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A REMARK ON HODGE ALGEBRAS AND GRÖBNER BASES

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0. INTRODUCTION

Let K be a field and let A be a finitely generated commutative K-algebra, given by a finite set E of generators and their ideal I of relations (i.e. $A = K[E]/I$, where $K[E]$ is the commutative polynomial ring over K). In order to do computations in A, it is very useful to have

(1) a good K-vector space basis of A, consisting of the residue classes of all power products outside a monomial ideal Σ ("standard monomials")

and

(2) "rewriting rules" allowing to write any monomial modulo I in a finite number of steps as a finite K-linear combination of standard monomials.

There are basically two concepts of such rewriting rules for polynomial ideals. The first one is the Hodge algebra concept, starting from a partial order on E. In many cases several properties of the ring $K[E]/I$ can be read off easily from the structure of the ordered set E. Important examples of Hodge algebras occur in constructive invariant theory (see for example Bruns & Vetter (1988), DeConcini et al. (1982), Sturmfels & White (1988), or example 4 below).

The second one is the Gröbner basis method (due to Buchberger (1965)), starting from a strict order on the monoid of power products in $K[E]$. There are powerful implementations of this method in modern computer algebra systems. We asume the reader to be familiar with this technique (see for example Buchberger (1985) or Pauer & Pfeifhofer (1989)).

In this note we show how one can use Gröbner bases to verify condition (1). The connection between Hodge algebras and Gröbner bases was first intensively studied in Gräbe (1986) and Sturmfels & White (1988). Here we use slightly generalized results of Gräbe (1989) to establish a criterion for the verification of (1). The criterion works for a wide class of Hodge algebras, which includes graded ones.

Let $K[E]$ be the commutative polynomial ring over a field K, generated by a finite set E. We denote by M the monoid of power products in $K[E]$. For $m = \prod_{e \in E} e^{p(e)} \in M$ set $\deg_e(m) := p(e)$.

A *grading* on M is a map $|\cdot|: M \to \mathbf{R}$ defined by its weight vector $d \in \mathbf{R}^E$ via

$$|m| = \sum_{e \in E} \deg_e(m) \cdot d(e).$$

A partial order $\leqslant$ on M is

monotone iff $m \leqslant n$ implies $mp \leqslant np$, for all $m, n, p \in M$;

noetherian iff every strictly decreasing chain in M is finite.

Let $\Sigma \subseteq M$ be a monomial ideal (i.e. $\Sigma \cdot M \subseteq M$) with the finite minimal generating set L. The monomials in $S := M \setminus \Sigma$ we call *standard* (with respect to Σ).

In the following we consider conditions posed by several authors to get handy algorithms to perform calculations with polynomial ideals. The starting points all of them are

- an ideal $I \subseteq K[E]$, given by a finite set of generators;
- a monomial ideal $\Sigma \subseteq M$ such that
  (1) the family $(\bar{s})_{s \in S}$ of residue classes of standard monomials with respect to Σ is a K-basis of $K[E]/I$;
- rewriting rules for the minimal set L of generators of Σ in terms of S, this means: for all $m \in L$ there are given (uniquely determined) finite K-linear-combinations

$$r(m) := \sum_{s \in S} c(m, s)s$$

  of elements in S such that $m - r(m) \in I$. $R := \{m - r(m) \mid m \in L\}$ is the set of *(basic) straightening relations* of L.

Of course these rewriting rules have to fulfill some "finiteness conditions" to guarantee that computations will terminate. We shall consider the following three conditions:

(2a) Let $\leqslant$ be a partial order on E. For all $m \in L$ and $s \in S$ with $c(m, s) \neq 0$ we have:

if $e \in E$ divides m there exists $f \in E$, $f < e$, dividing s;

(2b) There exists a grading $|\cdot|_d = |\cdot|: M \to \mathbf{R}_+$ with positive weight vector d, such that $|m| \geqslant |s|$ for all $m \in L$, $s \in S$ with $c(m, s) \neq 0$ (i.e. $K[E]/I$ is "affine graded" in the sense of Gräbe (1988)).

(2c) Let $\leqslant$ be a monotone noetherian order on M. For all $m \in L$ and $s \in S$ with $c(m, s) \neq 0$ we have $m \geqslant s$.

Example 1. Consider $E := \{X, Y, Z\}$; $Z \leqslant X, Z \leqslant Y$; $L_1 := \{XY\}$, $L_2 := \{X^2Y, XY^2\}$; $R_1 := \{XY - X^2Z - Y^2Z\}$, $R_2 := \{X^2Y - X^4Z, XY^2 - Y^4Z\}$; $S_i :=$ the set of standard monomials with respect to L_i, $i = 1, 2$.

Then $(E, L_1, \langle R_1\rangle, \leqslant)$ and $(E, L_2, \langle R_2\rangle, \leqslant)$ fulfill (2a) but not (2b).

The subset $\{\bar{s} \mid s \in S_1\}$ of $K[E]/\langle R_1\rangle$ is linear independent, but $\overline{XY}^2$ is not in its linear span (cf. Eisenbud (1980), 2, Example 1).

The linear span of $\{\bar{s} \mid s \in S_2\}$ is $K[E]/\langle R_2\rangle$, but this set is not linear independent, since $X^6Z^2 \in S_2$, $Y^6Z^2 \in S_2$ and $X^6Z^2 - Y^6Z^2 = (XY^2 - Y^4Z)(Y^2Z + X) - (X^2Y - X^4Z)(X^2Z + Y)$, so $\overline{X}^6\overline{Z}^2 = \overline{Y}^6\overline{Z}^2$. Hence neither $(E, L_1, \langle R_1\rangle)$ nor $(E, L_2, \langle R_2\rangle)$ fulfill (1).

Remark 1. If a solution $(d(e))_{e\in E} \in \mathbf{R}_+^E$ of the linear system of inequalities

$$|m|_d - |s|_d \geqslant 0 \quad \text{for} \quad m \in L,\ s \in S \quad \text{with} \quad c(m, s) \neq 0,$$

exists, it can be found by the usual simplex method. So condition (2b) can be checked in a finite number of steps.

A quadruple $(E, L, I \leqslant)$ with the properties (1) and (2a) is a *Hodge algebra* (cf. DeConcini et al. (1982)). Hodge algebra techniques are intensively used to study determinantal ideals, see DeConcini et al. (1982), Bruns & Vetter (1988) and others. All these ideals are graded. In such cases (2a) can be combined with (2b). Some results in DeConcini et al. (1982), e.g. Proposition 1.1, are stated only for graded Hodge algebras. Recently Trung (1990) pointed out that Hodge algebras without property (2b) may behave quite badly and gave counterexamples to an earlier statement in Eisenbud (1980) that the straightening relations give a presentation of *any* Hodge algebra. He showed that the straightening relations need not generate the whole ideal I. I may have "hidden relations".

If the triple (E, L, I) fulfills the properties (1) and (2c) for a *strict* monotone noetherian order then the straightening relations are a *Gröbner basis* of I.

While for Hodge algebra concept the main idea is to test whether a *given* basis has good properties and in general (1) is hard to check, the Gröbner basis method transforms a given basis into a "better one" and (2) is hard to control if your basis should have some additional property.

(2b) is the connecting bridge between (2a) and (2c) as was shown in Gräbe (1988). For a Hodge algebra satisfying (2b) the straightening relations give a Gröbner basis in the degreewise reverse lexicographic order (see proposition 2). In particular they generate I (see corollary 1) and the powerful Gröbner basis techniques (deformations, resolutions, ...) can be applied immediately.

Recall the following results from Gräbe (1989):
If $d \in \mathbb{R}^E$ defines a grading on M then

$$a <_d b \ :\Leftrightarrow\ |a| < |b|$$

is a monotone partial order on M, the *associated degreewise order*. If the weight vector d is positive, then any refinement of $<_d$ is noetherian.

Every partial monotone noetherian order can be refined to a strict monotone noetherian one (ibid. Satz 1.2).

For a strict monotone ordering $<$ on M, "noetherian" is equivalent to "for all $m \in M \setminus \{1\}$, $1 < m$" (ibid. Satz 1.1). A strict monotone noetherian order we call *admissible*.

Definition. Let $<_E$ be a strict order on E and let d be a map from E to $\mathbb{R}_+$. The *reverse lexicographic order associated to* $(d, <_E)$ is defined by $m < n$ iff $(|m| < |n|)$ or $(|m| = |n|$ and there exists an element $e \in E$ such that $\deg_e(m) > \deg_e(n)$ and $\deg_f(m) = \deg_f(n)$, for all $f \in E$ with $e <_E f)$.

This order is admissible and a strict refinement of the degreewise order associated to d.

Proposition 1. *Let $\leqslant$ be a (partial) monotone noetherian order on M, such that (E, L, I) satisfies (2c). Then the following conditions are equivalent:*

(a) *(E, L, I) satisfies (1), i.e. it is a ring with straightening law in the sense of Gräbe (1989).*

(b) *There is an admissible order on M, satisfying (2c), such that R is a Gröbner basis of I.*

(c) *For every admissible order on M satisfying (2c) the set R is a Gröbner basis of I.*

Proof. Using the remarks above and since every admissible refinement of $\leqslant$ satisfies (2c), the proof is obvious. □

Remark 2. The implication (b) $\Rightarrow$ (a) can be used to verify (1), the implication (a) $\Rightarrow$ (c) to falsify (1).

Proposition 2. *Let $\leqslant$ be a partial order on E such that (2a) holds for $(E, L, I, \leqslant)$. Then the following conditions are equivalent:*

(a) *(E, L, I) satisfies (1) and (2b), i.e. it is an affine graded Hodge algebra.*

(b) *There is an admissible order on M satisfying (2c), such that R is a Gröbner* basis of I.

(c) (E, L, I) *satisfies* (2b) *and* R *is a Gröbner basis of* I *with respect to every admissible order on* M *satisfying* (2c).

Proof. (a) $\Rightarrow$ (b): Let d be a positive weight vector as in (2b) and choose a strict refinement $<_E$ of $\leqslant$ on E. Then the reverse lexicographic order associated to $(d, <_E)$ is an order on M with the desired properties.

(b) $\Rightarrow$ (c): Follows from Gräbe (1989), Satz 2.1.

(c) $\Rightarrow$ (a): Obvious. □

Corollary 1 (Trung (1990)). *If* $(E, L, I, \leqslant)$ *is an affine graded Hodge algebra, then* R *generates* I.

Proof. Every Gröbner basis of I generates I. □

Corollary 2. *Let* $|\cdot|$ *be a grading with positive weight vector and* I *homogeneous with respect to this grading. Let* $(E, L, I, \leqslant)$ *satisfy* (2a). *Then* $(E, L, I, \leqslant)$ *is a Hodge algebra iff* R *is a Gröbner basis of* I *with respect to some (resp. every) admissible order on* M *satisfying* (2c).

Algorithm. Here is an algorithmic version of proposition 2:

We abbreviate $(E, L, I, \leqslant)$ by H. We want to decide whether H is a Hodge algebra or not.

Step 1: Check whether (2a) is fulfilled. If not, then H is not a Hodge algebra.

Step 2: Check whether (2b) is fulfilled (cf. remark 1). If not, then stop.

Step 3: If (2a) and (2b) are fulfilled, choose a solution $d \in \mathbf{N}^E$ of (2b) and choose a strict order $<_E$ on E, which refines the partial order $\leqslant$ on E. Check whether R is a Gröbner basis of $\langle R \rangle$ with respect to the reverse lexicographic order associated to $(d, <_E)$. If it is, then H is a Hodge algebra. If not, then H is not a Hodge algebra.

Example 2. Consider $E := \{X, Y, Z\}$; $Z \leqslant X$, $Z \leqslant Y$; $L := \{XY\}$; $R := \left\{XY - \sum_{i=1}^{p} \sum_{j=1}^{q} c_{ij} X^i Z^j - \sum_{k=1}^{r} d_k Y Z^k\right\}$; $I := \langle R \rangle$.

Conditions (2a) and (2b) are fulfilled (for example, choose $d(X) = r$, $d(Y) = (p-1)r + q$, $d(Z) = 1$). Therefore $(E, L, I, \leqslant)$ is Hodge algebra (see Eisenbud (1980), 2, Example 1).

Example 3. Consider $E := \{X, Y, Z\}$; $Z \leqslant X$, $Z \leqslant Y$; $L := \{XY^3, X^3Y, X^2Y^2\}$; $R := \{XY^3 - XZ - Y^3Z, X^2Y^2 - XYZ^2 - XY^2Z^3, X^3Y - X^2YZ - Y^2Z^2\}$; $I := \langle R \rangle$.

Conditions (2a) an (2b) are fulfilled.

Choose $d(X) = 5$, $d(Y) = 1$, $d(Z) = 1$ and $Z <_E X <_E Y$.

Then R is not a Gröbner basis of I, hence $(E, L, I, \leqslant)$ is not a Hodge algebra.

Example 4. Let E be the set of triples $[a_1a_2a_3]$ with $1 \leqslant a_1 < a_2 < a_3 \leqslant 5$ and let $\leqslant$ be the natural partial order on E

$$([a_1a_2a_3] \leqslant [b_1b_2b_3] :\Leftrightarrow a_i \leqslant b_i, \quad \text{for} \quad i = 1, 2, 3).$$

We set $L := \{ef \mid e \not\leqslant f \text{ and } f \not\leqslant e\}$,

$$\begin{aligned} R := \{&[125][134] - [124][135] + [123][145], [125][234] - [124][235] + [123][245], \\ &[135][234] - [134][235] + [123][345], [145][234] - [134][245] + [124][345], \\ &[145][235] - [135][245] + [125][345]\} \quad \text{and} \quad I := \langle R\rangle\,. \end{aligned}$$

(The algebra $K[E]/I$ is isomorphic to the algebra of $SL(3, K)$-invariant polynomial functions on $(K^3)^5$, the elements of R are deduced from the "Plücker-relations", see for example Bruns & Vetter (1988)).

Then $(E, L, I, \leqslant)$ fulfills conditions (2a) and (2b). Refine the order $\leqslant$ on E to the lexicographic order on the given symbols. By an easy computation we verify that R is a Gröbner basis of I with respect to the reverse lexicographic order associated to the usual grading, hence $(E, L, I, \leqslant)$ is a Hodge algebra.

3. Hodge algebras not satisfying 2(b)

Trung (1990) showed by two examples that such Hodge algebras exist but may behave quite badly. We will give here two further examples. Of course, Gröbner basis methods as developed so far can't be used for testing the Hodge algebra property (1), so we shall verify it "by hand". Another verification method will be the object of a forthcoming paper.

Example 5. Consider $E := \{W, X, Y, Z\}$; $W \leqslant X$, $W \leqslant Y$, $W \leqslant Z$; $L := \{XY, XZ^2, YZ^2\}$; $R := \{XY - WZ(X^2 + Y^2), XZ^2, YZ^2\}$; $I := \langle R\rangle$. (2a) is obviously satisfied. (2b) can't be satisfied, since the first element of R yields

$$|X| + |Y| \geqslant |WZ| + 2\max(|X|, |Y|).$$

No such grading with positive weight vector exists.

The residue classes $\overline{X}^m\overline{Y}^n$ are 0 for $m, n \geqslant 2$ and equal to $\overline{Y}^{n+1}\overline{WZ}$ (resp. $\overline{X}^{m+1}\overline{WZ}$) for $m = 1$, $n \geqslant 2$ (resp. $m \geqslant 2$, $n = 1$). Therefore, to verify (1), we only have to check the linear independence of the standard monomials modulo I. Assume

$$\sum_{s\in S} c_s s = A(XY - WZ(X^2+Y^2)) + BXZ^2 + CYZ^2$$

for appropriate polynomials A, B, C and $c_s \in K$. Substituting $Z = 0$ shows that Z divides A (since otherwise XY would divide standard monomials). But then the right hand side is a sum of non-standard monomials. Hence $A = B = C = 0$.

Example 6. Consider $E := \{W, X, Y, Z\}$; $W \leqslant X$, $W \leqslant Y$, $W \leqslant Z$; $L := \{XY, WX^3, WY^3\}$; $R := \{XY - WZ(X^2+Y^2), WX^3, WY^3\}$. Then rewriting X^2Y we get subsequently

$$X^2Y \to XWZ(X^2+Y^2) \to (WZ)XY^2 \to (WZ)^2X^2Y \to \dots$$

and analogously

$$XY^2 \to YWZ(X^2+Y^2) \to (WZ)X^2Y \to (WZ)^2XY^2 \to \dots.$$

Both expressions tend to zero in the corresponding power series ring. Let $I := \langle R \cup \{X^2Y, XY^2\}\rangle$. Then $(E, L, I, \leqslant)$ is a Hodge algebra with three basic straightening relations. Indeed, (2a) is obviously satisfied and (1) can be proved in the following way: If

$$\sum_{s\in S} c_s s = A(XY - WZ(X^2+Y^2)) + BWX^3 + CWY^3 + DX^2Y + EXY^2$$

for appropriate polynomials A, B, C, D, E, $A = A_0 + XA_1 + YA_2$ ($A_0 \in K[W, Z]$) and $c_s \in K$, then comparing coefficients of standard monomials only we get

$$\sum_{s\in S} c_s s = -A_0WZ(X^2+Y^2).$$

On the other hand comparing coefficients of XY over $K[W, Z]$ on both sides we get $A_0 = 0$. Hence standard monomials are linearly independent. Since $\overline{X^2Y} = \overline{XY^2} = 0$ by assumption and

$$\overline{X^2Y^2} = \overline{(WZ)XY^3} = 0, \quad \overline{X^mY} = \overline{(WZ)^2X^mY} = 0, \quad \overline{XY^m} = \overline{(WZ)^2XY^m} = 0$$

for $m \geqslant 3$, we see immediatly that residues of non-standard monomials can be expressed by residues of standard ones. Hence $(E, L, I, \leqslant)$ is a Hodge algebra.

We see the surprising effect that without (2b) the basic straightening relations $R = \{m - r(m) \mid m \in L\}$ need not further to generate the ideal I. There may be some "hidden relations" corresponding to infinite reduction loops tending to zero. This was first discovered in Trung (1990) who gave another example of even an ordinal Hodge algebra with "hidden relations".

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