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Svatoslav Staněk

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# ON THE BOUNDEDNESS AND PERIODICITY OF SOLUTIONS OF SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH A PARAMETER

SVATOSLAV STANĚK, Olomouc

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## 1. PROBLEM

Let $t_0 \in \mathbb{R}$, let $X_1$, $X_2$ be subsets of $C^0(\mathbb{R})$ and $I = \langle a, b\rangle$ $(-\infty < a < b < \infty)$. Consider the functional differential equation

$$y'' - q(t)y = F[y, y', \mu] \tag{1}$$

where $F\colon X_1 \times X_2 \times I \to C^0(\mathbb{R})$, $q(t) > 0$ for $t \in \mathbb{R}$, containing the parameter $\mu$. The problems considered are to determine sufficient conditions on $q$ and $F$ that would make possible to choose the parameter $\mu$ so that there exist

a) a solution of (1) vanishing at the point $t_0$ and such that $y$ and $y'$ are bounded on $\mathbb{R}$,

b) a periodic solution of (1) vanishing at the point $t_0$.

The same problems are considered for equation (1) where $F[y, z, \mu]$ does not depend on $z$.

In the special case with $F[y, z, \mu](t) = f(t, y(t), z(t), \mu)$, where $f(t, y, z, \mu)\colon \mathbb{R}^3 \times I \to \mathbb{R}$, the above formulated problems have been considered in [2] and [3].

## 2. NOTATION, LEMMAS

Let $t_0 \in \mathbb{R}$ and let $u, v$ be solutions of the equation

$$y'' = q(t)y, \quad q \in C^0(\mathbb{R}), \quad q(t) > 0 \quad \text{for } t \in \mathbb{R}, \tag{q}$$

$u(t_0) = 0$, $u'(t_0) = 1$, $v(t_0) = 1$, $v'(t_0) = 0$. Setting

$$r(t, s) = u(t)v(s) - u(s)v(t) \ \bigl( = -r(s, t)\bigr),$$

$$r_1'(t, s) = u'(t)v(s) - u(s)v'(t) \ \Bigl( = \frac{\partial r}{\partial t}(t, s)\Bigr)$$

for $(t,s) \in \mathbf{R}^2$, then $r(t,s) > 0$ for $t > s$, $r(t,s) < 0$ for $t < s$, $r_1'(t,s) > 1$ for $t \neq s$ and $r_1'(t,t) = 1$ for $t \in \mathbf{R}$ (see Lemma 1, [1]).

Denote by $Y_1(Y_0)$ the Fréchet space of all continuously differentiable (continuous) functions on $\mathbf{R}$ with the usual metric topology, and let $X_1(X_0)$ be the subset of $Y_1(Y_0)$ defined by

$$X_1 = \{y;\ y \in Y_1, y \text{ and } y' \text{ bounded on } \mathbf{R}\}$$
$$(X_0 = \{y;\ y \in Y_0, y \text{ bounded on } \mathbf{R}\}).$$

Let $F\colon X_1 \times X_0 \times I \to Y_0$, $F\colon [y,z,\mu] \to F[y,z,\mu](t)$ be an operator satisfying some of the following assumptions: there exist positive constants $r_0$, $r_1$ such that

(i) $F$ is a continuous operator on $D \times I$, where $D = \{(y,y');\ y \in Y_1,\ |y^{(i)}(t)| \leqslant r_i$ for $t \in \mathbf{R}$ and $i = 0,1\}$, that is, if $\{y_n\}$, $\{\mu_n\}$, $(y_n, y_n') \in D, \mu_n \in I$ are convergent sequences and $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} \mu_n = \mu_0$, then $\lim_{n\to\infty} F[y_n, y_n', \mu_n] = F[y, y', \mu_0]$;

(ii) $|F[y,z,\mu](t)| \leqslant q(t)r_0$ for $(y,z,\mu) \in H \times I$ and $t \in \mathbf{R}$, where $H = \{(y,z);\ y \in X_1, z \in X_0,\ |y(t)| \leqslant r_0,\ |z(t)| \leqslant r_1$ for $t \in \mathbf{R}\}$;

(iii) $F[y,y',\mu_1](t) < F[y,y',\mu_2](t)$ for $(y,y') \in D$, $t \in \mathbf{R}$, $\mu_1, \mu_2 \in I$, $\mu_1 < \mu_2$;

(iv) $F[y,y',a](t).F[y,y',b](t) \leqslant 0$ for $(y,y') \in D$, $t \in \mathbf{R}$;

(v) $2\sqrt{r_0}\sqrt{A + Qr_0} \leqslant r_1$, where $Q = \sup\{q(t);\ t \in \mathbf{R}\}, A = \sup\{|F[y,y',\mu](t)|;\ (y,y',\mu) \in D \times I,\ t \in \mathbf{R}\}$ $(\leqslant Qr_0)$;

(vi) $F[y,y',\mu](t)$ is an $\omega$-periodic function for every $(y,y') \in D$, $\mu \in I$, where $y$ is $\omega$-periodic.

When $F[y,z,\mu] = G[y,\mu]$ does not depend on $z$ we assume that $G$ satisfies some of the following assumptions:

there exists a positive constant $r_0$ such that

(j) $G$ is a continuous operator on $P \times I$, where $P = \{y; y \in X_0, |y(t)| \leqslant r_0 \text{ for } t \in \mathbf{R}\}$, that is, if $\{y_n\}$, $\{\mu_n\}$, $y_n \in P$, $\mu_n \in I$ are convergent sequences, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} \mu_n = \mu_0$, then $\lim_{n\to\infty} G[y_n, \mu_n] = G[y, \mu_0]$;

(jj) $|G[y,\mu](t)| \leqslant q(t)r_0$ for $(y,\mu) \in P \times I$ and $t \in \mathbf{R}$;

(jjj) $G[y,\mu_1](t) < G[y,\mu_2](t)$ for $y \in P$, $t \in \mathbf{R}$, $\mu_1, \mu_2 \in I$, $\mu_1 < \mu_2$;

(ju) $G[y,a](t).G[y,b](t) \leqslant 0$ for $y \in P$, $t \in \mathbf{R}$;

(u) $G[y,\mu](t)$ is an $\omega$-periodic function for every $\omega$-periodic function $y \in P$ and $\mu \in I$.

**Lemma 1.** *Let* $t_1$, $t_2 \in \mathbf{R}$, $t_1 < t_0 < t_2$. *If assumptions* (i)–(v) *hold for positive constants* $r_0$, $r_1$, *then for every* $\varphi$, $(\varphi, \varphi') \in D$ *there exists a unique* $\mu_0 \in I$ *such that the equation*

$$y'' - q(t)y = F[\varphi, \varphi', \mu](t) \tag{2}$$

*with $\mu = \mu_0$ has on the interval $\langle t_1, t_2\rangle$ a solution $y$ (which is then unique) satisfying*

$$y(t_1) = y(t_0) = y(t_2) = 0. \tag{3}$$

*Moreover, $|y^{(i)}(t)| \leqslant r_i$ for $t \in \langle t_1, t_2\rangle$ and $i = 0, 1$.*

P r o o f. Let $(\varphi, \varphi') \in D$. Setting $h(t, \mu) = F[\varphi, \varphi', \mu](t)$ for $(t, \mu) \in \langle t_1, t_2\rangle \times I$, Lemma 1 follows from Lemma 4 [1]. □

R e m a r k 1. The solution $y$ in the assertion of Lemma 1 may be written in the form

$$y(t) = \frac{r(t, t_0)}{r(t_0, t_1)} \int_{t_0}^{t_1} r(t_1, s) F[\varphi, \varphi', \mu_0](s)\, \mathrm{d}s + \int_{t_0}^{t} r(t, s) F[\varphi, \varphi', \mu_0](s)\, \mathrm{d}s.$$

**Lemma 2.** *Let $t_1, t_2 \in \mathbf{R}$, $t_1 < t_0 < t_2$. If assumptions* (j)–(ju) *hold for a positive constant $r_0$, then for every $\varphi \in P$ there exists a unique $\mu_0 \in I$ such that the equation*

$$y'' - q(t)y = G[\varphi, \mu](t) \tag{4}$$

*with $\mu = \mu_0$ has on the interval $\langle t_1, t_2\rangle$ a solution $y$ (which is then unique) satisfying* (3). *Moreover, $|y(t)| \leqslant r_0$ for $t \in \langle t_1, t_2\rangle$.*

P r o o f. Let $\varphi \in P$. Setting $h(t, \mu) = G[\varphi, \mu](t)$ for $(t, \mu) \in \langle t_1, t_2\rangle \times I$, Lemma 2 follows from Lemma 5 [1]. □

For $x$, $t_1$, $t_2 \in \mathbf{R}$, $t_1 < t_2$, define functions $\chi_{t_1, t_2}$, $\nu_x$, $\tau_x \colon \mathbf{R} \to \mathbf{R}$ by

$$\chi_{t_1,t_2}(t) = \begin{cases} 1 & \text{for } t \in \langle t_1, t_2\rangle, \\ 0 & \text{for } t \in \mathbf{R} - \langle t_1, t_2\rangle; \end{cases}$$

$$\nu_x(t) = \begin{cases} 0 & \text{for } t \in (-\infty, x\rangle, \\ 1 & \text{for } t \in (x, \infty); \end{cases}$$

$$\tau_x(t) = \begin{cases} 1 & \text{for } t \in (-\infty, x), \\ 0 & \text{for } t \in \langle x, \infty). \end{cases}$$

Let $t_1$, $t_2 \in \mathbf{R}$, $t_1 < t_0 < t_2$ and let $\varphi \in Y_1$, $\psi \in Y_0$, $(\varphi, \varphi') \in D$, $\psi \in P$, where $D$ and $P$ are defined in (i) and (j), respectively. Consider the equations

$$y'' - q(t)y = \chi_{t_1,t_2}(t) F[\varphi, \varphi', \mu](t) - q(t)\big(1 - \chi_{t_1,t_2}(t)\big)y - \Big(\frac{\alpha}{r_0}\Big)^2 \nu_{t_2}(t)y - \Big(\frac{\beta}{r_0}\Big)^2 \tau_{t_1}(t)y \tag{5}$$

and

$$y'' - q(t)y = \chi_{t_1,t_2}(t) G[\psi, \mu](t) - q(t)\big(1 - \chi_{t_1,t_2}(t)\big)y - \Big(\frac{\alpha}{r_0}\Big)^2 \nu_{t_2}(t)y - \Big(\frac{\beta}{r_0}\Big)^2 \tau_{t_1}(t)y, \tag{6}$$

which depend on the parameters $\mu$, $\alpha$, $\beta$; $\mu \in I$, $\alpha$, $\beta \in \mathbf{R}$. We say that $z$ is a solution of (5) ((6)) on $\mathbf{R}$ if $z \in C^1(\mathbf{R}) \cap C^2(\mathbf{R} - \{t_1, t_2\})$ and for $y = z(t)$ the equality (5) ((6)) holds for all $t \in \mathbf{R} - \{t_1, t_2\}$.

**Lemma 3.** *Let $t_1$, $t_2 \in \mathbf{R}$, $t_1 < t_0 < t_2$, and let assumptions* (i)–(v) *hold for positive constants $r_0$, $r_1$. Then for every $\varphi \in Y_1$, $(\varphi, \varphi') \in D$ there exist a unique $\mu_0 \in I$, $0 \leqslant \alpha_0 \leqslant r_1$, $0 \leqslant \beta_0 \leqslant r_1$ such that equation* (5) *with $\mu = \mu_0$, $\alpha = \alpha_0$, $\beta = \beta_0$ has a solution $y$ (which is then unique) satisfying* (3) *and*

$$\limsup_{t \to -\infty} y(t) = r_0 \operatorname{sign} \beta_0, \quad \limsup_{t \to \infty} y(t) = r_0 \operatorname{sign} \alpha_0.$$

Proof. For $t \in J = \langle t_1, t_2 \rangle$ equation (5) is of the form

$$y'' - q(t)y = F[\varphi, \varphi', \mu](t), \quad t \in J, \tag{8}$$

and by Lemma 1 there exists a unique $\mu_0 \in I$ such that equation (8) with $\mu = \mu_0$ has a solution $z$ (which is then unique), $z(t_1) = z(t_0) = z(t_2) = 0$. Moreover, $|z^{(i)}(t)| \leqslant r_i$ for $t \in J$, $i = 0$, 1.

For $t \in (-\infty, t_1)$ and $t \in (t_2, \infty)$ equation (5) is of the form

$$y'' = -\left(\frac{\beta}{r_0}\right)^2 y \tag{9}$$

and

$$y'' = -\left(\frac{\alpha}{r_0}\right)^2 y, \tag{10}$$

respectively. We see that equation (9) ((10)) has on the interval $(-\infty, t_1\rangle$ ($\langle t_2, \infty)$) a solution $y_1(y_2)$ satisfying $y_1^{(i)}(t_1) = z^{(i)}(t_1)$ for $i = 0$, 1 and $\limsup_{t \to -\infty} y_1(t) = r_0 \operatorname{sign} |z'(t_1)|$ ($y_2^{(i)}(t_2) = z^{(i)}(t_2)$ for $i = 0, 1$ and $\limsup_{t \to \infty} y_2(t) = r_0 \operatorname{sign} |z'(t_2)|$) if and only if $y_1(t) = r_0 \sin\left(\frac{z'(t_1)}{r_0}(t - t_1)\right)$ $\left(y_2(t) = r_0 \sin\left(\frac{z'(t_2)}{r_0}(t - t_2)\right)\right)$. Setting $\beta_0 = |z'(t_1)|$, $\alpha_0 = |z'(t_2)|$ we have $0 \leqslant \beta_0 \leqslant r_1$, $0 \leqslant \alpha_0 \leqslant r_1$ and the function

$$y(t) = \begin{cases} z(t) & \text{for } t \in J, \\ r_0 \operatorname{sign} z'(t_1) \sin\left(\frac{\beta_0}{r_0}(t - t_1)\right) & \text{for } t \in (-\infty, t_1), \\ r_0 \operatorname{sign} z'(t_2) \sin\left(\frac{\alpha_0}{r_0}(t - t_2)\right) & \text{for } t \in (t_2, \infty), \end{cases} \tag{11}$$

is the unique solution of (5) with $\mu = \mu_0$, $\alpha = \alpha_0$, $\beta = \beta_0$ having the properties demanded in the lemma. □

**Lemma 4.** *Let $t_1$, $t_2 \in \mathbb{R}$, $r_1 < t_0 < t_2$, $t_2 - t_1 \geqslant 2$ and let $Q_0 = \max\{q(t)\,;\, t \in \langle t_1, t_2\rangle\}$. Assume that assumptions* (j)-(ju) *hold for a positive constant $r_0$. Then for every $\psi \in P$ there exist a unique $\mu_0 \in I$, $0 \leqslant \alpha_0 \leqslant 2r_0(1+Q_0)$, $0 \leqslant \beta_0 \leqslant 2r_0(1+Q_0)$ such that equation* (6) *with $\mu = \mu_0$, $\alpha = \alpha_0$, $\beta = \beta_0$ has a solution $y$ (which is then unique) satisfying* (3) *and* (7).

Proof. Since for $t \in J = \langle t_1, t_2\rangle$ we may write equation (6) in the form

$$y'' - q(t)y = G[\psi, \mu](t), \quad t \in J, \tag{12}$$

there exists (by Lemma 2) a unique $\mu_0 \in I$ such that equation (12) with $\mu = \mu_0$ has a solution $z$ (which is then unique), $z(t_1) = z(t_0) = z(t_2) = 0$. Moreover, $|z(t)| \leqslant r_0$ for $t \in J$. Since $|G[\varphi, \mu](t)| \leqslant r_0 Q_0$ for $t \in J$ (by (jj)) we have $|z''(t)| \leqslant 2r_0 Q_0$ for $t \in J$. Next, $z(t_1 + 1) - z(t_1) = z'(\xi)$, $z'(\xi) - z'(t_1) = z''(\tau)(\xi - t_1)$, where $\xi \in (t_1, t_1 + 1)$, $\tau \in (t_1, \xi)$, thus $|z'(t_1)| \leqslant |z(t_1+1) - z(t_1)| + |z''(\tau)|(\xi - t_1) \leqslant 2r_0(1+Q_0)$. Analogously $|z'(t_2)| \leqslant 2r_0(1 + Q_0)$. Setting $\beta_0 = |z'(t_1)|$, $\alpha_0 = |z'(t_2)|$ as in the proof of Lemma 3 we can verify that the function $y$ defined by (11) is the unique solution of (6) with $\mu = \mu_0$, $\alpha = \alpha_0$, $\beta = \beta_0$ satisfying (3) and (7). □

Remark 2. From the proofs of Lemmas 3 and 4 we see that the solution $y$ of (5) ((6)) in Lemma 3 (Lemma 4) satisfies $|y^{(i)}(t)| \leqslant r_i$ for $t \in \mathbb{R}$ and $i = 0, 1$ ($|y(t)| \leqslant r_0$, $|y'(t)| \leqslant 2r_0(1 + Q_0)$ for $t \in \mathbb{R}$).

**Lemma 5.** *Let $t_1$, $t_2 \in \mathbb{R}$, $t_1 < t_0 < t_2$. Assume that assumptions* (i)–(v) *hold for positive constants $r_0$, $r_1$. Then there exist $\mu_0 \in I$, $0 \leqslant \alpha_0 \leqslant r_1$, $0 \leqslant \beta_0 \leqslant r_1$ such that the equation*

(12)

$$y'' - q(t)y = \chi_{t_1,t_2}(t)F[y, y', \mu] - q(t)(1 - \chi_{t_1,t_2}(t))y - \left(\frac{\alpha}{r_0}\right)^2 \nu_{t_2}(t)y - \left(\frac{\beta}{r_0}\right)^2 \tau_{t_1}(t)y$$

*with $\mu = \mu_0$, $\alpha = \alpha_0$, $\beta = \beta_0$ has a solution $y$ satisfying* (3) *and $(y, y') \in D$.*

Proof. Let $S = \{y;\, y \in Y_1, (y, y') \in D\}$ and let $J = \langle t_1, t_2\rangle$. By Lemma 3 for every $\varphi \in S$ there exist a unique $\mu_0 \in I$ and unique $\alpha_0$, $\beta_0$, $0 \leqslant \alpha_0 \leqslant r_1$, $0 \leqslant \beta_0 \leqslant r_1$ such that equation (5) with $\mu = \mu_0$, $\alpha = \alpha_0$, $\beta = \beta_0$ has a solution $y$ (which is then unique) satisfying (3), (7) and $(y, y') \in D$ (see Remark 2). Setting $T(\varphi) = y$ we obtain an operator $T\colon S \to S$. $S$ is evidently a closed convex and bounded subset of the Fréchet space $Y_1$.

To prove that $T$ is a continuous operator let $\{y_n\}$, $y_n \in S$ be a convergent sequence, $\lim\limits_{n\to\infty} y_n = y$, and let $z_n = T(y_n)$, $z = T(y)$. Then, by Lemma 3 and its proof and by

Remark 1, there exist a sequence $\{\mu_n\}$, $\mu_n \in I$ and $\mu_0 \in I$ such that

$$z_n(t) = \begin{cases} \dfrac{r(t_0,t)}{r(t_1,t_0)} \displaystyle\int_{t_0}^{t_1} r(t_1,s)F[y_n,y_n',\mu_n](s)\,\mathrm{d}s & \\ + \displaystyle\int_{t_0}^{t} r(t,s)F[y_n,y_n',\mu_n](s)\,\mathrm{d}s & \text{for } t \in J, \\ r_0 \operatorname{sign} z_n'(t_1) \sin\left(\frac{\beta_n}{r_0}(t-t_1)\right) & \text{for } t \in (-\infty,t_1), \\ r_0 \operatorname{sign} z_n'(t_2) \sin\left(\frac{\alpha_n}{r_0}(t-t_2)\right) & \text{for } t \in (t_2,\infty), \end{cases}$$

$$z(t) = \begin{cases} \dfrac{r(t_0,t)}{r(t_1,t_0)} \displaystyle\int_{t_0}^{t_1} r(t_1,s)F[y,y',\mu_0](s)\,\mathrm{d}s & \\ + \displaystyle\int_{t_0}^{t} r(t,s)F[y,y',\mu_0](s)\,\mathrm{d}s & \text{for } t \in J, \\ r_0 \operatorname{sign} z'(t_1) \sin\left(\frac{\beta_0}{r_0}(t-t_1)\right) & \text{for } t \in (-\infty,t_1), \\ r_0 \operatorname{sign} z'(t_2) \sin\left(\frac{\alpha_0}{r_0}(t-t_2)\right) & \text{for } t \in (t_2,\infty), \end{cases}$$

where

$$(|z_n'(t_1)| =) \quad \beta_n = \left| \frac{r_1'(t_1,t_0)}{r(t_0,t_1)} \int_{t_0}^{t_1} r(t_1,s)F[y_n,y_n',\mu_n](s)\,\mathrm{d}s + \int_{t_0}^{t_1} r_1'(t_1,s)F[y_n,y_n',\mu_n](s)\,\mathrm{d}s \right|,$$

$$(|z_n'(t_2)| =) \quad \alpha_n = \left| \frac{r_1'(t_2,t_0)}{r(t_0,t_1)} \int_{t_0}^{t_1} r(t_1,s)F[y_n,y_n',\mu_n](s)\,\mathrm{d}s + \int_{t_0}^{t_2} r_1'(t_2,s)F[y_n,y_n',\mu_n](s)\,\mathrm{d}s \right|,$$

$$(|z'(t_1)| =) \quad \beta_0 = \left| \frac{r_1'(t_1,t_0)}{r(t_0,t_1)} \int_{t_0}^{t_1} r(t_1,s)F[y,y',\mu_0](s)\,\mathrm{d}s + \int_{t_0}^{t_1} r_1'(t_1,s)F[y,y',\mu_0](s)\,\mathrm{d}s \right|,$$

$$(|z'(t_2)| =) \quad \alpha_0 = \left| \frac{r_1'(t_2,t_0)}{r(t_0,t_1)} \int_{t_0}^{t_1} r(t_1,s)F[y,y',\mu_0](s)\,\mathrm{d}s + \int_{t_0}^{t_2} r_1'(t_2,s)F[y,y',\mu_0](s)\,\mathrm{d}s \right|,$$

If $\{\mu_n\}$ is not a convergent sequence then $\lim_{n\to\infty}\mu_{k_n} = \lambda_1$, $\lim_{n\to\alpha}\mu_{r_n} = \lambda_2$, $\lambda_1 < \lambda_2$ for subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$ of $\{\mu_n\}$ and by (i)

$$\lim_{n\to\infty} z_{k_n}(t) = \frac{r(t_0,t)}{r(t_1,t_0)}\int_{t_0}^{t_1} r(t_1,s)F[y,y',\lambda_1](s)\,ds + \int_{t_0}^{t} r(t,s)F[y,y',\lambda_1](s)\,ds,$$

$$\lim_{n\to\infty} z_{r_n}(t) = \frac{r(t_0,t)}{r(t_1,t_0)}\int_{t_0}^{t_1} r(t_1,s)F[y,y',\lambda_2](s)\,ds + \int_{t_0}^{t} r(t,s)F[y,y',\lambda_2](s)\,ds$$

uniformly on $J$. Since $\frac{r(t_0,t_2)}{r(t_1,t_0)}r(t_1,s) < 0$ for $s \in (t_1,t_0\rangle$, $r(t_2,s) > 0$ for $s \in \langle t_0,t_2)$ and $F[y,y',\lambda_1](s) < F[y,y',\lambda_2](s)$ for $s \in J$ (by (iii)) we get

$$\lim_{n\to\infty}\big(z_{k_n}(t_2) - z_{r_n}(t_2)\big) = \frac{r(t_0,t_2)}{r(t_1,t_0)}\int_{t_0}^{t_1} r(t_1,s)\{F[y,y',\lambda_1](s) - F[y,y',\lambda_2](s)\}\,ds + \int_{t_0}^{t_2} r(t_2,s)\{F[y,y',\lambda_1](s) - F[y,y',\lambda_2](s)\}\,ds < 0,$$

which contradicts $z_n(t_2) = 0$ for all $n \in N$. Consequently, $\{\mu_n\}$ is convergent and we may write $\lim_{n\to\infty}\mu_n = \mu^*$. Then

$$(z^*(t) =)\quad \lim_{n\to\infty} z_n(t) = \frac{r(t_0,t)}{r(t_1,t_0)}\int_{t_0}^{t_1} r(t_1,s)F[y,y',\mu^*](s)\,ds + \int_{t_0}^{t} r(t,s)F[y,y',\mu^*](s)\,ds$$

uniformly on $J$ and $z^*$ is the unique solution of the equation

$$z'' - q(t)z = F[y,y',\mu^*](t), \quad t \in J,$$

$z^*(t_1) = z^*(t_0) = z^*(t_2) = 0$. Consequently, by Lemma 1 $\mu^* = \mu_0$ and $z^*(t) = z(t)$ for $t \in J$, hence $\lim_{n\to\infty}\alpha_n = \alpha_0$, $\lim_{n\to\infty}\beta_n = \beta_0$ and $\lim_{n\to\infty} z_n(t) = z(t)$ locally uniformly on $(-\infty,t_1\rangle \cup \langle t_2,\infty)$.

Next, the equalities

$$z_n'(t) = \begin{cases} \dfrac{r_1'(t,t_0)}{r(t_0,t_1)}\displaystyle\int_{t_0}^{t_1} r(t_1,s)F[y_n,y_n',\mu_n](s)\,ds \\ \quad + \displaystyle\int_{t_0}^{t} r_1'(t,s)F[y_n,y_n',\mu_n](s)\,ds & \text{for } t \in J, \\ z_n'(t_1)\cos\left(\frac{\beta_n}{r_0}(t-t_1)\right) & \text{for } t \in (-\infty,t_1), \\ z_n'(t_2)\cos\left(\frac{\alpha_n}{r_0}(t-t_2)\right) & \text{for } t \in (t_2,\infty), \end{cases}$$

$$z'(t)=\begin{cases}\dfrac{r_1'(t,t_0)}{r(t_0,t_1)}\displaystyle\int_{t_0}^{t_1} r(t_1,s)F[y,y',\mu_0](s)\,\mathrm{d}s \\ +\displaystyle\int_{t_0}^{t} r_1'(t,s)F[y,y',\mu_0](s)\,\mathrm{d}s & \text{for } t\in J,\\ z'(t_1)\cos\left(\frac{\beta_0}{r_0}(t-t_1)\right) & \text{for } t\in(-\infty,t_1),\\ z'(t_2)\cos\left(\frac{\alpha_0}{r_0}(t-t_2)\right) & \text{for } t\in(t_2,\infty)\end{cases}$$

imply $\lim\limits_{n\to\infty} z_n'(t)=z'(t)$ locally uniformly on $\mathbb{R}$. Consequently, $\lim\limits_{n\to\infty} T(y_n)=T(y)$ and $T$ is a continuous operator.

Let $z\in T(S)$ and let $Q_0=\max\{q(t);\, t\in J\}$, $B=\max\{\frac{r_1^2}{r_0}, r_0Q_0+A\}$. Then $z=T(y)$ for some $y\in S$ and since $|z''(t)|=|q(t)z(t)+F[y,y',\mu_0](t)|\leqslant r_0Q_0+A$ for $t\in J$, where $\mu_0\in I$ is an appropriate number, $|z''(t)|=\left|\frac{\beta_0^2}{r_0}\sin\left(\frac{\beta_0}{r_0}(t-t_1)\right)\right|\leqslant\frac{r_1^2}{r_0}$ for $t\in(-\infty,t_1)$,$|z''(t)|=\left|\frac{\alpha_0^2}{r_0}\sin\left(\frac{\alpha_0}{r_0}(t-t_2)\right)\right|\leqslant\frac{r_1^2}{r_0}$ for $t\in(t_2,\infty)$, we have $|z''(t)|\leqslant B$ for $t\in\mathbb{R}$. Then $T(S)\subset K=\{y;\, y\in Y_1\cap C^2(\mathbb{R}-\{t_1,t_2\}), (y,y')\in D,\ |y''(t)|\leqslant B$ for $t\in\mathbb{R}-\{t_1,t_2\}\}$ and since $K$ is a compact subset of $Y_1$, $T(S)$ is a relative compact subset of $Y_1$.

Therefore by the Schauder-Tychonoff fixed point theorem there exists a fixed point $y\in S$ of $T$ satisfying the conclusion of Lemma 5. □

**Lemma 6.** *Let the assumptions of Lemma 4 hold. Then there exist* $\mu_0\in I$, $0\leqslant\alpha_0\leqslant 2r_0(1+Q_0)$, $0\leqslant\beta_0\leqslant 2r_0(1+Q_0)$ *such that the equation*

$$(13)\quad y''-q(t)y=\chi_{t_1,t_2}(t)G[y,\mu]-q(t)(1-\chi_{t_1,t_2}(t))y-\left(\frac{\alpha}{r_0}\right)^2\nu_{t_2}(t)y-\left(\frac{\beta}{r_0}\right)^2\tau_{t_1}(t)y$$

*with* $\mu=\mu_0$, $\alpha=\alpha_0$, $\beta=\beta_0$ *has a solution* $y$ *satisfying* (3) *and*

$$(14)\qquad |y(t)|\leqslant r_0,\quad |y'(t)|\leqslant 2r_0(1+Q_0)\quad \text{for } t\in\mathbb{R}.$$

Proof. Let $S=\{y;\, y\in Y_1\cap P,\ |y'(t)|\leqslant 2r_0Q_0(t_2-t_1)$ for $t\in\mathbb{R}\}\subset Y_0$. $S$ is evidently a closed convex bounded subset of the Fréchet space $Y_0$. By Lemma 4 (see also Remark 2) for every $\psi\in S$ there exist a unique $\mu_0\in I$ and unique $0\leqslant\alpha_0\leqslant 2r_0(1+Q_0)$, $0\leqslant\beta_0\leqslant 2r_0(1+Q_0)$ such that equation (6) with $\mu=\mu_0$, $\alpha=\alpha_0$, $\beta=\beta_0$ has a solution $y$ (which is then unique) satisfying (3), (7) and (14). Setting $T(\psi)=y$ we obtain an operator $T\colon S\to S$. Proceeding analogously to the proof of Lemma 5, with evident modifications, we can prove that $T$ is a continuous operator and $T(S)$ is a relative compact subset of $Y_0$. By the Schauder-Tychonoff fixed point theorem there exists a fixed point $y\in S$ of $T$, and from the definition of $T$ we see that Lemma 6 holds. □

Lemma 4 [3] yields

**Lemma 7.** *Assume that assumptions* (i)–(vi) *hold for positive constants* $r_0$, $r_1$, *and* $q$ *is* $\omega$*-periodic. Then for every* $\omega$*-periodic function* $\varphi \in Y_1$, $(\varphi, \varphi') \in D$ *there exists a unique* $\mu_0 \in I$ *such that equation* (2) *with* $\mu = \mu_0$ *has an* $\omega$*-periodic solution* $y$ *satisfying*

$$y(t_0) = 0. \tag{15}$$

*This solution* $y$ *is unique and* $(y, y') \in D$.

Using the method of the proof of Lemma 4 [3] we can easily prove

**Lemma 8.** *Assume that assumptions* (j)–(u) *hold for a positive constant* $r_0$ *and* $q$ *is* $\omega$*-periodic. Then for every* $\omega$*-periodic function* $\varphi \in P$ *there exists a unique* $\mu_0 \in I$ *such that equation* (4) *with* $\mu = \mu_0$ *has an* $\omega$*-periodic solution* $y$ *satisfying* (15). *This solution* $y$ *is unique and*

$$|y(t)| \leqslant r_0, \quad |y'(t)| \leqslant 2r_0\omega Q_1 \quad \text{for } t \in \mathbb{R},$$

*where* $Q_1 = \max\{q(t);\, t \in \langle t_0, t_0 + \omega\rangle\}$.

## 3. Boundedness of solutions

**Theorem 1.** *Assume that assumptions* (i)–(v) *hold for positive constants* $r_0$, $r_1$. *Then there exists* $\mu_0 \in I$ *such that equation* (1) *with* $\mu = \mu_0$ *has a solution* $y$, $y(t_0) = 0$ *and* $(y, y') \in D$.

Proof. Let $\{t_n\}$, $\{x_n\}$ be sequences, $\ldots < t_{n+1} < t_n < \ldots < t_1 < t_0 < x_0 < x_1 < \ldots < x_n < x_{n+1} < \ldots$, $\lim\limits_{n\to\infty} t_n = -\infty$, $\lim\limits_{n\to\infty} x_n = \infty$, and let $Q = \sup\{q(t);\, t \in \mathbb{R}\}$, $B = \max\{\frac{r_1^2}{r_0}, r_0Q + A\}$. By Lemma 5 and its proof, the equation

$$y'' - q(t)y = \chi_{t_n,x_n}(t)F[y, y', \mu] - q(t)(1 - \chi_{t_n,x_n}(t))y - \left(\frac{\alpha}{r_0}\right)^2 \nu_{x_n}(t)y - \left(\frac{\beta}{r_0}\right)^2 \tau_{t_n}(t)y$$

has a solution $y_n$, $y_n(t_n) = y_n(t_0) = y_n(x_n) = 0$, $(y_n, y_n') \in D$, $|y_n''(t)| \leqslant B$ for $t \in \mathbb{R} - \{t_n, x_n\}$ with $\mu = \mu_n$, $\alpha = \alpha_n$, $\beta = \beta_n$, where $\mu_n \in I$, $0 \leqslant \alpha_n \leqslant r_1$, $0 \leqslant \beta_n \leqslant r_1$. Consider the sequence $\{y_n(t)\}$. Using the Ascoli theorem and Cauchy's diagonal method we may assume, without loss of generality, that $\{y_n(t)\}$, $\{y_n'(t)\}$ are locally uniformly convergent on $\mathbb{R}$. Since $\{\mu_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ are bounded sequences, we may also assume that they are convergent, $\lim\limits_{n\to\infty} \mu_n = \mu_0$, $\lim\limits_{n\to\infty} \alpha_n = \alpha_0$, $\lim\limits_{n\to\infty} \beta_n = \beta_0$.

Let $y(t) = \lim_{n\to\infty} y_n(t)$ for $t \in \mathbb{R}$ and let $J \subset \mathbb{R}$ be a compact interval. Then $y(t_0) = 0$, $(y, y') \in D$ and by letting $n \to \infty$ in the equalities

$$\begin{aligned} y_n''(t) - q(t)y_n(t) = {} & \chi_{t_n,x_n}(t)F[y_n, y_n', \mu_n](t) - q(t)\big(1 - \chi_{t_n,x_n}(t)\big)y_n(t) \\ & - \Big(\frac{\alpha_n}{r_0}\Big)^2 \nu_{x_n}(t)y_n(t) - \Big(\frac{\beta_n}{r_0}\Big)^2 \tau_{t_n}(t)y_n(t), \\ & t \in \mathbb{R} - \{t_n, x_n\}, n \in N, \end{aligned}$$

we obtain

$$y''(t) - q(t)y(t) = F[y, y', \mu_0](t) \quad \text{for } t \in J.$$

Since $J$ is an arbitrary interval, we see that $y$ is a solution of (1) with $\mu = \mu_0$, $y(t_0) = 0$, $(y, y') \in D$. □

Example 1. Consider the equation

$$(16) \qquad y'' - q(t)y = \frac{1}{\pi} \int_{-\operatorname{ch} t}^{t^2} \frac{y(s)}{1+s^2}\, ds + \cos\Big(y'\big(\psi(t)\big) + t\Big) \exp\Big(y\big(\varphi(t)\big) - 1\Big) + \mu,$$

where $\varphi$, $\psi$, $q \in C^0(\mathbb{R})$, $4 \leqslant q(t) \leqslant Q$ for $t \in \mathbb{R}$. Assumptions (i)–(v) hold with $r_0 = 1$, $r_1 = 2\sqrt{Q+4}$ and $I = \langle -2, 2 \rangle$. Therefore by Theorem 1 there exists $\mu_0 \in \langle -2, 2 \rangle$ such that equation (16) with $\mu = \mu_0$ has a solution $y$, $y(t_0) = 0$, $|y(t)| \leqslant 1$, $|y'(t)| \leqslant 2\sqrt{Q+4}$ for $t \in \mathbb{R}$.

**Theorem 2.** *Assume that assumptions* (j)–(ju) *hold for a positive constant* $r_0$ *and* $Q = \sup\{q(t); t \in \mathbb{R}\} < \infty$. *Then there exists* $\mu_0 \in I$ *such that the equation*

$$(17) \qquad y'' - q(t)y = G[y, \mu]$$

*with* $\mu = \mu_0$ *has a solution* $y$, $y(t_0) = 0$ *and* $|y(t)| \leqslant r_0$ *for* $t \in \mathbb{R}$.

Proof. Let $\{t_n\}$, $\{x_n\}$ be as in the proof of Theorem 1, $x_1 - t_1 \geqslant 2$. By Lemma 6 the equation

$$\begin{aligned} y'' - q(t)y = {} & \chi_{t_n,x_n}(t)G[y, \mu] - q(t)\big(1 - \chi_{t_n,x_n}(t)\big)y \\ & - \Big(\frac{\alpha}{r_0}\Big)^2 \nu_{x_n}(t)y - \Big(\frac{\beta}{r_0}\Big)^2 \tau_{t_n}(t)y \end{aligned}$$

has a solution $y_n$, $y_n(x_n) = y_n(t_0) = y_n(x_n) = 0$, $|y_n(t)| \leqslant r_0$, $|y_n'(t)| \leqslant 2r_0(1+Q)$ for $t \in \mathbb{R}$ with $\mu = \mu_n$, $\alpha = \alpha_n$, $\beta = \beta_n$ where $\mu_n \in I$, $0 \leqslant \alpha_n \leqslant 2r_0(1+Q)$, $0 \leqslant \beta_n \leqslant 2r_0(1+Q)$. As in the proof of Theorem 1 we may assume that $\{y_n(t)\}$ is locally uniformly convergent on $\mathbb{R}$, $\lim_{n\to\infty} \mu_n = \mu_0$, $\lim_{n\to\infty} \alpha_n = \alpha_0$, $\lim_{n\to\infty} \beta_n = \beta_0$. Setting $y(t) = \lim_{n\to\infty} y_n(t)$ for $t \in \mathbb{R}$ we have $y(t_0) = 0$, $|y(t)| \leqslant r_0$ for $t \in \mathbb{R}$ and it is obvious that $y$ is a solution of (17) with $\mu = \mu_0$. □

E x a m p l e 2. Consider the equation

$$(18)\qquad y'' - q(t)y = \arctan t\bigl(1 + \sup_{0\leqslant s\leqslant |t|} y(s)\bigr)\mathrm{e}^{y(\varphi(t))} \int_{t^3}^{\ln(1+|t|)} \mathrm{e}^{-|s|} y(s)\,\mathrm{d}s + \mathrm{e}^{\sin t}\mu,$$

where $q,\ \varphi \in C^0(\mathbb{R})$, $3\pi\mathrm{e}^2(1+\mathrm{e}^2) \leqslant q(t)$, $\sup\{\dot q(t);\ t\in\mathbb{R}\} < \infty$. Since the assumptions of Theorem 2 are satisfied for $\mu \in \langle -6\pi\mathrm{e}^3, 6\pi\mathrm{e}^3\rangle$ and $r_0 = 2$ there exists $\mu_0 \in \langle -6\pi\mathrm{e}^3, 6\pi\mathrm{e}^3\rangle$ such that equation (18) with $\mu = \mu_0$ has a solution $y$, $y(t_0) = 0$, $|y(t)| \leqslant 2$ for $t \in \mathbb{R}$.

## 4. Periodicity of solutions

**Theorem 3.** *Let assumptions* (i)–(vi) *be satisfied for positive constants* $r_0$, $r_1$ *and let* $q$ *be* $\omega$*-periodic. Then there exists* $\mu_0 \in I$ *such that equation (1) with* $\mu = \mu_0$ *has an* $\omega$*-periodic solution* $y$, $(y, y') \in D$ *and* $y(t_0) = 0$.

P r o o f. By Lemma 7 for every $\omega$-periodic $\varphi \in Y_1$, $(\varphi,\varphi') \in D$ there exists a unique $\mu_0 \in I$ such that equation (2) with $\mu = \mu_0$ has an $\omega$-periodic solution $y$, $y(t_0) = 0$ and $(y, y') \in D$. This solution $y$ is unique and we may write it in the form

$$y(t) = \frac{r(t,t_0)}{r(t_0,t_0+\omega)} \int_{t_0}^{t_0+\omega} r(t_0+\omega, s) F[\varphi,\varphi',\mu_0](s)\,\mathrm{d}s + \int_{t_0}^{t} r(t,s)F[\varphi,\varphi',\mu_0](s)\,\mathrm{d}s, \quad t\in\mathbb{R}.$$

Setting $T(\varphi) = y$ we obtain an operator $T\colon S \to S$ with $S = \{y;\ y\in Y_1,\ (y,y')\in D,$ $y$ is $\omega$-periodic$\}$. To complete the proof of Theorem 3 it is sufficient to prove that $T$ has a fixed point.

We will prove that $T$ is a completely continuous operator. Let $\{y_n\}$, $y_n \in S$ be a convergent sequence, $\lim_{n\to\infty} y_n = y$, and let $z_n = T(y_n)$, $z = T(y)$. Then there exist $\{\mu_n\}$, $\mu_n \in I$ and $\mu_0 \in I$ such that

$$z_n(t) = \frac{r(t,t_0)}{r(t_0,t_0+\omega)} \int_{t_0}^{t_0+\omega} r(t_0+\omega, s) F[y_n, y_n', \mu_n](s)\,\mathrm{d}s + \int_{t_0}^{t} r(t,s)F[y_n,y_n',\mu_n](s)\,\mathrm{d}s, \quad t\in\mathbb{R},$$

and

$$z(t) = \frac{r(t, t_0)}{r(t_0, t_0 + \omega)} \int\limits_{t_0}^{t_0+\omega} r(t_0 + \omega, s) F[y, y', \mu_0](s) \, \mathrm{d}s + \int_{t_0}^{t} r(t, s) F[y, y', \mu_0](s) \, \mathrm{d}s, \quad t \in \mathbb{R}.$$

Obviously

$$z_n'(t) = \frac{r_1'(t, t_0)}{r(t_0, t_0 + \omega)} \int\limits_{t_0}^{t_0+\omega} r(t_0 + \omega, s) F[y_n, y_n', \mu_n](s) \, \mathrm{d}s + \int_{t_0}^{t} r_1'(t, s) F[y_n, y_n', \mu_n](s) \, \mathrm{d}s, \quad t \in \mathbb{R}.$$

If $\{\mu_n\}$ is not a convergent sequence then there exist convergent subsequences $\{\mu_{k_n}\}$, $\{\mu_{r_n}\}$, $\lim\limits_{n\to\infty} \mu_{k_n} = \lambda_1$, $\lim\limits_{n\to\infty} \mu_{r_n} = \lambda_2$, $\lambda_1 < \lambda_2$, and consequently

$$\lim_{n\to\infty} z_{k_n}'(t) = \frac{r_1'(t, t_0)}{r(t_0, t_0 + \omega)} \int\limits_{t_0}^{t_0+\omega} r(t_0 + \omega, s) F[y, y', \lambda_1](s) \, \mathrm{d}s + \int_{t_0}^{t} r_1'(t, s) F[y, y', \lambda_1](s) \, \mathrm{d}s,$$

$$\lim_{n\to\infty} z_{r_n}'(t) = \frac{r_1'(t, t_0)}{r(t_0, t_0 + \omega)} \int\limits_{t_0}^{t_0+\omega} r(t_0 + \omega, s) F[y, y', \lambda_2](s) \, \mathrm{d}s + \int_{t_0}^{t} r_1'(t, s) F[y, y', \lambda_2](s) \, \mathrm{d}s$$

uniformly on $\mathbb{R}$. Since $z_n$ are $\omega$-periodic, we have

(19)
$$0 = \lim_{n\to\infty} \left(z_{k_n}'(t_0 + \omega) - z_{k_n}'(t_0)\right) = \int\limits_{t_0}^{t_0+\omega} k(s) F[y, y', \lambda_1](s) \, \mathrm{d}s,$$

$$0 = \lim_{n\to\infty} \left(z_{r_n}'(t_0 + \omega) - z_{r_n}'(t_0)\right) = \int\limits_{t_0}^{t_0+\omega} k(s) F[y, y', \lambda_2](s) \, \mathrm{d}s,$$

where $k(t) = \frac{r_1'(t_0+\omega,t_0)-1}{r(t_0,t_0+\omega)} r(t_0+\omega,t) + r_1'(t_0+\omega,t)$ for $t \in \langle t_0, t_0+\omega\rangle$. Since $k(t) > 0$ on $\langle t_0, t_0+\omega\rangle$ by Lemma 2 [3] and $F[y,y',\lambda_1](t) < F[y,y',\lambda_2](t)$ for $t \in \langle t_0, t_0+\omega\rangle$ (by (iii)) we have

$$\int_{t_0}^{t_0+\omega} k(s)\{F[y,y',\lambda_1](s) - F[y,y',\lambda_2](s)\}\,\mathrm{d}s < 0,$$

which contradicts (19). Therefore $\{\mu_n\}$ is convergent and we may write $\lim_{n\to\infty} \mu_n = \mu^*$. Then

$$(z^*(t) =)\ \lim_{n\to\infty} z_n(t) = \frac{r(t,t_0)}{r(t_0,t_0+\omega)} \int_{t_0}^{t_0+\omega} r(t_0+\omega,s)F[y,y',\mu^*](s)\,\mathrm{d}s + \int_{t_0}^{t} r(t,s)F[y,y',\mu^*](s)\,\mathrm{d}s$$

and

$$\lim_{n\to\infty} z_n'(t) = \frac{r_1'(t,t_0)}{r(t_0,t_0+\omega)} \int_{t_0}^{t_0+\omega} r(t_0+\omega,s)F[y,y',\mu^*](s)\,\mathrm{d}s + \int_{t_0}^{t} r_1'(t,s)F[y,y',\mu^*](s)\,\mathrm{d}s \quad (= {z^*}'(t))$$

uniformly on $\mathbf{R}$. Hence $z^*$ is an $\omega$-periodic solution (which is then unique) of the equation

$$z'' - q(t)z = F[y,y',\mu^*](t),$$

$z^*(t_0) = 0$, $(z^*, {z^*}') \in D$. By Lemma 7 $\mu_0 = \mu^*$ and $z = z^*$ and therefore $\lim_{n\to\infty} T(y_n) = T(y)$, thus $T$ is a continuous operator.

Let $y \in S$ and $z = T(y)$. Then $z''(t) = q(t)z(t) + F[y,y',\mu_0](t)$ for $t \in \mathbf{R}$, where $\mu_0 \in I$ is an appropriate number, and thus $|z''(t)| \leqslant r_0 Q + A$ $(= B)$ for $t \in \mathbf{R}$. Since $T(S) \subset L = \{y;\ y \in C^2(\mathbf{R}) \cap S,\ |y''(t)| \leqslant B$ for $t \in \mathbf{R}\}$ and $L$ is a compact subset of $Y_1$, $T(S)$ is a relative compact subset of $Y_1$. By Schauder's fixed point theorem there exists a fixed point of $T$. This completes the proof. □

Using Lemma 8 we may prove

**Theorem 4.** *Let assumptions* (j)–(u) *be satisfied for a positive constant $r_0$ and let $q$ be $\omega$-periodic. Then there exist $\mu_0 \in I$ such that equation* (17) *with $\mu = \mu_0$ has an $\omega$-periodic solution $y$, $y(t_0) = 0$, $|y(t)| \leqslant r_0$ and $|y'(t)| \leqslant 2r_0\omega Q_1$ for $t \in \mathbf{R}$, where $Q_1$ is defined as in Lemma* 8.

E x a m p l e 3. Consider the equation

$$y'' - q(t)y = \exp\big(-|y'(t+\sin t)| + 1\big)\operatorname{ch}\big(|y(t+1)|^n\big) + \mu\exp(\cos t), \tag{20}$$

where $q \in C^0(\mathbf{R})$ is a $2\pi$-periodic function, $q(t) \geqslant e(1+e^2)\operatorname{ch} 1$ for $t \in \mathbf{R}$ and $n$ is a positive integer. The assumptions of Theorem 3 are satisfied with $I = \langle -e^2 \operatorname{ch} 1, 0\rangle$, $r_0 = 1$ and $r_1 = 2\sqrt{e(1+e^2)\operatorname{ch} 1 + Q}$, where $Q = \max\{q(t);\ t \in \langle 0, 2\pi\rangle\}$. Thus there exists $\mu_0 \in \langle -e^2 \operatorname{ch} 1, 0\rangle$ such that equation (20) with $\mu = \mu_0$ has a $2\pi$-periodic solution $y$, $y(t_0) = 0$, $|y(t)| \leqslant 1$, $|y'(t)| \leqslant 2\sqrt{e(1+e^2)\operatorname{ch} 1 + Q}$ for $t \in \mathbf{R}$.

E x a m p l e 4. Consider the equation

$$y'' - q(t)y = \cos(2\pi t)\ln\Big[y^{2n}\big(y(t)+t\big) + e\Big] + \mu \tag{21}$$

where $q \in C^0(\mathbf{R})$ is a 1-periodic function, $q(t) \geqslant 2\ln(1+e)$ for $t \in \mathbf{R}$ and $n$ is a positive integer. The assumptions of Theorem 4 are satisfied with $I = \langle -\ln(1+e), \ln(1+e)\rangle$ and $r_0 = 1$. Therefore there exists $\mu_0 \in \langle -\ln(1+e), \ln(1+e)\rangle$ such that equation (21) with $\mu = \mu_0$ has a 1-periodic solution $y$, $y(t_0) = 0$, $|y(t)| \leqslant 1$ and $|y'(t)| \leqslant 2Q_1$ for $t \in \mathbf{R}$, where $Q_1 = \max\{q(t);\ t \in \langle 0, 1\rangle\}$.

## *References*


[1] *S. Staněk*: Three-point boundary value problem for nonlinear second-order differential equation with parameter, Czech. Math. J. *42(117)* (1992), 241–256.

[2] *S. Staněk*: On the boundedness of solutions of nonlinear second-order differential equations with parameter, to appear.

[3] *S. Staněk*: Periodic solutions of nonlinear second-order differential equations with parameter, to appear.



*Author's address*: 772 00 Olomouc, třída Svobody 26, Czechoslovakia (PF UP).