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GENERALIZED GREEN'S RELATIONS

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1. INTRODUCTION

We wish to extend the concept of Green's relation which plays an important role in the algebraic theory of semigroups [cf. [2], [3], and [4]). Let T be a set and W a monoid whose identity is denoted by 1 or 1_w if necessary. We say that *W acts on T from the left (right)* iff there is a map ϕ: $W \times T \to T$ such that for all $t \in T$ and $w_1, w_2 \in W$ we have

$$\phi(1, t) = t,$$
$$\phi(w_1 w_2, t) = \phi(w_1, \phi(w_2, t)) \quad [\phi(w_1 w_2, t) = \phi(w_2, \phi(w_1, t))].$$

If W is a group this definition reduces to the usual concept of a group acting on a set [5, p. 70]. It is convenient to denote $\phi(w, t)$ by wt (tw) if W acts on T from the left (right) and to call the operation *left (right) multiplication of t by w*. If two monoids U, V act on the same set T from the left and right, respectively, then for $t \in T$, $u_i \in U$, $v_i \in V (i = 1, 2)$ we have

(1.1) $\quad 1_U t = t = t 1_V ,$

(1.2) $\quad (u_1 u_2)\, t = u_1(u_2 t) ,$

(1.3) $\quad t(v_1 v_2) = (tv_1)\, v_2 .$

Further, if

(1.4) $\quad u(tv) = (ut)\, v , \quad u \in U , \quad v \in V , \quad t \in T ,$

then we say that U and V act *associatively* on T, and we call T a *$U - V$ combine*. There are numerous examples of this kind of algebraic structure. For instance,

(a) Any monoid M is obviously an $M - M$ combine.

(b) Let $M_{s,t}(R)$ denote the set of all $s \times t$ matrices with entries from a commutative ring R with unity. Then $M_s(R) = M_{s,s}(R)$ is a monoid under matrix multiplication, and for any positive integers m, n the set $M_{m,n}(R)$ is a $M_m(R) - M_n(R)$ combine if the left (right) action is defined as left (right) matrix multiplication.

(c) Let $Z[i] = \{a + bi |\, a, b \in Z\}$ be the ring of Gaussian integers. Then $Z[i]$ is a $Z[i] - Z[i]$ combine where the left multiplication is ordinary multiplication of complex numbers but the right multiplication is defined by

$$(a + bi)(v_1 + iv_2) = v_1 a + iv_1 b .$$

It is easy to verify that (1.1) through (1.4) hold.

(d) A matrix $S = [s_{ij}]$ in $M_{m,n}(\mathbb{R}^+)$ is called *substochastic* if $\sum_{j=1}^{n} s_{ij} \leqq 1$ $(i = 1, \dots, n)$ and stochastic is equality holds for all i. S is called *doubly substochastic* if both S and S^{T} are substochastic. The set of all (square) substochastic (resp. doubly substochastic) matrices in $M_n(\mathbb{R}^+)$ forms a compact Hausdorff semigroup which is denoted by $\mathscr{S}_n$ [$\mathscr{D}_n$ resp.] under matrix multiplication [cf. [7]]. Let $\mathscr{S}$ be the set of all substochastic matrices in $M_{m,n}(\mathbb{R}^+)$. It is easy to check that

$$SA \in \mathscr{S} \quad \text{for} \quad A \in \mathscr{S} \quad \text{and} \quad S \in \mathscr{S}_m ;$$

$$AS \in \mathscr{S} \quad \text{for} \quad A \in \mathscr{S} \quad \text{and} \quad S \in \mathscr{S}_n .$$

Therefore $\mathscr{S}$ is an $\mathscr{S}_m - \mathscr{S}_n$ combine. Similarly, the set of all doubly substochastic matrices in $M_{m,n}(\mathbb{R}^+)$ is a $\mathfrak{S}_m - \mathfrak{S}_n$ combine. If we consider the semigroup $\mathfrak{S}_n$ of stochastic matrices in $M_n(\mathbb{R}^+)$, then the set of all stochastic matrices in $M_{m,n}(\mathbb{R}^+)$ is an $\mathfrak{S}_m - \mathfrak{S}_n$ combine.

The following propositions indicate ways to construct new combines from given ones. Since the proofs are immediate, they are omitted.

Proposition 1.1. *If U and V are monoids and $T_1, \dots, T_k$ are $U - V$ combines, then the direct product $T = T_1 \times \dots \times T_k$ is a $U - V$ combine if the multiplications are defined coordinatewise.*

Proposition 1.2. *If the monoid acts from the left on a set T_1 and the monoid V acts from the right on a set T_2, then the direct product $T = T_1 \times T_2$ is a $U - V$ combine if the multiplications are defined by*

$$u(t_1, t_2) = (ut_1, t_2), \quad (t_1, t_2)\, v = (t_1, t_2 v)$$

for $t_1 \in T_1$, $t_2 \in T_2$, $u \in U$, and $v \in V$.

As an example let T_1 be the set of all m-dimensional stochastic column vectors, T_2 be the set of all n-dimensional stochastic row vectors, $V = \mathfrak{S}_n$, the set of all $n \times n$ (row) stochastic matrices, and $V = \mathfrak{S}^{\mathrm{T}}$ the set of all $m \times m$ column stochastic matrices. Then $T_1 \times T_2$ is an $\mathfrak{S}_n^{\mathrm{T}} - \mathfrak{S}_n$ combine if the left and right multiplications are defined as

$$P(x, y) = (Px, y) \quad \text{and} \quad (x, y)\, Q = (x, yQ)$$

for $P \in \mathfrak{S}_n^{\mathrm{T}}$, $Q \in \mathfrak{S}_n$, $x \in T_1$, $y \in T_2$.

If T is a $U - V$ combine the Green's relations $\mathscr{R}, \mathscr{L}, \mathscr{J}, \mathscr{D}$, and $\mathscr{H}$ on T are defined as follows: for any two elements $a, b \in T$

(i) $a\mathscr{R}b$ iff $a = bv_1$ and $b = av_2$ for some $v_1, v_2 \in V$;
(ii) $a\mathscr{L}b$ iff $a = u_1 b$ and $b = u_2 a$ for some $u_1, u_2 \in U$;
(iii) $a\mathscr{J}b$ iff $a = u_1 b v_1$ and $b = u_2 a v_2$ for some $u_1, u_2 \in U$, $v_1, v_2 \in V$;

(iv) $a\mathscr{H}b$ iff $a\mathscr{R}b$ and $a\mathscr{L}b$;

(v) $a\mathscr{D}b$ iff $a\mathscr{R}c$ and $c\mathscr{L}b$ for some $c \in T$.

Again by way of example suppose that $T_{m,n}(\mathbb{F})$, the set of $m \times n$ matrices over a field $\mathbb{F}$ and that U, V are the general linear groups of the appropriate orders. Then $a\mathscr{R}b$ iff a and b are column equivalent. Similarly, $a\mathscr{L}b$ iff a and b are row equivalent, and $a\mathscr{J}b$ iff a and b are (row-column) equivalent.

In Section 2 we investigate the Green's relations on a $U - V$ combine T with special reference to the question "When does $\mathscr{D} = \mathscr{L}$?" In Section 3 we investigate the Green's relations on the set of $m \times n$ nonnegative matrices $M_{m,n}(\mathbb{R}^+)$ as an $M_m(\mathbb{R}^+) - M_n(\mathbb{R}^+)$ combine. In Section 4 we study the regular elements in $M_{m,n}(\mathbb{R}^+)$.

2. GREEN'S RELATIONS AND TOPOLOGY ON A GENERAL COMBINE

Throughout this section we assume U and V are monoids acting associatively on a set T, in other words T is a $U - V$ combine. The equality of $\mathscr{D}$ with $\mathscr{J}$ for the stochastic matrices (cf. [2]) or more generally for a compact topological semigroup (cf. [3]) is known. We transfer the latter development to the case of a combine, and refer to [3] for the notions of topological semigroups.

Definition 2.1. A $U - V$ combine T is *stable* iff

(a) $a \in T$, $v \in V$, and $Ua \subset Uav$ imply that $Ua = Uav$; and

(b) $a \in T$, $u \in U$, and $aV \subset uaV$ imply that $aV = uaV$.

Lemma 2.2. *Let T be a stable $U - V$ combine, and let $a, b \in T$. Then*

(a) *$aV \subset bV \subset UaV$ implies $aV = bV$; and*

(b) *$Ua \subset Ub \subset UaV$ implies $Ua = Ub$.*

Proof. If $aV \subset bV \subset UaV$, then $b = uav$ for some $u \in U$, $v \in V$. Thus $aV \subset bV = uavV \subset uaV$. Since T is stable we have $aV = uaV$, whence $aV = bV$. Thus (a) holds. The proof of (b) is analogous.

Theorem 2.3. *If T is a stable $U - V$ combine, then $\mathscr{D} = \mathscr{J}$ in T.*

Proof. It suffices to prove that for any $a, b \in T$, $a\mathscr{J}b$ implies $a\mathscr{D}b$. If $a\mathscr{J}b$, then $UaV = UbV$, and $a = ubv$ for some $u \in U$, $v \in V$. Hence $aV = ubV$ by Lemma 2.2(a). So $a\mathscr{R}(ub)$. On the other hand we have

$$Uub \subset Ub \subset UbV = UaV = UubV \subset UubV,$$

whence $Uub = Ub$ by Lemma 2.2(b). The latter equality yields $(ub)\mathscr{L}b$. Therefore, $a\mathscr{J}b$ implies that $a\mathscr{R}(ub)$ and $(ub)\mathscr{L}b$, or $a\mathscr{D}b$.

Theorem 2.4. *Let T be a $U - V$ combine. If U is a compact monoid such that for any $a, b \in T$, $\{x \in U \mid bV \subset xaV\}$ is a closed subset of U, and if V is a compact monoid such that for any $a, b \in T$, $\{y \in V \mid Ub \subset Uay\}$ is a closed subset of V, then T is stable and $\mathscr{D} = \mathscr{J}$ in T.*

Proof. Suppose $aV \subset uaV$ for some $a \in T$ and $u \in U$. By hypothesis

$$A = \{x \in U \mid uaV \subset xaV\}$$

is a closed, hence compact, subset of U. For any $x, y \in A$ we have

$$uaV \subset xaV \subset xuaV \subset xyaV,$$

which yields $xy \in A$. Thus A is a compact subsemigroup of U, and so (cf. Theorem 1.8 of [3, p. 13]) there exists an idempotent $e \in A$. So from the definition of A we obtain that $aV \subset uaV \subset eaV$, or for any $v \in V$ there exists a $v' \in V$ such that $av = eav'$. Now $eav = e^2av' = eav' = av$, whence $aV = eaV$. Thus $aV = uaV$. This proves that $aV \subset uaV$ implies $aV = uaV$. Similarly, we can show that $Ua \subset Uav$ implies $Ua = Uav$. Therefore the $U - V$ combine T is stable, and the remaining assertion follows from Theorem 2.3.

Let us return to example (d) of Section 1 of the $\mathscr{S}_m - \mathscr{S}_n$ combine

$$T = \{a \in M_{m,n}(\mathbb{R}^+) \mid \text{a is substochastic}\} .$$

We wish to show that T is stable. As noted previously $\mathscr{S}_m$ and $\mathscr{S}_n$ are compact monoids. For any $a, b \in T$, the set

$$X = \{x \in \mathscr{S}_m \mid b\mathscr{S}_n \subset xa\mathscr{S}_n\}$$

is closed. To see this observe that if $\{x_n\} \subset X$ is a sequence which converges to $x \in \mathscr{S}_m$, then for a fixed $z \in \mathscr{S}_n$ and for each k, there is a $v_k \in \mathscr{S}_n$ such that

$$(*) \qquad bz = x_k a v_k .$$

Since $\{v_k\}$ is a sequence in the compact set $\mathscr{S}_n$ it has a subsequence which we again denote by $\{v_k\}$ which converges to $v \in \mathscr{S}_n$. Pass to the limit in (*) to obtain

$$(**) \qquad bz = xav .$$

But $z \in \mathscr{S}_n$ is arbitrary so that $b\mathscr{S}_n \subset xa\mathscr{S}_n$. Thus X is closed. Similarly, the set $\{y \in \mathscr{S}_n \mid \mathscr{S}_m b \subset \mathscr{S}_m ay\}$ is closed. Thus the hypotheses of Theorem 2.4 are satisfied.

Clearly $a\mathscr{D}b$ in a general combine T is equivalent with $a\mathscr{J}b$ plus some other condition. Such a condition is given in the next theorem.

Theorem 2.5. *Let a and b be elements of the $U - V$ combine T. Then $a\mathscr{D}b$ iff there $u, u' \in U$, $v, v' \in V$ such that*

$$\text{(i)} \qquad a = ubv , \quad b = u'av'$$

and

$$\text{(ii)} \qquad av'v = a .$$

Proof. If $a\mathscr{D}b$, then for some $c \in T$ we have $a\mathscr{R}c$ and $c\mathscr{L}b$. Thus there exists $u, u' \in U$, $v, v' \in V$ such that

$$a = cv , \quad c = av' , \quad c = ub , \quad b = u'c ,$$

whence

$$a = ubv , \quad b = u'av' , \quad \text{and} \quad av'v = a .$$

Conversely, (i) and (ii) imply

$$u'a = u'av'v = bv\,, \quad \text{and} \quad b = u'av' = bvv'\,.$$

Therefore

$$av' = ubvv' = ub \quad \text{and} \quad b = u'(av')\,,$$

while

$$a = (av')\,v \quad \text{and} \quad (av') = a(v')\,.$$

Consequently, $a\mathscr{R}(av')$ and $(av')\mathscr{L}b$. Thus $a\mathscr{D}b$.

Remark. Condition (i) is of course the statement that $a\mathscr{J}b$. The additional condition (ii) could be replaced in Theorem 2.5 by any one of the five equalities

$$uu'a = a\,, \quad u'ub = b\,, \quad bvv' = b\,, \quad ub = av'\,, \quad \text{or} \quad u'a = bv\,.$$

In fact if (i) and any one of these six equalities holds, the remaining are true.

Corollary 2.6. *The relation $\mathscr{D}$ in T is an equivalence relation.*

Proof. We consider only transitivity. By Theorem 2.5, $a\mathscr{D}b$ and $b\mathscr{D}c$ in T imply

$$a = u_1bv_1\,, \quad b = u_2av_2\,, \quad av_2 = u_1b\,;$$

$$b = u_3cv_3\,, \quad c = u_4bv_4\,, \quad \text{and} \quad bv_4v_3 = b\,,$$

whence

$$a = u_1u_3cv_3v_1\,, \quad c = u_4u_2av_2v_4\,, \quad \text{and}$$

$$a(v_2v_4v_3v_1) = u_1(bv_4v_3)\,v_1 = u_1bv_1 = a\,.$$

Therefore $a\mathscr{D}c$.

Corollary 2.7. *$a\mathscr{D}b$ in T iff*

(iii) $\qquad av' = ub\,, \quad av'v = a\,,$

(iv) $\qquad u'a = bv\,, \quad bvv' = b\,,$

where $u, u' \in U$ and $v, v' \in V$.

Proof. Use Theorem 2.5 and the observations that (iii) implies $a = ubv$, while (iv) implies $b = u'av'$.

Note that we can, of course, replace $av'v = a$ by $uu'a = a$ and $bvv' = b$ by $u'ub = b$.

A $U - V$ combine T has several kinds of subobjects. If a subset T_1 of T is a $U - V$ combine we call it a $U - V$ subcombine of T. If $U_1(V_1)$ is a submonoid of $U(V)$ then the $U - V$ combine T is also a $U_1 - V_1$ combine, the latter is called a *sub $U - V$ combine of the former.* When a and b in T have some Green's relation relative to a sub $U - V$ combine, they obviously have the same relation in the original $U - V$ combine. Since each monoid has a special submonoid — its maximal subgroup, which is the set of all invertibl elements, each $U - V$ combine has a special sub $U - V$ combine, namely a $U^0 - V^0$ combine where $U^0(V^0)$ is the maximal subgroup of $U(V)$. Denote the Green's relation on the $U^0 - V^0$ combine by $\mathscr{R}^0$, $\mathscr{L}^0$, $\mathscr{J}^0$, $\mathscr{H}^0$, and $\mathscr{D}^0$. We have the following summary.

Proposition 2.8.

(a) $\mathscr{R}^0 \subset \mathscr{R}$, $\mathscr{L}^0 \subset \mathscr{L}$, $\mathscr{J}^0 \subset \mathscr{J}$, $\mathscr{H}^0 \subset \mathscr{H}$, $\mathscr{D}^0 \subset \mathscr{D}$.

(b) $a\mathscr{R}^0 b$ *iff* $a = bv$ *for some* $v \in V^0$.

(c) $a\mathscr{L}^0 b$ *iff* $a = ub$ *for some* $u \in U^0$.

(d) $a\mathscr{J}^0 b$ *iff* $b = uav$ *for some* $u \in U^0$, $v \in V^0$.

(e) $a\mathscr{H}^0 b$ *iff* $a = bv$ *and* $a = ub$ *for some* $u \in U^0$ *and* $v \in V^0$.

(f) $a\mathscr{D}^0 b$ *iff* $b = uav$ *for some* $u \in U^0$, $v \in V^0$.

Proof. (a)–(e) are immediate. For (f) note that if $b = uav$, then $a\mathscr{R}^0(av)$ and $(av)\mathscr{L}^0 b$ by (b) and (c), whence $a\mathscr{D}^0 b$. Conversely, $a\mathscr{D}^0 b$ implies $a\mathscr{J}^0 b$ by Theorem 2.5, so that $b = uav$ by (d).

Corollary 2.9. $\mathscr{D}^0 = \mathscr{J}^0$.

3. GREEN'S RELATIONS ON $M_{m,n}(\mathbb{R}^+)$

In the remainder of this paper we shall concentrate on a particularly important combine, namely the $\mathscr{N}_m - \mathscr{N}_n$ combine $M_{m,n}(\mathbb{R}^+)$, where $\mathscr{N}_k = M_k(\mathbb{R}^+)$ is the multiplicative monoid of $k \times k$ nonnegative matrices and the left and right actions are the usual matrix multiplications. We shall investigate the generalized Green's relations on $M_{m,n}(\mathbb{R}^+)$

First let us note how we are employing the terms *nonsingular* and *invertible*. If $A \in \mathscr{N}_k$, then A is *nonsingular* iff $\det A \neq 0$. However, A is *invertible* (in $\mathscr{N}_k$) iff A^{-1} exists and is an element of $\mathscr{N}_k$. If A is invertible, then A is a monomial matrix (cf. [2, p. 67]), that is

$$A = P \operatorname{diag}(a_1, \ldots, a_k)$$

where $a_j > 0$ $(j = 1, \ldots, k)$ are the nonzero entries of a diagonal matrix and P is a permutation matrix.

Following [2] and [7] we shall say that a (finite) set S of vectors in $(\mathbb{R}^+)^n$ is *cone independent* iff no vector in S lies in the polyhedral cone generated by the remaining ones. Equivalently, S is cone independent iff no vector of S is a nonnegative linear combination of the remaining. If S consists of the columns of $A \in M_{m,n}(\mathbb{R}^+)$, then we denote by $d(A)$ the maximum number of cone independent columns of A. Consequently, $d(A^{\mathrm{T}})$ is the maximum number of cone independent rows of A. Let A' denote an $m \times d(A)$ submatrix of A with cone independent columns; $\tilde{A}'$ denotes a $d(A^{\mathrm{T}}) \times n$ submatrix of A with cone independent rows; and A_0 denotes the $d(A^{\mathrm{T}}) \times$ $\times\, d(A)$ submatrix of A which is a submatrix of both A' and $\tilde{A}'$. Such an $A'(\tilde{A}')$ is called *a greatest column (row) cone independent submatrix of* A, while A_0 is called *a greatest cone independent submatrix of* A. An important fact is that each $A \in$ $\in M_{m,n}(\mathbb{R}^+)$ is uniquely determined by A' and $\tilde{A}'$ (cf. [8, p. 97]).

It is easily seen that $A\mathscr{R}B$ in $M_{m,n}(\mathbb{R}^+)$ iff the polyhedral cone $G(A)$ in $\mathbb{R}^m$ generated by the columns of A coincides with the polyhedral cone $G(B)$ generated by the columns of B. Equivalently, $A\mathscr{R}B$ iff $d(A) = d(B)$ and $A' = B'M$ where $A'(B')$ is a greatest cone independent submatrix of $A(B)$ and M is a $d(A) \times d(A)$ (nonnegative) monomial matrix. Therefore the next two results concerning the structure of $\mathscr{R}$, $\mathscr{L}$, $\mathscr{H}$ classes in $M_{m,n}(\mathbb{R}^+)$ follow.

Theorem 3.1. *The following statements are equivalent:*

(i) $A\mathscr{R}B$ $[A\mathscr{L}B]$ *in* $M_{m,n}(\mathbb{R}^+)$;

(ii) $G(A) = G(B)$ $[G(A^\mathsf{T}) = G(B^\mathsf{T})]$;

(iii) *There exists an invertible matrix* M *in* $\mathcal{N}_{d(A)}$ $[\mathcal{N}_{d(A^\mathsf{T})}]$ *such that*

$$A' = B'M \ [\tilde{A}' = \tilde{M}B'] \ ;$$

(iv) $A'\mathscr{R}^0B'$ $[\tilde{A}'\mathscr{L}^0\tilde{B}']$ *in* $M_{n,d(A)}(\mathbb{R}^+)$ [*in* $M_{d(A^\mathsf{T}),n}(\mathbb{R}^+)$] .

Theorem 3.2. *The following are equivalent:*

(i) $A\mathscr{H}B$ *in* $M_{m,n}(\mathbb{R}^+)$;

(ii) $G(A) = G(B)$ *and* $G(A^\mathsf{T}) = G(B^\mathsf{T})$;

(iii) *There exist invertible matrices* $M \in \mathcal{N}_{d(A)}$ *and* $N \in \mathcal{N}_{d(A^\mathsf{T})}$ *such that*

$$B' = A'M \ , \quad \tilde{B}' = N\tilde{A}' \ , \quad B_0 = A_0M = NA_0 \ ;$$

(iv) $A'\mathscr{R}^0B'$, $\tilde{A}'\mathscr{L}^0\tilde{B}'$, *and* $A_0\mathscr{H}^0B_0$.

These two results are generalizations of Theorem 2.2 and Theorem 3.1 of [8] on which the other results in [8] are based. Therefore all the results obtained in [8] are true for the generalized Green's relations on $M_{m,n}(\mathbb{R}^+)$. For instance we have

(a) $d(A) = d(B)$ and $d(A^\mathsf{T}) = d(B^\mathsf{T})$ if $A\mathscr{D}B$ in $M_{m,n}(\mathbb{R}^+)$.

(b) Let V_k be the maximal subgroup of $\mathcal{N}_k$ whose elements are all the invertible (monomial) matrices; let $W = V_{d(A)} \times V_{d(A^\mathsf{T})}$ be the group direct product of $V_{d(A)}$ and $V_{d(A^\mathsf{T})}$. The set

$$W_{A_0} = \{(M, N) \in W | \ A_0M = NA_0\}$$

is a subgroup of W. The $\mathscr{H}$ class containing A, $\mathscr{H}_A$, consists of all matrices $B \in$ $\in M_{m,n}(\mathbb{R}^+)$ such that

$$B' = A'M \ , \quad \tilde{B}' = N\tilde{A}' \ , \quad \text{and} \quad (M, N) \in W_{A_0} \ .$$

Finally, the mapping $f\colon W_{A_0} \to \mathscr{H}_A$ with $f(M, N) = B$ is bijective.

The following two theorems concerning the structure of $\mathscr{J}$ and $\mathscr{D}$ classes in the combine $M_{m,n}(\mathbb{R}^+)$ are generalizations of Theorem 3.2, Proposition 3.3 and Corollary 3.4 of [1]. We can prove them by almost the same arguments as used in [1].

Theorem 3.3. $A\mathscr{J}B$ *in* $M_{m,n}(\mathbb{R}^+)$ *iff there exist nonnegative matrices* X_1, Y_1, X'_1, Y'_1 *of sizes* $d(A^\mathsf{T}) \times d(B^\mathsf{T})$, $d(B) \times d(A)$, $d(B^\mathsf{T}) \times d(A^\mathsf{T})$, *and* $d(A) \times d(B)$ *respectively such that*

$$A_0 = X_1B_0Y_1 \ , \quad B_0 = X'_1A_0Y'_1 \ .$$

Theorem 3.4. *The following are equivalent:*

(i) $A\mathscr{D}B$ *in* $M_{m,n}(\mathbb{R}^+)$;

(ii) $A_0 \mathscr{D} B_0$ *in* $M_{d(A^{\mathrm{T}}),d(B^{\mathrm{T}})}(\mathbb{R}^+)$;

(iii) *There exist* $X_1, X_1' \in \mathscr{N}_{d(A^{\mathrm{T}})}$, $Y_1, Y_1' \in \mathscr{N}_{d(A)}$ *such that*

$$A_0 = X_1 B_0 Y_1 , \quad B_0 = X_1' A_0 Y_1' ,$$

and any one of the following equalities holds:

$$X_1 X_1' A_0 = A_0 , \quad A_0 Y_1' Y_1 = A_0 , \quad X_1' X_1 B_0 = B_0 ,$$

$$B_0 Y_1 Y_1' = B_0 , \quad X_1 B_0 = A_0 Y_1' , \quad X_1' A_0 = B_0 Y_1 ;$$

(iv) $A_0 \mathscr{D} B_0$ *in* $M_{d(A^{\mathrm{T}}),d(A)}(\mathbb{R}^+)$;

(v) *There exist invertible matrices* $X \in \mathscr{N}_{d(A^{\mathrm{T}})}$, $Y \in \mathscr{N}_{d(A)}$ *such that*

$$B_0 = X A_0 Y .$$

Remark. If A_0 is a greatest cone independent submatrix of $A \in M_{m,n}(\mathbb{R}^+)$ then any greatest cone independent submatrix A_0^* of A can be expressed as

$$A_0^* = M_1 A_0 M_2$$

where $M_1 \in \mathscr{N}_{d(A^{\mathrm{T}})}$ and $M_2 \in \mathscr{N}_{d(A)}$ are monomial (cf. Theorem 3.1 of [1]). Thus $A_0^* \mathscr{J}^0 A_0$ or $A_0^* \mathscr{D}^0 A_0$ in $M_{d(A^{\mathrm{T}}),d(A)}(\mathbb{R}^+)$. Therefore Theorems 3.3 and 3.4 remain true if A_0, B_0 there are replaced by any other greatest cone independent submatrices A_0^*, B_0^* respectively.

Proposition 3.5. *If* $A \in M_{m,n}(\mathbb{R}^+)$ *and* rank $A = n \leqq m$, *then*

(i) $A \mathscr{R} B$ *iff* $A \mathscr{R}^0 B$;

(ii) *if* A *has a nonnegative left inverse, so does any* B *in* $\mathscr{R}_A$.

Proof. Since

$$n \geqq d(A) \geqq \text{rank } A = n ,$$

we have $d(A) = n$, $A' = A$, and $B' = B$. Then by Theorem 3.1, $A \mathscr{R} B$ implies $A' \mathscr{R}^0 B'$, or $A \mathscr{R}^0 B$. This proves (i).

Let $Z \in M_{m,n}(\mathbb{R}^+)$ be a left inverse of A so that $ZA = I_n$. Then for any $B \in \mathscr{R}_A$ there is by (i) an invertible matrix $M \in \mathscr{N}_{d(A)}$ such that $B = AM$. Therefore $M^{-1}Z$ is a nonnegative left inverse of B, and the proof is complete.

The next two results follow immediately.

Proposition 3.6. *If* $A \in M_{m,n}(\mathbb{R}^+)$ *and* rabk $A = m \leqq n$, *then*

(i) $A \mathscr{L} B$ *iff* $A \mathscr{L}^0 B$;

(ii) *If* A *has a nonnegative right inverse, so does any* B *in* $\mathscr{L}_A$.

Proposition 3.7. *If* $A \in \mathscr{N}_n$ *is nonsingular, then the following are equivalent*:

(i) $A \mathscr{D} B$;

(ii) $A \mathscr{D}^0 B$;

(iii) *There exist invertible matrices* X, Y *in* $\mathscr{N}_n$ *such that* $B = XAY$.

Note that Proposition 3.7 contains the known result given in Corollary (3.4.7) of [2, p. 73].

Recall that an element a of a semigroup T is regular iff $axa = a$ is solvable for some $x \in T$. Regularity is an important concept in the theory of semigroups, especially in the study of Green's relations. Regularity in $\mathcal{N}_n$ has been studied in [1] and [2]. We restate the main results as:

Theorem 4.1. *Let $A \in \mathcal{N}_n$ be of rank r. The following are equivalent:*

(a) *A is regular in $\mathcal{N}_n$.*

(b) *A has a semi-inverse in $\mathcal{N}_n$ of the form $D_1 A^T D_2$, where $D_1, D_2 \in \mathcal{N}_n$ are diagonal.*

(c) *A has a semi-inverse in $\mathcal{N}_n$ which is r – monomial, that is, the largest nonzero submatrix of the semi-inverse is a monomial matrix of order r.*

(d) *A has a monomial submatrix of order r.*

(e) *$A \mathcal{D} E_r$, where E_r is the canonical idempotent of rank r given by*

$$E_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = I_r \oplus 0 .$$

(f) *$A \mathcal{J} E_r$.*

(g) *$d(A) = d(A^T) = r$ and A_0 is regular in $\mathcal{N}_r$, where A_0 is a greatest cone independent submatrix of A.*

To formulate regularity in a general $U - V$ combine we need to add to the structure, specifically, we require T to have a conjugate combine, that is, a $V - U$ combine T' which satisfies the following condition: there exist surjective maps λ: $T \times T' \to U$ and μ: $T' \times T \to V$ such that

$$(t_1 t_1') t_2 = t_1(t_1' t_2), \quad (t_1' t_1) t_2' = t_1'(t_1 t_2')$$

for any $t_1, t_2 \in T$, $t_1', t_2' \in T'$, where $t_1 t_1'$ and $t_1' t_1$ denote $\lambda(t_1, t_1')$ and $\mu(t_1', t_1)$ respectively. The $V - U$ combine T' is called a *conjugate* of the $U - V$ combine T. As examples we may take $M_{n,m}(\mathbb{R}^+)$ as a conjugate of $M_{m,n}(\mathbb{R}^+)$, and each semigroup which is considered as a combine may be considered as a conjugate of itself.

We now define regular elements in a combine T which has a conjugate T'. This definition reduces to the original one when T is a self conjugate semigroup.

Definition 4.2. The element $a \in T$ is *regular* iff $axa = a$ is solvable for some $x \in T'$. Further, if $axa = a$ and $xax = x$ for some $x \in T'$ and $a \in T$, then a and x are said to be *semi-inverses* of each other.

It is easily seen that each regular element in a general combine has a semi-inverse. It can be shown that in a general combine if one element of a $\mathcal{D}$ class is regular, then all the elements in the $\mathcal{D}$ class are regular (cf. exercises (3.6.1) and (3.6.3) of [2, p. 83]). On the other hand elements in a general combine T which has a conjugate T' may have one sided invertibility. If $ax = 1_U$ [$xa = 1_V$] is solvable for some $a \in T$ and $x \in T'$, then $x \in T'$ is said to be a *right* [*left*] *inverse of* $a \in T$. An element in T

is *half invertible* if it has a right or left inverse. For example,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \in M_{2,3}(\mathbb{R}^+)$$

is half invertible because it has a right inverse

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_{3,2}(\mathbb{R}^+) .$$

Proposition 4.3. *If $U = V$ and an element a in a $U - V$ combine T has a right inverse x and a left inverse y, then $x = y$; that is, the half inverse in unique.*

Suppose a $U - U$ combine T is self cpnjugate. An element $a \in T$ is said to be *invertible* iff there exists an $x \in T$ such that

(4.1) $$ax = xa = 1_U .$$

By Proposition 4.3 the element satisfying (4.1) is unique. We call this *unique* x the inverse of a. When a semigroup T is considered as a $T - T$ combine, the concept of invertibility comforms to the common one.

The next result is immediate.

Proposition 4.4. *If an element $a \in T$ has a left [right] inverse, and if $b\mathscr{R}^0 a$ [$b\mathscr{L}^0 a$] in T, then b has a left [right] inverse.*

We return to considetation of the $\mathcal{N}_m - \mathcal{N}_n$ combine $M_{m,n}(\mathbb{R}^+)$ whose conjugate we take to be $M_{n,m}(\mathbb{R}^+)$. In the remainder of this paper we assume, without loss of generality, that $m \geqq n$.

Lemma 4.5. *Let $A \in M_{m,n}(\mathbb{R}^+)$. Then*

(i) *A is regular in $M_{m,n}(\mathbb{R}^+)$ iff $[A\ 0]$ is regular in $\mathcal{N}_m$, where 0 denotes the $m \times (m - n)$ zero matrix.*

(ii) *A has a semi-inverse in $M_{n,m}(\mathbb{R}^+)$ iff $[A\ 0]$ has a semiinverse in $\mathcal{N}_m$. Further A has a semi-inverse which is r − monomial, where $r =$ rank A, iff $[A\ 0]$ has a semi-inverse which is r − monomial.*

Proof. By Theorem 4.1 it suffices to prove (ii). If $X_1 \in M_{n,m}(\mathbb{R}^+)$ is a semi-inverse of A, that is, $AX_1A = A$ and $X_1AX_1 = X_1$, then the two $m \times m$ nonnegative matrices

$$[A\ 0] \quad \text{and} \quad \begin{bmatrix} X_1 \\ 0 \end{bmatrix}$$

satisfy

(4.2) $$[A\ 0]\begin{bmatrix} X_1 \\ 0 \end{bmatrix}[A\ 0] = [A\ 0],$$

and

(4.3) $$\begin{bmatrix} X_1 \\ 0 \end{bmatrix}[A\ 0]\begin{bmatrix} X_1 \\ 0 \end{bmatrix} = \begin{bmatrix} X_1 \\ 0 \end{bmatrix}.$$

Therefore $[A\ 0]$ has a semi-inverse $\begin{bmatrix} X_1 \\ 0 \end{bmatrix}$ in $\mathcal{N}_m$. It is clear that if X_1 is r – monomial, then so is

$$\begin{bmatrix} X_1 \\ 0 \end{bmatrix}.$$

On the other hand if $[A\ 0]$ has a semi-inverse

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

in $\mathcal{N}_m$, where X_1 is $n \times m$ and X_2 is $(m - n) \times n$, then

$$[A\ 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [A\ 0] = [A\ 0], \tag{4.4}$$

and

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [A\ 0] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{4.5}$$

Since (4.4) and (4.5) obviously imply (4.2) and (4.3), $X_1 \in M_{n,m}(\mathbb{R}^+)$ is therefore a semi-inverse of $A \in M_{m,n}(\mathbb{R}^+)$. If

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

is r-monomial, then X_1 must be r-monomial, otherwise $\operatorname{rank} X_1 < r = \operatorname{rank} A$, which contradicts $A = AX_1A$.

Lemma 4.6. *$A \in M_{m,n}(\mathbb{R}^+)$ has a semi-inverse in $M_{n,m}(\mathbb{R}^+)$ of the form $\operatorname{diag}(c_1, \ldots, c_n)\, A^{\mathrm{T}} \operatorname{diag}(d_1, \ldots, d_m)$ iff $[A\ 0] \in \mathcal{N}_m$ has a semi-inverse of the form $\operatorname{diag}(s_1, \ldots, s_m)\, [A\ 0]^{\mathrm{T}} \operatorname{diag}(t_1, \ldots, t_m)$, where all the diagonal matrices are non-negative.*

Proof. If $X = \operatorname{diag}(c_1, \ldots, c_n)\, A^{\mathrm{T}} \operatorname{diag}(d_1, \ldots, d_n)$ satisfies

$$AXA = A \quad \text{and} \quad XAX = X, \tag{4.6}$$

then we have

$$[A\ 0] \begin{bmatrix} X \\ 0 \end{bmatrix} [A\ 0] = [A\ 0] \quad \text{and} \quad \begin{bmatrix} X \\ 0 \end{bmatrix} [A\ 0] \begin{bmatrix} X \\ 0 \end{bmatrix} = \begin{bmatrix} X \\ 0 \end{bmatrix}, \tag{4.7}$$

where

$$\begin{bmatrix} X \\ 0 \end{bmatrix} = \operatorname{diag}(c_1, \ldots, c_n, 0, \ldots, 0)\ [A\ 0]^{\mathrm{T}} \operatorname{diag}(d_1, \ldots, d_m)$$

is a semi-inverse of $[A\ 0]$ in $\mathcal{N}_m$. Conversely, if $[A\ 0]$ hsa a semi-inverse

$$\operatorname{diag}(s_1, \ldots, s_m)\ [A\ 0]^{\mathrm{T}} \operatorname{diag}(t_1, \ldots, t_m) = \begin{bmatrix} X \\ 0 \end{bmatrix}$$

in $\mathcal{N}_m$ where X denotes $\operatorname{diag}(s_1, \ldots, s_n)\ A\ \operatorname{diag}(t_1, \ldots, t_m) \in M_{n,m}(\mathbb{R}^+)$, then (4.7).

holds. Since (4.6) is implied by (4.7), the matrix X is a semi-inverse of A which satisfies the desired conditions.

Lemma 4.7. *Let $A \in M_{m,n}(\mathbb{R}^+)$, $r = \operatorname{rank} A$, and $I_r \in \mathcal{N}_r$ be the identity matrix. Then*

(i) $$A \mathscr{D} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ in } M_{m,n}(\mathbb{R}^+) \text{ iff } [A\ 0]\ \mathscr{D} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ in } \mathcal{N}_m;$$

(ii) $$A \mathscr{J} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ in } M_{m,n}(\mathbb{R}^+) \text{ iff } [A\ 0]\ \mathscr{J} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ in } \mathcal{N}_m.$$

Proof. (i) By Theorem 3.4(ii), $A\mathscr{D}(I_r \oplus 0)$ implies that $d(A) = r = d(A^T)$ and $A_0 \mathscr{D} I_r$ in $\mathcal{N}_r$, where A_0, I_r are greatest cone independent submatrices of A and $I_r \oplus 0$ respectively. But A_0, I_r are also greatest cone independent submatrices of $[A\ 0]$ and $I_r \oplus 0$ in $\mathcal{N}_m$, whence $(A \oplus 0)\mathscr{J}(I_r \oplus 0)$ are in $\mathcal{N}_m$ by Theorem 3.4. This proves the „only if" statement. The „if" statement is proved similarly.

(ii) By Theorem 3.3, $A\mathscr{J}(I_r \oplus 0)$ is equivalent to

$$A_0 = X_1 I_r Y_1 \quad \text{and} \quad I_r = X_1' A_0 Y_1', \tag{4.8}$$

where X_1, Y_1, X_1', Y_1' are nonnegative of respective sizes $d(A^T) \times r$, $r \times d(A)$, $r \times d(A^T)$, and $d(A) \times r$. It is clear from Theorem 3.3 that (4.8) is equivalent to $(A \oplus 0)\mathscr{J}(I_r \oplus 0)$ in $\mathcal{N}_m$.

Lemma 4.8. *If $A \in M_{m,n}(\mathbb{R}^+)$ has a monomial of order $r = \operatorname{rank} A$, then this submatrix is a greatest cone independent submatrix of A.*

Proof. We have

$$PAQ = \begin{bmatrix} M & B_2 \\ B_3 & B_4 \end{bmatrix},$$

where P and Q are permutation matrices, $M \in \mathcal{N}_r$ is monomial with $M^{-1} \in \mathcal{N}_r$. Let $B_2 = MC_2$, $B_3 = C_3M$; then $C_2 = M^{-1}B_2$ and $C_3 = B_3M^{-1}$ are nonnegative. Since $r = \operatorname{rank} A = \operatorname{rank}(PAQ)$ we have

$$[B_3,\ B_4] = X[M,\ B_2],$$

where X is some real but not necessarily nonnegative matrix. Now $B_3 = XM$ and $B_3 = C_3M$ yield $X = C_3MM^{-1} = C_3$, whence

$$PAQ = \begin{bmatrix} M & MC_2 \\ C_3M & C_3MC_2 \end{bmatrix}.$$

This shows that M is a greatest cone independent submatrix of A.

Finally Theorem 4.1 and the lemmas of this section imply the following generalization of Theorem 4.1.

Theorem 4.9. *Let $A \in M_{m,n}(\mathbb{R}^+)$ be of rank r and let A_0 be a greatest cone independent submatrix of A. The following are equivalent.*

(a) *A is regular in $M_{m,n}(\mathbb{R}^+)$.*

(b) $[A\ 0]$ *is regular in* $\mathcal{N}_m$.

(c) *A has a semi-inverse in* $M_{n,m}(\mathbb{R}^+)$ *of the form* $D_1A^TD_2$, *where* $D_1 \in \mathcal{N}_n$ *and* $D_2 \in \mathcal{N}_m$ *are diagonal.*

(d) *A has a semi-inverse in* $M_{n,m}(\mathbb{R}^+)$ *which is r-monomial.*

(e) *A has a monomial submatrix of order r.*

(f) $A\mathcal{D}E_r$, *where*

$$E_r = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \in M_{m,n}(\mathbb{R}^+) \quad (E_0 = 0).$$

(g) $A\mathcal{J}E_r$.

(h) $d(A) = d(A^T) = r$ *and* A_0 *is regular in* $\mathcal{N}_r$.

(i) $A_0\mathcal{D}^0I_r$ *in* $\mathcal{N}_r$.

(j) $A_0\mathcal{J}^0I_r$ *in* $\mathcal{N}_r$.

Remark. Using the same argument as stated in the remark after Theorem 3.4, we claim that if $A \in M_{m,n}(\mathbb{R}^+)$ of rank r is regular, then any greatest cone independent submatrix A_0 is regular in $\mathcal{N}_r$, is monomial, and satisfies $A_0\mathcal{D}^0I_r$ and $A_0\mathcal{J}^0I_r$ in $\mathcal{N}_r$.

Corollary 4.10. *If* $A \in M_{m,n}(\mathbb{R}^+)$ *is regular, then the* $\mathcal{D}$ *class containing A and the* $\mathcal{J}$ *class containing A are the same; that is* $\mathcal{D}_A = \mathcal{J}_A$. *Further, all the elements of* $\mathcal{D}_A = \mathcal{J}_A$ *are regular.*

We call a $\mathcal{D}(\mathcal{J})$ class in a combine a regular $\mathcal{D}(\mathcal{J})$ class iff all its elements are regular.

Corollary 4.11. *Let* $b = \min\{m, n\}$. *The combine* $M_{m,n}(\mathbb{R}^+)$ *has exactly* $b + 1$ *regular* $\mathcal{J}$ *classes:* $\mathcal{J}_{E_r}$ $(r = 0, 1, \ldots, b)$ *and hence* $b + 1$ *regular* $\mathcal{D}$ *classes.*

The next theorem shows that half invertibility and regularity for a matrix in $M_{m,n}(\mathbb{R}^+)$ of full rank are actually the same.

Theorem 4.12. *Let* $A \in M_{m,n}(\mathbb{R}^+)$ *be of rank* $\min\{m, n\}$. *Then A is regular iff A has a nonnegative left inverse when* $m > n$, *or a nonnegative right inverse when* $m < n$, *or a nonnegative inverse when* $m = n$.

Proof. It suffices to prove this when $m > n$. If A is regular, then A has a monomial submatrix M of order $n = \operatorname{rank} A$ by Theorem 4.9(e). Then there is an $n \times n$ permutation matrix P such that

$$PA = \begin{bmatrix} M \\ A_1 \end{bmatrix},$$

whence $X = [M^{-1}\ 0]\,P^{-1} \in M_{n,m}(\mathbb{R}^+)$ is obviously a left inverse of A.

Conversely, if A has a left inverse $X \in M_{n,m}(\mathbb{R}^+)$ so that $XA = I_n$, then $AXA = A$, and A is regular.

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