functions for which $\int_a^b f(x)\,\mathrm{d}x = 0$, and $f(a) = f(b) = 0$ yields the estimate

$$|f(x)| \leqslant \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right](b-a)\|f'\|_\infty.$$

This is far from being best possible since we will eventually show for example, that

$$|f(x)| \leqslant \frac{x-a}{b-a}\|f'\|_\infty \quad \text{if } 0 \leqslant \frac{x-a}{b-a} \leqslant \frac{1}{4}.$$

In a similar way, suppose we consider the class of functins f such that $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$. Then Theorem A gives

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant K(n,p,x)\|(f^{(n)}\|_p.$$

This inequality is no longer best possible since the case of inequality in (1) does not allow a primitive to satisfy the boundary conditions. It is instructive to consider the simplest case as an example. Suppose $f(a) = f(b) = 0$ is required. For $n = p = 1$, Theorem A gives

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant \frac{\|f'\|_\infty}{2}\frac{(x-a)^2 + (v-x)^2}{b-a} \tag{12}$$

with equality at x if

$$f^{-1}(t) = \operatorname{sgn} k(t,x) = \begin{cases} 1, & t \leqslant x, \\ -1, & t \geqslant x. \end{cases}$$

Then

$$f(t) = \begin{cases} f(a) + (t-a), & t \leqslant x, \\ f(a) + (x-a) + (x-t), & t \geqslant x. \end{cases} \tag{13}$$

If f is required to have a zero at a and b, then from (13) we must have $x = \dfrac{a+b}{2}$. We can remedy this impasse in the following way. If g is a function with a primitive that is zero at a and b then $\int_a^b g = 0$ is necessary. It is as well sufficient for such a primitive. Indeed $G(x) = \int_a^x g(t)\,\mathrm{d}t$ is zero at a and b if and only if $\int_a^b g(t)\,\mathrm{d}t = 0$. The counterpart to equation (1) is

$$f(x)(b-a) = \int_a^b f(y)\,\mathrm{d}y + \int_a^b f'(t)k(t,x)\,\mathrm{d}t.$$

If $f(a) = f(b) = 0$ then $\int_a^b f'(t)\,\mathrm{d}t = 0$ and we may equally as well write

$$f(x) = \frac{1}{b-a}\int_a^b f'(t)[k(t,x) - \alpha]\,\mathrm{d}t$$

for any constant α. Select $\hat{\alpha}$ so that

$$\int_a^b \operatorname{sgn}[k(x,t) - \hat{\alpha}]\,\mathrm{d}t = 0, \tag{14}$$

then

$$\left|\int_a^b f'(t)[k(t,x) - \alpha]\,\mathrm{d}t\right| = \|f'\|_\infty \int_a^b |k(t,x) - \hat{\alpha}|\,\mathrm{d}t,$$

if $f'(t) = \operatorname{sgn}[k(t,x) - \hat{\alpha}]$. But this is also the criteria so that $\int\limits_a^x f'(t)\,\mathrm{d}t$ satisfies $f(a) = f(b) = 0$. If $f(a) = f(b) = 0$ then

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant \frac{\|f'\|_\infty}{b-a}\int_a^b |k(t,x) - \hat{\alpha}|\,\mathrm{d}t \tag{15}$$

is a best possible inequality in the strong sense that for each x, there is an f for which equality holds at x. To compute the integral $\int_a^b |k(t,x) - \hat{\alpha}|\,\mathrm{d}t$ we normalize by letting $t = sb + (1-s)a$ to get

$$\int_a^b |k(x,t) - \hat{\alpha}|\,\mathrm{d}t = (b-a)^2 \int_0^1 |g(s,s_0) - \alpha^1|\,\mathrm{d}s,$$

where

$$g(s,s_0) = \begin{cases} s, & s \leqslant s_0, \\ s-1, & s_0 \leqslant s, \end{cases} \qquad s_0 = \frac{x-a}{b-a};$$

and $\alpha^1 = \dfrac{\hat{\alpha}}{b-a}$. It is readily verified that $\alpha^1 = s_0 - \frac{1}{2}$ gives the minimum. The extremal has

$$f'(t) = \begin{cases} 1, & a \leqslant t \leqslant x - \dfrac{a+b}{2} \quad \text{on } x \leqslant t \leqslant b, \\[2ex] -1, & x - \dfrac{a+b}{2} < t < x, \end{cases}$$

if $x > \dfrac{b+a}{2}$. A similar statement if $x < \dfrac{b+a}{2}$. Furthermore $\int_0^1 |g(s,s_0) - \alpha^1|\,\mathrm{d}s = \frac{1}{4}$ independent of s_0. We have the best possible inequality

$$(16)\qquad \left|f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant \frac{(b-a)^1}{4}\|f'\|_\infty,$$

if $f(a) = f(b) = 0$. This is a vast improvement over (12). The condition (14) is central to the above argument. This condition is a necessary and sufficient that

$$\min_\beta \int_a^b |k(t,x) - \beta|\,\mathrm{d}t = \int_a^b |k(t,x) - \hat{\alpha}|\,\mathrm{d}t,$$

see Lorentz [8, page 112].

We recall the more general theorem that if π_n is the set of polynomials of degree at most n then

$$\min_{p\in\pi_n} \|g(t) - p(t)\|_p = \|g(t) - p_0(t)\|_p$$

if and only if $|g(t) - p_0(t)|^{p-1}\operatorname{sgn}(g(t) - p_0(t)) \perp \pi_n$. A function $h(t) \perp \pi_n$ if $\int_a^b h(t)p(t)\,\mathrm{d}t = 0$ for all $p \in \pi_n$. This holds also for $p = 1$ in that the condition is always sufficient but necessary only if $g(t) - p_0(t) = 0$ on at most a set of measure zero. This was one fact used in the proof of (16). The second fact is given in the next lemma.

Lemma 1. *Consider the boundary conditions:*

$$(17)\qquad f^{(j)}(a) = f^{(j)}(b) = 0, \quad j = 0,\dots,n-1$$

and

$$(18)\qquad f \perp \pi_0.$$

Then

(i) *g is the n^{th} derivative of a function that satisfies* (17) *if and only if $g \perp \pi_{n-1}$;*

(ii) *g is the n^{th} derivative of a function that satisfies* (17) *and* (18) *if and only if $g \perp \pi_n$*

Proof. Let $g = f^{(n)}$ where f satisfies (17). Then

$$(19)\qquad f(x) = \frac{1}{(n-1)!}\int_a^b (x-t)_+^{n-1} g(t)\,\mathrm{d}t,$$

since $f^{(j)}(a) = 0$, $j = 0, \ldots, n-1$. We derive

$$(20) \qquad f^{(j)}(b) = \frac{1}{(n-1-j)!} \int_a^b (b-t)_+^{n-1-j} g(t)\,\mathrm{d}t, \quad j = 0, \ldots, n-1.$$

By (17), $g \perp (b-t)^j$, $j = 0, \ldots, n-1$. We conclude that $g \perp \pi_{n-1}$ where each polynomial is written in its Taylor expansion about b. Conversely, if $f \perp \pi_{n-1}$, define $f(t)$ by (19). We see that $f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$ by (20). This completes the proof of (i).

To prove (ii) we integrate (19) to arrive at

$$\int_a^b f(x)\,\mathrm{d}x = \frac{1}{(n-1)!} \int_a^b g(t)\,\mathrm{d}t \int_a^b (x-t)_+^{n-1}\,\mathrm{d}x = \frac{1}{n!} \int_a^b g(t)(b-t)^n\,\mathrm{d}t.$$

This shows that $f \perp 1$ if and only if $g \perp (b-t)^n$ (given that (17) holds). Combining this with (i) finishes the proof. $\square$

We are now ready to get a variant of our generalization of Ostrowski's inequality.

Theorem B. *Let $f^{(n-1)}$ be absolutely continuous and $f^{(n)} \in L^p(a,b)$ with $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$. Then*

$$(21) \qquad \left| \frac{1}{n} f(x) - \frac{1}{b-a} \int_a^b f(y)\,\mathrm{d}y \right| \leqslant \frac{M(n,p,x)}{n!} \|f^{(n)}\|_p,$$

where (see (9))

$$(22) \qquad M(n,p,x) = \min_{q \in \pi_{n-1}} \frac{\|(x-t)^{n-1} k(t,x) - q(t)\|_{p'}}{b-a}.$$

If in addition $\int_a^b f(y)\,\mathrm{d}y = 0$, then

$$(23) \qquad |f(x)| \leqslant \frac{C(n,p,x)}{(n-1)!} \|f^{(n)}\|_p,$$

where

$$(24) \qquad C(n,p,x) = \min_{q \in \pi_n} \frac{\|(x-t)^{n-1} k(t,x) - q(t)\|_{p'}}{b-a}.$$

All of these inequalities are best possible if $1 < p < \infty$. If $p = \infty$ (21) is always best possible.

P r o o f. If f satisfies the zero conditions at a and b then see (10)

$$\frac{1}{n}f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y = \frac{1}{n!(b-a)}\int_a^b f^{(n)}(t)(x-t)^{n-1}k(t,x)\,\mathrm{d}t. \tag{25}$$

In view of the Lemma the last integral may be replaced by

$$\int_a^b f^{(n)}(t)[(x-t)^{n-1}k(t,x) - q(t)]\,\mathrm{d}t \tag{26}$$

for any $g \in \pi_{n-1}$. In particular we select q_0 to minimize $\|(x-t)^{n-1}k(t,x) - q(t)\|_{p'}$ (for fixed x). This means that for $1 < p' < \infty$.

$$|(x-t)^{n-1}k(t,x) - q_0(t)|^{p'-1}\operatorname{sgn}\{(x-t)^{n-1}k(t,x) - q_0(t)\} \perp \pi_{n-1}. \tag{27}$$

With this choice of q we use Hölder's inequality on the integral in (26) to derive (21)–(22). If $\int_a^b f(y)\,\mathrm{d}y = 0$ then we replace π_{n-1} by π_n in the above proof. Equality will hold in Hölder's inequality when

$$f^{(n)} = A|(x-t)^{n-1}k(t,x) - q_0(t)|^{n-1}\operatorname{sgn}\{(x-t)^{n-1}k(t,x) - q_0(t)\}$$
$$\text{for } 1 < p < \infty.$$

For $p = \infty$ one caveat must be made. The inequality will be best possible only if there is a polynomial q_0 for which (27) holds. This in turn is necessary if $(x-t)^{n-1}k(t,x) - q_0(t)$ is zero only on a set of measure zero. If $q_0 \in \pi_{n-1}$ this must be the case, but if $q_0 \in \pi_n$ we cannot rule out the possibility that $q_0(t) = (x-t)^{n-1}(t-a)$, for example.

We have given formuli for the best possible constants but it is unikely that many of these can be computed explicitly. We can get upper bounds for the constants $C(n,p,x)$ and $M(n,p,x)$ by judicious choices of q. We begin with the constants $C(n,p,x)$. □

Corollary 1. *If $f \perp \pi_0$ and $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$ then for $1 \leqslant p \leqslant \infty$,*

$$|f(x)| \leqslant \frac{\|f^{(n)}\|_p}{(n-1)![(n-1)p'+1]^{\frac{1}{p'}}}\min\{(x-a)^{n-\frac{1}{p}}, (b-x)^{n-\frac{1}{p}}\}. \tag{28}$$

P r o o f. For $x \leqslant \dfrac{a+b}{2}$ we tale $q(t) = (x-t)^{n-1}(t-b)$ $(= (x-t)^{n-1}k(t,x)$ on $t \geqslant x)$. Then

$$C(n,p,x) \leqslant \frac{\left(\int_a^b [(x-t)^{n-1}[(t-a)-(t-b)]^{p'}\,\mathrm{d}t]\right)^{\frac{1}{p'}}}{b-a} = \frac{(x-a)^{n-\frac{1}{p}}}{[(n-1)p'+1]^{\frac{1}{p'}}}. \tag{29}$$

Similarly, if $x > \dfrac{a+b}{2}$ we take $q(t) = (x-t)^{n-1}(t-a)$ and we get the bound (29) with $(x-a)$ replaced by $(b-x)$.

This result improves the estimate one would get from Theorem A especially since this bound goes to zero as x approaches the end points. Moreover Corollary 1 can be used to get a Mahajani type inequality (see (2)). $\square$

Corollary 2. *If* $1 \leqslant p \leqslant \infty$, $f \perp \pi_0$ *and* $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$, *then*

$$\left|\int_a^x f(y)\,\mathrm{d}y\right| \leqslant \frac{\|f^{(n)}\|_p \min\{(x-a)^{\frac{n+1}{p'}}, (b-x)^{\frac{n+1}{p'}}\}}{(n-1)![(n-1)p'+1]^{\frac{1}{p'}}(n+\frac{1}{p'})}.$$

Proof. If $a \leqslant x \leqslant \dfrac{a+b}{2}$ we integrate the estimate (28) from a to x. If $\dfrac{a+b}{2} \leqslant x \leqslant b$ then we observe that $|\int_a^x f| = |\int_x^b f|$ since $f \perp \pi_0$, so we integrate (28) from x to b.

This is a particular neat result when $p = \infty$ for we get a bound

$$\frac{\|f^{(n)}\|_\infty}{(n+1)!} \min\{(x-a)^{n-1}, (b-x)^{n-1}\}$$

which reflects the minimum number of zeros at the end points.

Bounds for $M(n,p,x)$ are not as easy to derive as no apparent choice of $q_0 \in \pi_{n-1}$ yields a function that can be integrated. However one can use Corollary 1 since if f satisfies the zero conditions then $f' \perp 1$ and satisfies the zero conditions for $n-1$ replacing n. $\square$

Corollary 3. *If* $n \geqslant 2$ *and* $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$, *then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(a)\,\mathrm{d}y\right| \leqslant \frac{\|f^{(n)}\|_p}{(n-2)![(n-2)p'+1]^{\frac{1}{p'}}} \frac{(b-a)^{n-\frac{1}{p}}}{2^{n-\frac{1}{p}}(n-\frac{1}{p})}$$

that is, $M(n,p,x) \leqslant \dfrac{n(n-1)}{[(n-2)p'+1]^{\frac{1}{p'}}} \dfrac{(b-a)^{n-\frac{1}{p}}}{2^{n-\frac{1}{p}}(n-\frac{1}{p})}$ *for* $n \geqslant 2$.

Proof. If f satisfies the hypothesis, then $f' \perp \pi_0$ and $(f')^{(j)}(a) = (f')^{(j)}(b) = 0$ $j = 0, \ldots, n-2$ so

$$(29) \qquad |f'(t)| \leqslant \frac{\|f^{(n)}\|_p}{(n-2)![(n-2)p'+1]^{\frac{1}{p'}}} \min\{(t-a)^{n-1-\frac{1}{p}}, (b-t)^{n-1-\frac{1}{p}}\}.$$

By (8)

$$f(x)-\frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y=\frac{1}{b-a}\int_a^b k(t,x)f'(t)\,\mathrm{d}t. \tag{30}$$

If $a \leqslant x \leqslant \dfrac{a+b}{2}$, then we must estimate $\int_a^b k(t,x)(t-a)^{n-1-\frac{1}{p}}\,\mathrm{d}t$, (putting the constant aside for the moment) this estimate is

$$\int_a^x (t-a)^{n-\frac{1}{p}}\,\mathrm{d}t+\int_x^{\frac{a+b}{2}}(b-t)(t-a)^{n-1-\frac{1}{p}}+\int_{\frac{a+b}{2}}^b (b-t)^{n-\frac{1}{p}}\,\mathrm{d}t.$$

We can do this integral explicitly but it is simplest at its maximum. The derivative with respect to x is $(x-a)^{n-1-\frac{1}{p}}[2x-a-b]\leqslant 0$. We therefore take $x=a$ where this integral is $\dfrac{(b-a)^{n+1-\frac{1}{p}}}{2^{n-\frac{1}{p}}(n-\frac{1}{p})}$. We are done.

Corollary 3 gives upper bounds for $M(n,p,x)$ when $n\geqslant 2$. In next section we will compute $M(1,p,x)$ explicitly and obtain other bounds for $M(n,p,x)$. □

4. Explicit constants and estimates for special cases

In this section, we compute some of the constants explicitly. We begin by looking at $M(1,p,x)$. We computed $M(1,\infty,x)=\dfrac{(b-a)^2}{4}$ (see (16)) as an introduction to Theorem B. We may now generalize that computation

Theorem C. *For $1\leqslant p\leqslant\infty$ and $f(a)=f(b)=0$, we have the best possible inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right|\leqslant\frac{(b-a)^{\frac{1}{p'}}}{2(1+p')^{\frac{1}{p'}}}\|f'\|_p$$

that is, $M(1,p,x)=\dfrac{(b-a)^{\frac{1}{p'}}}{2(1+p')^{\frac{1}{p'}}}$, and $M(1,1,x)=\dfrac{1}{2}$. For $x\geqslant\dfrac{a+b}{2}$ an extremal is given by $(1<p)$

$$p'f(t)=\begin{cases}\left(x-\dfrac{a+b}{2}\right)^{p'}-\left(x-t-\dfrac{b-a}{2}\right)^{p'}, & a\leqslant t\leqslant x-\dfrac{b-a}{2};\\[2ex] \left(x-\dfrac{a+b}{2}\right)-\left(t-x+\dfrac{b-a}{2}\right)^{p'}, & x-\dfrac{b-a}{2}\leqslant t\leqslant x;\\[2ex] \left(x-\dfrac{a+b}{2}\right)^{p'}-\left(x-t-\dfrac{b-a}{2}\right)^{p'}, & x\leqslant t\leqslant b.\end{cases}$$

A similar formula holds when $x < \dfrac{a+b}{2}$. Any other extremal is a multiple of this one.

P r o o f. We begin with the formula (22) and compute $\min \int_a^b |k(t,x) - \hat{\alpha}|^{p'} dt$ and as in (15) we transfer to the interval $[0,1]$ by $t = sb + (1-s)a$. Then

$$\int_a^b |k(t,x) - \hat{a}|^{p'} dt = (b-a)^{1+p'} \int_0^1 |g(s,s_0) - \alpha^1|^{p'} ds$$

where

$$g(x,s_0) = \begin{cases} s, & s \leqslant s_0, \\ s-1, & s_0 \leqslant s. \end{cases}$$

For $1 < p \leqslant \infty$ the condition for α^1 is that

$$\int_0^1 |g(s,s_0) - \alpha^1|^{p'-1} \operatorname{sgn}(g(s,s_0) - \alpha^1)\, ds = 0.$$

If $s_0 > \frac{1}{2}$ this is ($\alpha' > 0$)

$$\int_0^{\alpha^1} (\alpha^1 - s)^{p'-1} ds - \int_{\alpha^1}^{s_0} (s - \alpha^1)^{p'-1} ds + \int_{s_0}^1 (\alpha^1 + 1 - s)^{p'-1} ds = 0,$$

or $(\alpha^1 + 1 - s_0)^{p'} = (s_0 - \alpha^1)^{p'}$ which makes $\alpha^1 = s_0 - \frac{1}{2}$. Then $\int_0^1 |g(s,s_0) - \alpha^1|^{p'} = \dfrac{1}{2^{p'}(1+p')}$ and the theorem follows if $p > 1$. For $p = 1$ we must find the least infinity norm. This clearly is also at $\alpha^1 = s_0 - \frac{1}{2}$ with the norm being $\frac{1}{2}$. Extremals may be computed from the recipe

$$f'(t) = |k(t,x) - \hat{\alpha}|^{p'-a} \operatorname{sgn}(k(t,x) - \hat{\alpha}),$$

where $\hat{\alpha} = (b-a)\alpha^1 = (b-a)\left[\dfrac{x-a}{b-a} - \dfrac{1}{2}\right] = x - \dfrac{a+b}{2}$.

We next look at $C(1,p,x)$. Corollary 1 gives the estimate

$$\frac{\min\{(x-a)^{\frac{1}{p'}}, (b-x)^{\frac{1}{p'}}\}}{b-a}.$$

Here we get a different estimate that is better than this one. If $f \perp \pi_0$ and $f(a) = f(b) - 0$ then since $f' \perp \pi_1$

$$f(x) \int_a^x f'(t)\, dt = \int_a^b I_{[a,x]}(t) f'(t)\, dt = \int_a^b [I_{[a,x]}(t) - \alpha t - \beta] f'(t)\, dt,$$

where $I_{[a,x]}(t)$ is the indicator function of the interval $[a,x]$ and α and β are arbitrary constants. Minimization of $\|I_{[a,x]}(t) - \alpha t - \beta\|_{p'}$ over choices of α, β leads to a pair of non-linear equations that are not solvable explicitly. We are content with an estimate for general p and get explicit numbers for $p = 1$ and ∞. □

Theorem D. *If $f(a) = f(a) = f(b)$ and $f \perp \pi_0$ then we have the best possible inequalities*

$$|f(x)| \leqslant \frac{1}{2}\|f'\|_1, \quad \textit{and}$$

$$|f(x)| \leqslant (b-a)g(x)\|f'\|_\infty, \quad \textit{where}$$

$$g(x) = \begin{cases} \dfrac{x-a}{b-a}, & a \leqslant x \leqslant \frac{3}{4}a + \frac{1}{4}b; \\ \left(\dfrac{1}{2} + \dfrac{(x - \frac{a+b}{2})^2}{(b-a)^2}\right)^{\frac{1}{2}} - \dfrac{1}{2}, & \frac{3}{4}a + \frac{1}{4}b \leqslant x \leqslant \frac{1}{4}a + \frac{3}{4}b; \\ \dfrac{b-x}{b-a}, & \frac{1}{4}a + \frac{3}{4}b \leqslant x \leqslant b. \end{cases}$$

For $1 < p < \infty$ we have

$$|f(x)| \leqslant \frac{(x-a)^{\frac{1}{p'}}(b-x)^{\frac{1}{p'}}}{[(x-a)^{p-1} + (b-x)^{p-1}]^{\frac{1}{p}}} \quad \|f'\|_p.$$

That is $C(1,1,x) = \frac{1}{2}$, $C(1,\infty,x) = (b-a)g(x)$ and

$$C(1,p,x) \leqslant \frac{(x-a)^{\frac{1}{p'}}(b-x)^{\frac{1}{p'}}}{[(x-a)^{p-1} + (b-x)^{p-1}]^{\frac{1}{p'}}}.$$

Proof. We must approximate $I_{[a,x]}$ by a linear function. If $p = 1$ then we must aproximate $I_{[a,x]}$ in the supremum norm. Since the discontinuity at x is a jump of 1, the least norm of any function is at least $\frac{1}{2}$. It is $\frac{1}{2}$ for a variety of choices, eg. the constant function $\frac{1}{2}$. We have $C(1,1,x) = \frac{1}{2}$. To compute $C(1,\infty,x)$ we must approximate $I_{[a,x]}$ with a linear function in $L_1(a,b)$. By change of scale we consider the interval $[0,1]$ with $\dfrac{x-a}{b-a}$ replaced by x. We use the sufficient condition that $\operatorname{sgn}(I_{[0,x]} - \alpha t - \beta) \perp \pi_1$, which is also necessary if $\alpha \neq 0$. We parameterize such lines $\alpha t + \beta$, $\alpha \neq 0$ by having then pass through $(t_0, 1)$ and $(t_1, 0)$ with $t_0 < x < t_1$. We allow $t_0 < 0$ and $t_1 > 1$. In the first case if $0 \leqslant t_0 \leqslant x \leqslant t_1 \leqslant 1$ the orthogonal conditions are $t_0 + t_1 = \frac{1}{2} + x$ and $t_0^2 + t_1^2 = x^2 + \frac{1}{2}$. The solutions are $t_0 = \frac{1}{4} + \frac{1}{2}x - \frac{1}{2}\sqrt{\frac{1}{2} + (x - \frac{1}{2})^2}$ and $t_1 = \frac{1}{4} + \frac{1}{2}x + \frac{1}{2}\sqrt{\frac{1}{2} + (x - \frac{1}{2})^2}$. But these satisfy $0 \leqslant t_0 \leqslant x \leqslant t \leqslant 1$ only for $\frac{1}{4} \leqslant x \leqslant \frac{3}{4}$ since they are the roots of $y^2 - (\frac{1}{2} + x)y + \frac{1}{2}(x - \frac{1}{4}) = 0$. The second case is $t_0 < 0 < x \leqslant t_1 \leqslant 1$. The orthogonal conditions are $t_1 = x + \frac{1}{2}$ and $t_1^2 = x + \frac{1}{2}$. This is $x = \frac{1}{4}$. Similarly, $0 \leqslant t_0 \leqslant x \leqslant 1 \leqslant t_1$ leads to $x = \frac{3}{4}$. When

$t_0 \leqslant 0 \leqslant x \leqslant 1 \leqslant t_1$, the orthogonal conditions are $x = \frac{1}{2}$ and $x^2 = \frac{1}{2}$. The summary is that for $\frac{1}{4} \leqslant x \leqslant \frac{3}{4}$ we get the sufficient conditions satisfied for a line $\alpha t + \beta$ with $\alpha \neq 0$, and the norm is $\pm(2t_0 - x) = \pm\left(\frac{1}{2} - \sqrt{\frac{1}{2} + (x - \frac{1}{2})^2}\right)$. For $0 \leqslant x \leqslant \frac{1}{4}$ and $\frac{3}{4} \leqslant x \leqslant 1$ the best approximation must be a constant. The constant is clearly in $[0, 1]$ and the norm is thus $(1 - \beta)x + p(1 - x)$ which is minimized by $\beta = 0$ or 1. This completes the proof of the computation of $C(1, \infty, x)$.

For $1 < p < \infty$, $\|I_{[a,x]} - \alpha t - \beta\|_{p'} = (b - a)^{\frac{1}{p'}} \|I_{[0,x]} - \alpha^1 t - \beta^1\|_{p'}$ and we estimate the latter with a best constant β^1. We assume that $0 \leqslant \beta^1 \leqslant 1$. Then $\int_0^1 |I_{[0,x]} - \beta^1|^{p'} \mathrm{d}s = g(\beta^1) = (1 - \beta^1)^{p'} x + (\beta^1)^{p'}(1 - x)$ which is minimized at $\hat{\beta} = \dfrac{x^{p-1}}{x^{p-1} + (1 - x)^{p-1}}$ and $g(\hat{\beta}) = \dfrac{x(1 - x)}{[x^{p-1} - (1 - x)^{p-1}]^{p'-1}}$. This gives the indicated estimate and completes the proof. □

The estimate of $C(1, p, x)$ in Theorem D is superior to that in Corollary 1.

The constants $M(n, p, x)$ for $n \geqslant 2$ are estimated in Corollary 3. The argument used the estimates for $C(n - 1, p, x)$ since f' satisfied these hypothesis. For $n = 1$, the conditions for f are $f(a) = f(b) = 0$. This implies that $f' \perp \pi_0$. We do not have an inequality with these boundary conditions alone. We remedy this.

Theorem E. *Suppose $f \perp \pi_0$. Then the best possible inequalities are*

$$|f(x)| \leqslant \|f'\|_p \frac{[(x - a)^{p'+1} + (b - x)^{p'+1}]^{\frac{1}{p'}}}{(b - a)(1 + p')^{\frac{1}{p'}}} \qquad 1 < p \leqslant \infty$$

and

$$|f(x)| \leqslant \|f'\|_1 \min\left\{\frac{x - a}{b - a}, \frac{b - x}{b - a}\right\}.$$

Moreover, we also have best possible inequalities

$$\left|\int_a^x f(t)\,\mathrm{d}t\right| \leqslant \frac{(b - x)(x - a)}{(b - a)^{\frac{1}{p}}} \frac{\|f'\|_p}{(1 + p')^{\frac{1}{p'}}} \qquad 1 < p \leqslant \infty$$

and

$$\left|\int_a^x f(t)\,\mathrm{d}t\right| \leqslant \frac{(b - x)(x - a)}{b - a} \|f'\|_1.$$

P r o o f. From

$$f(x) = f(a) + \int_a^x f'(t)\,\mathrm{d}t = f(a) + \int_a^b (x - t)_+^0 f'(t)\,\mathrm{d}t$$

we get

$$0=\int_a^b f(x)\,\mathrm{d}x = f(a)(b-a)+\int_a^b f'(t)\,\mathrm{d}t\int_a^b (x-t)_+^0\,\mathrm{d}x$$
$$= f(a)(b-a)+\int_a^b (b-t)f'(t)\,\mathrm{d}t.$$

We solve this for $f(a)$ and insert in the first equation to write

$$f(x)=\int_a^b g(x,t)f'(t)\,\mathrm{d}t, \tag{31}$$

where

$$g(x,t)=\begin{cases}\dfrac{t-a}{b-a}, & t\leqslant x,\\[2mm] \dfrac{t-b}{b-a}, & x\leqslant t.\end{cases}$$

We apply Hölder's inequality and compute the norm of g. These are best possible since equality holds when $|f'|^p = K|g(x,t)|^{p'}$.

To prove the second inequalities we integrate (31) to get

$$\int_a^x f(t)\,\mathrm{d}t=\int_a^b K(x,t)f'(t)\,\mathrm{d}t,$$

where

$$K(x,t)=\int_a^x g(s,t)\,\mathrm{d}t=\begin{cases}\dfrac{(t-b)(x-a)}{b-a}, & x\leqslant t,\\[2mm] \dfrac{(b-x)(a-t)}{b-a}, & t\leqslant x.\end{cases}$$

We complete the proof as above.

We note that the last inequalities of Theorem E are generalizations of (2). In particular for $p=\infty$ we get

$$\left|\int_a^x f(t)\,\mathrm{d}t\right|\leqslant \frac{(b-x)(x-a)}{2}\|f'\|_\infty,$$

which is superior to (2) except when $x=\dfrac{a+b}{2}$. We now look at the second of Mahajani's inequality (2) when $f\perp\pi_0$ and $f(a)=f(b)=0$. □

Theorem F. *Let $f \perp \pi_0$, $f(a) = f(b) = 0$, and $p = 1$, or ∞. Then*

$$\left|\int_a^x f(t)\,\mathrm{d}t\right| \leqslant h_p(x)\|f'\|_p$$

are best possible inequalities where

$$h_\infty(x) = \begin{cases} \dfrac{(x-a)^2}{2}, & a \leqslant x \leqslant \frac{3}{4}a + \frac{1}{4}b; \\ \dfrac{1}{16} - \dfrac{1}{2}\dfrac{(x - \frac{a+b}{2})^2}{(b-a)^2}, & \frac{3}{4}a + \frac{1}{4}b \leqslant x \leqslant \frac{1}{4}a + \frac{3}{4}b; \\ \dfrac{(b-x)^2}{2}, & \frac{1}{4}a + \frac{3}{4}b \leqslant x \leqslant b. \end{cases}$$

and $h_1(x) = \dfrac{(x-a)(x-b)}{b-a}$.

For $1 < p < \infty$

$$\left|\int_a^x f(t)\,\mathrm{d}t\right| \leqslant \frac{1}{(1+p')^{\frac{1}{p'}}} \min\{(x-a)^{1+\frac{1}{p'}}, (b-x)^{1+\frac{1}{p'}}\}\|f'\|_p.$$

P r o o f. We use Taylor's Theorem to write

$$\int_a^x f(t)\,\mathrm{d}t = \int_a^x (x-t)f'(t)\,\mathrm{d}t = \int_a^b (x-t)_+ f'(x)\,\mathrm{d}t$$

and

$$\int_a^x f(t)\,\mathrm{d}t = -\int_x^b f(t)\,\mathrm{d}t = \int_a^b (t-x)_+ f'(t)\,\mathrm{d}t$$

so that

$$\int_a^x f(t)\,\mathrm{d}t = \frac{1}{2}\int_a^b |t-x| f'(t)\,\mathrm{d}t. \tag{32}$$

Let $p = \infty$. If we rescale to $[0,1]$ and apply the sufficient condition for $\min \int_0^1 ||t - | - \alpha t - \beta|\,\mathrm{d}t$ with $\frac{1}{4} \leqslant x \leqslant \frac{3}{4}$ then $\operatorname{sgn}[|t-x| - (\alpha t + \beta)] \perp \pi_1$. To do this, take $\alpha t + \beta = |t - x|$ at $t = \frac{1}{4}$ and $\frac{3}{4}$.

If $x \notin [\frac{1}{4}, \frac{3}{4}]$ then the conditions cannot be satisfied so the best approximation is equal to $|t - x|$ on a set of positive measure. The choice $\alpha t + \beta = t - x$ results in the norm $2x^2$.

If $\alpha t + \beta = x - t$ we get $2(1-x)^2$ so that h_∞ is explained for $x \notin [\frac{1}{4}, \frac{3}{4}]$. If $x \in [\frac{1}{4}, \frac{3}{4}]$, h_∞ is computed with the choice indicated above. For h_1 the choice of best approximation is dictated by the values at $t = 0,\ x,\ 1$. The equi-oscillation property requires $x-\beta = \alpha x + \beta = 1 - x - \alpha - \beta$. Then $\alpha = 1-2x$ and $\beta = x^2$ so $x - \beta = x - x^2 = x(1-x)$. This computes h_1. For $1 < p < \infty$ we take $\alpha t + \beta = t - x$ or $x - t$ and compute the norm $\| \, |t-x| - (\alpha t + \beta) \, \|_{p'}$ after using Hölder's inequality in (32). This completes the proof.

Notice that $h_\infty(x)$ has its maximum at $x = \dfrac{a+b}{2}$ where it is $\frac{1}{16}$. Hence this result is superior to Mahajani's second inequality.

Finally we consider some bounds for $p = 1$. First for $C(n,1,x)$. □

Theorem G. *If $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$ and $f \perp \pi_0$ then*

$$|f(x)| \leqslant (b-a)^n \frac{\sigma^{n-1}(1-\sigma)^{n-1}}{[1-\sigma^{\frac{n-1}{n-2}}]^{n-2}} \|f^{(n)}\|_1 \quad n > 2,$$

where $\sigma = \min\left(\dfrac{x-a}{b-a}, \dfrac{b-x}{b-a}\right)$ and

$$|f(x)| \leqslant (x-a)(b-x)\|f''\|_1.$$

Proof. We need to approximate $(x-t)^{n-1}k(t,x)$ by a polynomial of degree n. But $(x-t)^{n-1}k(t,x) = (x-t)^{n-1}t + (x-t)^{n-1}b - (b-a)(x-t)^{n-1}I_{[a,x]}(t)$. Since the first two terms are in π_n we may approximate $(b-x)(x-t)^{n-1}I_{[a,x]}(t)$. By a change of scale this is $(b-a)^n \cdot (x-t)^{n-1}I_{[0,x]}$ where the last function is considered on $[0,1]$. We choose $q(t) = x^{n-1}(1-t)^n$. Now $g(t) = (x-t)^{n-1}I_{[0,x]} - x^{n-1}(1-t)^n$ for $n \geqslant 2$ has $g'(0) \leqslant 0$, $g'(x) > 0$ and $g'(t_0) = 0$ where $t_0 = \dfrac{x - x^\alpha}{1 - x^\alpha}$, $\alpha = \dfrac{n-1}{n-2}$. Further $g(0) = 0$ and $g(x) \leqslant 0$. Thus $\|g\|_\infty$ is either $|g(t_0)|$ or $|g(1)|$. It is $|g(t_0)| = \dfrac{x^{n-1}(1-x)^{n-1}}{(1-x^\alpha)^{n-1}}$. This estimate is good for x near zero. However we note that $(x-t)^{n-1}k(t,x)$ differs from $(t-x)^{n-1}I_{[x,b]}$ by a polynomial of degree n so that we can effectively replace x by $1-x$ in this estimate. Thus we replace x by $\min\left\{\dfrac{x-a}{b-a}, \dfrac{b-x}{b-a}\right\}$ to get the best estimate. For $n = 2$, $g(t) = -t(1-x)$ on $[0,x]$ and $-x(1-t)$ on $[x,1]$. Then $\|g\|_\infty = x(1-x)$. □

Now we look at bounds for $M(n,1,x)$.

Theorem H. *For $n \geqslant 4$ and $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 0, \ldots, n-1$*

$$\left|\frac{1}{n}f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant$$

$$\leqslant (b-a)^3 \frac{2}{n-2}\left(\frac{n-3}{n-2}\right)^{n-3} \max\{(x-a)^{n-3}, (b-x)^{n-3}\}\|f^{(n)}\|_1,$$

moreover for $n = 3$

$$\left|\frac{1}{3}f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant (b-a)\,2\max(x-a, b-x)\|f'''\|_1$$

and for $n = 2$ we have the best possible bound

$$\left|\frac{1}{2}f(x) - \frac{1}{b-a}\int_a^b f(y)\,\mathrm{d}y\right| \leqslant \frac{b-a}{8}\max\{(x-a)^2, (b-x)^2\}\|f''\|_1.$$

Proof. For all n $M(n,1,x) = \dfrac{1}{(b-a)}\|(x-t)^{n-1}k(t,x) - q(t)\|_\infty$, for some $q \in \pi_{n-1}$. We normalize to the unit interval to get $M(n,1,\infty) = (b-a)^{n-1}\|(s_0 - s)^{n-1}g(s,s_0) - q\|_\infty$ where

$$g(s,s_0) = \begin{cases} s, & s \leqslant s_0, \\ s-1, & s \geqslant s_0, \end{cases} \qquad s_0 = \frac{x-a}{b-a}.$$

For $n \geqslant 3$ we select $q(s) = s(1-s)(s_0-s)^{n-3}$. The function whose infinity norm we seek is

$$(s_0 - s)^{n-3}\begin{cases} s((s_0-1)^2 + \sigma^2), & s \leqslant s_0, \\ (s-1)(s_0^2 + s^2), & s \geqslant s_0. \end{cases}$$

This function is majorized by $2|s_0 - s|^{n-3}\left\{\begin{matrix} s \\ 1-s \end{matrix}\right\}$. For $n \geqslant 4$, piecewise differentiation yields the norm $\dfrac{2s_0^{n-4}}{n-2}\left(\dfrac{n-3}{n-2}\right)^{n-3}$ on $[0, s_0]$ and $\dfrac{2(1-s_0)^{n-4}}{n-4}\left(\dfrac{n-3}{n-2}\right)^{n-3}$ on $[s_0, 1]$. This yields the first inequality of the theorem.

For $n = 3$, the majorizing function has norm $2\max(x, 1-x)$. We turn to $n = 2$ where we are approximating

$$h(s) = (s_0 - s)\begin{cases} s, & s \leqslant s_0, \\ s-1, & s \geqslant s_0. \end{cases}$$

This needs to be approximated by a linear function. Now $h(s)$ is positive except at $0, s_0$, and 1. For sake of argument, we assume $s_0 \leqslant \frac{1}{2}$. The maximum of $h(s)$ is at $\dfrac{s_0+1}{2} = \hat{s}$ and is $\dfrac{(1-s_0)^2}{4}$. Consider a line that best approximates. We claim that at $\hat{s}$ it must have value $\dfrac{(1-s_0)^2}{8}$. The horizontal line gives an approximating norm of $\dfrac{(1-s_0)^2}{8}$, so the best approximate $\ell(s)$ must have $\ell(\hat{s}) \geqslant \dfrac{(1-s_0)^2}{8}$. If

$\ell(s) > \dfrac{(1-s_0)^2}{8}$, then either $\ell(0)$ or $\ell(1)$ is larger thatn $\dfrac{(1-s_0)^2}{8}$. This gives an approximation norm $> (1-s_0)\dfrac{2}{8}$. Thus the best approximate passes through $\left(s, \dfrac{(1-s_0)^2}{8}\right)$, and the horizontal line is best. If $s_0 \geqslant \frac{1}{2}$ we make a symmetric argument. The norm is max $\dfrac{(s_0^2, (1-s_0)^2)}{8}$, and we have completed the proof. □

Corollary 4. *If $n \geqslant 3$ then for $p > 1$*

$$\begin{aligned} M(n,p,x) \leqslant\ & 2(b-a)^{-(1+\frac{1}{p'})} B(1+p'(n-3), 1+p')^{\frac{1}{p'}} \\ & \times \left[(x-a)^{p'(n-2)+1} + (b-x)^{p'(n-2)+1}\right]^{\frac{1}{p'}}. \end{aligned}$$

P r o o f. We pick up the proof of Theorem II at the majorizing function. We have

$$\begin{aligned} M(n,p,x) &\leqslant (b-a)^{n-1} \| 2|s_0 - s|^{n-3} g(s,s_0) \|_{p'} \\ &= (b-a)^{n-1} 2 \left[\int_0^{s_0} s^{p'} (s_0 - s)^{(n-3)p'} \,\mathrm{d}s + \int_{s_0}^1 (1-s)^{p'} (s-s_0)^{(n-3)p'} \,\mathrm{d}s \right]^{\frac{1}{p'}} \end{aligned}$$

In the first integral we take $s = s_0 u$ and in the second $1 - s = (1-s_0)u$. Both are multiples of the beta function and we get the result by substituting $s_0 = \dfrac{x-a}{b-a}$. □

5. Iyengar's Inequality.

We now turn our attention to inequalities that estimate the quantity (3)

$$\frac{1}{b-a} \int_a^b f(x) \,\mathrm{d}x - \frac{f(a)+f(b)}{2}$$

under the hypothesis $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 1, \ldots, n-1$. This condition is omitted if $n = 1$. We first derive a representation of this quantity. We start with Taylor's Theorem

$$f(x) = f(a) + \frac{1}{(n-1)!} \int_a^b (x-t)_+^{n-1} f^{(n)}(t) \,\mathrm{d}t$$

and

$$f(x) = f(b) + \frac{1}{(n-1)!} \int_b^x (x-t)^{n-1} f^{(n)}(t) \,\mathrm{d}t = f(b) + (-1)^n \int_a^b (t-x)_+^{n-1} f^{(n)}(t) \,\mathrm{d}t.$$

Combining we arrive at

$$f(x) = \frac{f(a)+f(b)}{2} + \frac{1}{2(n-1)!}\int_a^b [(x-t)_+^{n-1} + (t-x)_+^{n-1}(-1)^n] f^{(n)}(t)\,\mathrm{d}t.$$

Integration leads to

$$\text{(33)}\quad \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2} = \frac{1}{2n!(b-a)}\int_a^b f^{(n)}(t)[(b-t)^n + (a-b)^n]\,\mathrm{d}t.$$

Lemma 2. *Let $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 1, \ldots, n-1$, $n \geqslant 2$. Then $g = f^{(n)} \perp \pi_{n-2}$. Conversely, if $g \perp \pi_{n-2}$. Then there is an f such that $f^{(n)} = g$ and $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 1, \ldots, n-1$.*

The proof is similar to that of Lemma 1 so we omit it. We may now state our result.

Theorem I. *Let $f^{(j)}(a) = f^{(j)}(b) = 0$, $j = 1, \ldots, n-1$, (no condition if $n = 1$) then for $1 \leqslant p \leqslant \infty$*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x - \frac{f(a)+f(b)}{2}\right| \leqslant \frac{R(n,p)}{n!}\|f^{(n)}\|_p$$

are best possible inequalities where

$$\text{(34)}\qquad R(n,p) = \min_{q\in\pi_{n-2}} \frac{\|(b-t)^n - (a-t)^n - q(t)\|_{p'}}{2(b-a)}.$$

Proof. We use the representation (33). In view of Lemma 2 we may modify the integral in the right by writing $\int_a^b [(b-t)^n + (a-t)^n - q(t)] f^{(n)}(t)\,\mathrm{d}t$, $q \in \pi_{n-2}$. As before we apply Hölder's inequality and we get equality when q_0 is the minimizer in (34). The details are the same as in Theorems A and B. □

We now turn our attention to computing or estimating the constants $R(n,p)$.

Corollary 5. *For $1 < p \leqslant \infty$*

$$R(1,p) = \frac{(b-a)^{1-\frac{1}{p}}}{2(1+p')^{\frac{1}{p'}}}.$$

Also

$$R(1,1) = \frac{1}{2},\quad R(2,\infty) = \frac{(b-a)^2}{16},\quad R(2,1) = \frac{b-a}{8},\quad R(2,2) = \frac{(b-a)^{\frac{3}{2}}}{3\sqrt{5}}$$

and for any p

$$R(2,p) \leqslant \frac{(b-a)^{2-\frac{1}{p}}}{4(2p'+1)^{\frac{1}{p'}}}$$

For $p > 1$

$$R(3,p) \leqslant \frac{(b-a)^{3-\frac{1}{p}}2^{\frac{1}{p}}}{16} B\Big(\frac{p}{2}+\frac{1}{2}, p'+1\Big)^{\frac{1}{p'}}.$$

For $n \geqslant 4$ and $p > 1$

$$R(n,p) \leqslant \frac{(b-a)^{n-\frac{1}{p}}}{2^{\frac{1}{p'}}} B\Big(\Big(\frac{n}{2}-1\Big)p'+\frac{1}{2}, p'+1\Big)^{\frac{1}{p'}}.$$

Finally

$$R(3,\infty) = \frac{(b-a)^3}{32} \quad \textit{and}$$

$$R(n,1) \leqslant (b-a)^{n-1}\frac{2}{n}\Big(\frac{n-2}{n}\Big)^{\frac{n-2}{2}}.$$

P r o o f. We being by rescaling to get

$$R(n,p) = \frac{(b-a)^{n-\frac{1}{p}}}{2} \min_{q\in\pi_{n-2}} \|(1-s)^n + (-s)^n - q(s)\|_{p'}$$

where the norm is on $[0,1]$. For $n = 1$ there is no approximation and $R(1,p)$ is computed directly as $\|(1-2s)\|_{p'}$. For $n = 2$ we consider $\|s^2+(1-s)^2-\alpha\|_{p'}$. If $p = \infty$ and $p' = 1$, then take $\alpha = \frac{5}{8}$. Then $s^2+(1-s)^2-\frac{5}{8} = 0$ at $s = \frac{1}{4}, \frac{3}{4}$ and $\operatorname{sgn}(s^2+(1-s)^2-\frac{5}{8}) \perp 1$. It gives the best approximation. The integration $\int_0^1 [s^2+(1-s)^2]\operatorname{sgn}(s^2+(1-s)^2-\frac{5}{8})\,\mathrm{d}s = \frac{1}{8}$ gives $R(2,\infty)$. For $R(2,1)$ we take $\alpha = \frac{3}{4}$ to compute $\min\|s^2+(1-s)^2-\alpha\|_\infty = \frac{1}{4}$. For $R(2,2)$ the minimum can be computed by differentiation of the integral. The best $\alpha = \frac{2}{3}$ and $\|s^2+(1-s)^2-\frac{2}{3}\|_2 = \dfrac{1}{3\sqrt{5}}$. To estimate $R(2,p)$ for general p we take $\alpha = \frac{1}{2}$ so that we are computing $(\int_0^1 |2(s-\frac{1}{2})^2|^{p'}\mathrm{d}s)^{\frac{1}{p'}}$. This offers no difficulties.

For $n = 3$ we approximate $(1-s)^3 - s^3$ by a polynomial of degree 1. We begin by subtracting $1-2s$ to get $s(1-s)(2s-1)$. For $1 < p < \infty$ we use this to approximate $R(3,p)$. The integral is

$$\int_0^1 s^{p'}(1-s)^{p'}|2s-1|^{p'}\mathrm{d}s.$$

We set $s = \frac{1}{2}(t+1)$ to get

$$\int_{-1}^{1} \frac{|t|^{p'}(1-t^2)^{p'}\,\mathrm{d}t}{2^{2p'+1}} = \frac{1}{2^{2p'}} \int_0^1 (t)^{p'}(1-t^2)^{p'}\,\mathrm{d}t.$$

The further substitution $t = \sqrt{u}$ reduces this to a beta function. For $R(3,\infty)$ we again begin with $g(s) = s(1-s)(2s-1)$. The line that intersect the graph of $g(s)$ at $\frac{1}{2} \pm \frac{1}{4}\sqrt{2}$ and $\frac{1}{2}$ satisfies the condition that $\operatorname{sgn}[s(1-s)(2s-1)-(\alpha s+\beta)] \perp \pi_1$. We may compute the least norm (using symmetry) by $2\int_0^{\frac{1}{2}-\frac{\sqrt{2}}{4}} g(s)\,\mathrm{d}s - 2\int_{\frac{1}{2}-\frac{\sqrt{2}}{4}g(s)}^{\frac{1}{2}} \mathrm{d}s = \frac{1}{16}$. For $n \geqslant 3$ we approximate $(1-s)^n + (-s)^n$ by $(1-s)^{n-2} + (-s)^{n-2}$. By Minkowski's inequality $\|(1-s)^n + (-s)^n - [(1-s)^{n-2} + (-s)^{n-2}]\|_{p'} \leqslant \|(1-s)^n + (1-s)^{n-2}\|_{p'} + \|(-s)^n - (-s)^{n-2}\|_{p'} = 2(\int_0^1 |s^{n-2}(1-s^2)|^{p'})^{\frac{1}{p'}}$. Again we substitute $s = \sqrt{u}$ to get a beta function. For $R(n,1)$, when we compute the infinity norm, we begin as above to get $2\|s^{n-2}(1-s^2)\|_\infty$. An easy differentiation solves this problem. □

References

[1] *A. Ostrowski*: Über die Absolutabweichung einer differentiebaren Funktion von ihrem Integralmittelwert, Comment. Math. Helv. *10* (1938), 226–227.

[2] *G. S. Mahajani*: A note on an inequality, Math. Student *6* (1938), 125–128.

[3] *K. S. K. Iyengar*: Note on an inequality, Math. Student *6* (1938), 75–76.

[4] *G. V. Milovanovič*: On some integral inequalities, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. *498–541* (1975), 119–124.

[5] *G. V. Milovanovič*: On nekim funcionalnim nejednakostim, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. *599* (1977), 1–59.

[6] *G. V. Milovanovič* and *J. E. Pečarić*: On generalizations of the inequality of A. Ostrowski and some related applications, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. *544–576* (1976), 155–158.

[7] *G. V. Milovanovič* and *J. E. Pečarić*: Some considerations on Iyengar's inequality and some related applications, Univ. Beograd. Publ. Elektrotehn, Fak. Ser. Mat. Fiz. *544–576* (1976), 166–170.

[8] *G. C. Lorenz*: Approximation of Functions, Holt, Rinehart and Winson, New York, 1966.

Author's address: Iowa State University of Science and Technology, Department of Mathematics, 400 Carver Hall, Ames, Iowa 50011-2066, U.S.A.