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PARTIAL MONOUNARY ALGEBRAS WITH COMMON QUASI-ENDOMORPHISMS *)

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Homomorphisms and endomorphisms of monounary algebras were investigated in [2], [6]–[9]; for the case of partial monounary algebras, cf. [3]–[5].

The theory of partial algebras was systematically studied by B. Wojdylo and P. Burmeister in [1]. They investigated some types of mappings between partial algebras; in particular, they studied quasi-endomorphisms of partial algebras.

In the present paper we shall deal with quasi-endomorphisms of partial monounary algebras.

For a partial monounary algebra (A, f) let $Q(f)$ be the system of all quasi-endomorphisms of (A, f). We are interested in a constructive description of all partial mappings g of A into A with $Q(f) = Q(g)$; let us denote by $EQ(f)$ the system of all such mappings g. The desired construction is contained in Thm. 4.10. From this theorem it follows that there is at most one partial mapping $g \neq f$ belonging to $EQ(f)$, i.e., $\|EQ(f)\| \leqslant 2$.

An analogous question concerning endomorphisms of partial monounary algebras was investigated in [3] and [4].

1. PRELIMINARIES

Let $\mathcal{N}$ be the set of all positive integers, $\mathcal{N}_0 = \mathcal{N} \cup \{0\}$ and $\mathcal{Z}$ be the set of all integers.

The system of all monounary algebras will be denoted by $\mathcal{U}$ and for the denotation of the system of all partial monounary algebras we will use the symbol $\mathcal{U}_p$.

For a nonempty set A, the system of all partial mappings of A into A (i.e., of all mappings from a subset of a set A into the set A) will be denoted by the symbol $F(A)$.

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Let $(A, f) \in \mathcal{U}_p$. If B is a subset of A with the property $f(b) \in B$ for each $b \in \operatorname{dom} f$, then there is (uniquely determined) element $f \restriction B$ of $F(B)$ such that $B \cap \operatorname{dom}(f \restriction B) = B \cap \operatorname{dom} f$ and $f(b) = (f \restriction B)(b)$ for each $b \in B \cap \operatorname{dom} f$. In this case the pair $(B, f \restriction B)$ is said to be a subalgebra of a partial monounary algebra (A, f).

Let $(A, f) \in \mathcal{U}_p$. For $x \in A$ put $f^0(x) = x$. If $f^k(x)$ is defined for $k \in \mathcal{N}_0$ and $f^k(x) \in \operatorname{dom} f$, then $f^{k+1}(x) = f(f^k(x))$. An algebra (A, f) is called *connected*, if whenever $x, y \in A$, there exist $m, n \in \mathcal{N}_0$ such that $f^m(x) = f^n(y)$. A maximal connected subalgebra of (A, f) is said to be a component of (A, f). We shall say that algebras (A, f) and (A, g) have the same partition into components, if the following condition is satisfied: if $(B, g \restriction B)$ is a component of (A, g), then $(B, f \restriction B)$ is a component of (A, f) and conversely.

The system of all connected algebras belonging to $\mathcal{U}$ will be denoted by the symbol $\mathcal{U}_c$.

A nonempty set $C \subset A$ is called a *cycle* of $(A, f) \in \mathcal{U}_p$, if $(C, f \restriction C)$ is a connected subalgebra of (A, f) and there exists $k \in \mathcal{N}$ with $f^k(y) = y$ for each $y \in C$.

An algebra $(A, f) \in \mathcal{U}_p$ is said to be a chain if some of the following conditions is satisfied:

1. $A = \{a_1, a_2, \ldots, a_n\}$, $n \in \mathcal{N}$, $n > 1$ and $f(a_i) = a_{i+1}$ for $i = 1, 2, \ldots, n-1$, $a_n \notin \operatorname{dom} f$;
2. $A = \{a_i, i \in \mathcal{N}\}$ and $f(a_i) = a_{i+1}$ for each $i \in \mathcal{N}$;
3. $A = \{a_i, i \in \mathcal{Z}\}$ and $f(a_i) = a_{i+1}$ for each $i \in \mathcal{Z}$;
4. $A = \{a_i; i \in \mathcal{Z}, i \leqslant 1\}$ and $f(a_i) = a_{i+1}$ for each $i \in \mathcal{Z}, i \leqslant 0, a_1 \notin \operatorname{dom} f$.

Further, $(R, f_R) \in \mathcal{U}_p$ is a chain of $(A, f) \in \mathcal{U}_p$ if (R, f_R) is a chain and (R, f_R) is a subalgebra of (A, f).

Let $(A, f) \in \mathcal{U}_p$. Then $g \in F(A)$ is called an endomorphism of (A, f) if $\operatorname{dom} g = A$ and $x \in \operatorname{dom} f$ implies $g(x) \in \operatorname{dom} f$ and $g(f(x)) = f(g(x))$ for each $x \in A$. Further, $g \in F(A)$ is said to be a quasi-endomorphism of (A, f) if $x \in \operatorname{dom} f$ and $x, f(x) \in \operatorname{dom} g$ yield $g(x) \in \operatorname{dom} f$ and $g(f(x)) = f(g(x))$. If g is a quasi-endomorphism and there is no $x \in A$ such that $x \in \operatorname{dom} f$ and $x, f(x) \in \operatorname{dom} g$, then we shall say that g is a trivial quasi-endomorphism of (A, f).

For $(A, f) \in \mathcal{U}_p$ put

$$H(f) = \{g \in F(A) : g \text{ is an endomorphism of } (A, f)\},$$
$$Q(f) = \{g \in F(A) : g \text{ is a quasi-endomorphism of } (A, f)\}.$$

Remark. Let $(A, f) \in \mathcal{U}_p$.

a) Then $H(f) = \{g \in Q(f) : \operatorname{dom} g = A\}$.

b) If (B, f_B) is a component of (A, f), $g_B \in Q(f_B)$, then $g_B \in F(B)$ and $g_B \in Q(f)$.

We shall use the following notations

$$EH(f) = \{g \in F(A) : H(f) = H(g)\},$$
$$EQ(f) = \{g \in F(A) : Q(f) = Q(g)\},$$
$$EH_0(f) = EH(f) \cap H(f).$$

1.1. Lemma. *Let $(A, f) \in \mathcal{U}_p$. Then $EH_0(f) = \{g \in H(f)\colon H(f) = H(g)\}$ and $EQ(f) = \{g \in Q(f)\colon Q(f) = Q(g)\}$.*

Proof. Since $g \in Q(g)$ for each $g \in F(A)$ and $g \in H(g)$ for $g \in F(A)$ such that $\operatorname{dom} g = A$, we obtain that the assertion is valid. □

1.2. Lemma. *Let $(A, f) \in \mathcal{U}_p$. If $g \in EQ(f)$ or $g \in EH(f)$, then $EQ(f) = EQ(g)$ or $EH_0(f) = EH_0(g)$, $EH(f) = EH(g)$, respectively.*

Proof. Assume that $Q(f) = Q(g)$. Then $h \in EQ(f)$ if and only if $Q(h) = Q(f) = Q(g)$ and this relation holds if and only if $h \in EQ(g)$. This gives the desired conclusion $EQ(f) = EQ(g)$.

The remaining assertion can be shown similarly. □

1.3. Lemma. *Let $(A, f) \in \mathcal{U}_p$. Then $EQ(f) \subset EH(f)$.*

Proof. Take $g \in Q(f)$ with $Q(f) = Q(g)$. We have $h \in H(f)$ if and only if $\operatorname{dom} h = A$ and $h \in Q(f)$. Further this relation is valid if and only if $h \in H(g)$, since $Q(f) = Q(g)$. Thus $H(f) = H(g)$ and $g \in EH(f)$. □

1.4. Corollary. *Let $(A, f) \in \mathcal{U}_p$. Then $EQ(f) \cap H(f) \subset EH_0(f)$.*

1.5. Lemma. *Let $(A, f) \in \mathcal{U}$ and $g \in Q(f)$. Then $f \in Q(g)$ if and only if $x \in \operatorname{dom} g$ implies $f(x) \in \operatorname{dom} g$ for each $x \in A$.*

Proof. Let $f \in Q(g)$. Since $\operatorname{dom} f = A$, we get $f(x) \in \operatorname{dom} g$ for $x \in \operatorname{dom} g$.

On the other hand, assume that $x \in \operatorname{dom} g$ and $x, g(x) \in \operatorname{dom} f$. Then $f(x) \in \operatorname{dom} g$ and we have $g(f(x)) = f(g(x))$, because $g \in Q(f)$. This proves that $f \in Q(g)$. □

1.6. Corollary. *Suppose that $(A, f) \in \mathcal{U}_c$ has a cycle $C, g \in Q(f)$, $f \in Q(g)$ and $\operatorname{dom} g \neq \emptyset$. Then $C \subset \operatorname{dom} g$.*

Consider $(A, f) \in \mathcal{U}_p$. We put

$$K_d = \{a \in \operatorname{dom} f\colon (\{a\}, f \restriction \{a\}) \text{ is a component of } (A, f)\},$$

$$K_n = \{a \notin \operatorname{dom} f : (\{a\}, f \restriction \{a\}) \text{ is a component of } (A, f)\},$$

$$K = K_d \cup K_n.$$

Further we shall say that (A, f) is of type α, τ, π, γ or δ if it fulfils the following condition (α), (τ), (π), (γ) or (δ), respectively:

(α) $K \neq A$ and each component (B, f_B) of (A, f) such that $\|B\| > 1$ is a cycle or a chain;

(τ) $K \neq A$, $\operatorname{dom} f = A$ and there is $a \in A$ with $f(x) = a$ for each $x \in A$;

(π) $K = A$, $\|K_d\| = 1$;

(γ) $K_n = A$;

(δ) $K_d = A$.

Remark. If (A, f) is of type τ with $f(x) = a$ for each $x \in \operatorname{dom} f$, then we say that (A, f) is of type τ with a value a; analogously for the type π.

1.7. Lemma. *Let $(A, f) \in \mathcal{U}_p$ and let (A, f) be neither of type τ nor of type π. Further let $g \in F(A)$. If $g \in EQ(f)$, then (A, f) and (A, g) have the same partition into components.*

Proof. According to 1.3 we obtain that $g \in EH(f)$. Thus in view of Thm.4.6 of the paper [3], the algebras (A, f) and (A, g) have the same partition into components. □

1.8. Lemma. *Let $(A, f) \in \mathcal{U}_p$ be neither of type π nor of type τ, (B, f_B) be a component of (A, f) and $g \in EQ(f)$. Then $Q(g_B) = Q(f_B)$ where $g_B = g \restriction B$.*

Proof. Let us show that $Q(f_B) \subset Q(g_B)$. (The relation $Q(g_B) \subset Q(f_B)$ can be proved analogously since (A, f) and (A, g) have the same partition into components by 1.7.)

Let $h \in Q(f_B)$. Then $h \in F(B)$. Define $h_1 \in F(A)$ as follows: $\operatorname{dom} h_1 = \operatorname{dom} h$ and $h_1(x) = h(x)$ for each $x \in \operatorname{dom} h_1$. We have $h_1 \in Q(f)$ since $h \in Q(f_B)$. According to the assumption, $Q(f) = Q(g)$, hence $h_1 \in Q(g)$.

Now we shall prove that $h \in Q(g_B)$. Let $x \in \operatorname{dom} g \cap B$ and $x, g(x) \in \operatorname{dom} h$. Since $h_1 \in Q(g)$ and $\operatorname{dom} h_1 = \operatorname{dom} h$, we get $h_1(x) \in \operatorname{dom} g$ and $g(h(x)) = g(h_1(x)) = h_1(g(x)) = h(g(x))$. Further $h(x) \in B$, therefore $h \in Q(g_B)$. □

1.9. Lemma. *Let $(A, f) \in \mathcal{U}_p$. Then $\|EQ(f)\| \leqslant c$.*

Proof. The assertion is the consequence of the lemma 1.3 and Thm. 4.11 of the paper [4]. □

2. Components of algebras with common quasi-endomorphisms

In this section we shall suppose that $(A, f) \in \mathcal{U}_p, g \in F(A), Q(f) = Q(g)$ and that (B, f_B) is a component of (A, f) such that $\|B\| > 1$.

2.1. Lemma. *Let $y \in \operatorname{dom} g \cap \operatorname{dom} f$, $f(y) = y$. Then $g(y) = y$.*

Proof. Let us define a mapping $\varphi \in F(A)$ such that $\varphi(z) = y$ for each $z \in A$. We have $\varphi \in Q(f) = Q(g)$ and $g(y) = g(\varphi(y)) = \varphi(g(y)) = y$. □

2.2. Lemma. *The relation $f \in Q(g)$ is valid.*

Proof. The desired relation follows from the fact that $f \in Q(f)$ and from the assumption that $Q(f) = Q(g)$. □

2.3. Lemma. *If $y \in \operatorname{dom} f_B$ and $f(y) = y$, then $y \in \operatorname{dom} g$ and $g(y) = y$.*

Proof. Let $y \in \operatorname{dom} f_B - \operatorname{dom} g$ and $f(y) = y$. Since $\|B\| > 1$ we can choose $z \in \operatorname{dom} f_B, z \neq y$ such that $f(z) = y$. Let us construct $\psi \in F(A)$ such that $\psi = \{[y, z], [z, z]\}$.

According to 2.2 we have $f \in Q(g)$ and $g \in Q(f)$. Therefore either $z \notin \operatorname{dom} g$ or $g(z) \notin \operatorname{dom} f$. If $z \in \operatorname{dom} g$ and $g(z) \notin \operatorname{dom} f$, then $g(z) \notin \{y, z\}$, because $\{y, z\} \subset \operatorname{dom} f$. Consequently the mapping ψ is a trivial element of $Q(g)$. If $z \notin \operatorname{dom} g$, then ψ is a trivial element of $Q(g)$, too. But $f(\psi(z)) = f(z) = y, \psi(f(z)) = \psi(y)) = z$ and $y \neq z$. Thus $\psi \notin Q(f)$, a contradiction with $Q(f) = Q(g)$. The equality $g(y) = y$ follows from 2.1. □

2.4. Lemma. *Let $y \in \operatorname{dom} f$ and $f(y) \in \operatorname{dom} f$. If $f^2(y) = f(y)$ and $y \in \operatorname{dom} g$, then $g(y) = f(y)$.*

Proof. If $f(y) = y$, then $g(y) = y = f(y)$ by 2.3.

Assume that $f(y) \neq y$ and $g(y) \neq f(y)$. According to 2.3 (take $z = f(y)$) we have $g(f(y)) = f(y)$ and since $g \in Q(f)$, we get $g(y) \in \operatorname{dom} f$ and $f(g(y)) = g(f(y)) = f(y)$. Now suppose that $g(y) \neq y$. Let us define $\varphi \in F(A)$ such that $\varphi = \{[y, y], [g(y), y]\}$. Then $\varphi \in Q(f) - Q(g)$, a contradiction. Consequently $g(y) = y$ and in view of 2.1, by interchanging f and g, we conclude $f(y) = y$. Thus the assumption that $g(y) \neq f(y)$ is not tenable. □

2.5. Lemma. *Let (B, f_B) have a cycle C. Then $C \subset \operatorname{dom} g$ and $g(x) \in C$ for each $x \in C$.*

Proof. Assume that $p = \|C\|$. If $p = 1$, then the assertion is valid by 2.3.

Let $p > 1$. Then (A, f) is neither of type π nor of type τ. According to 1.7 and 1.8 (B, g_B) is a component of (A, g) and $Q(f_B) = Q(g_B)$ where $g_B = g \restriction B$. From this and the assertions 2.2 and 1.6 we obtain $C \subset \operatorname{dom} g$. Further if $x \in C$, then $g(x) = g(f^p(x)) = f(g(f^{p-1}(x))) = \ldots = f^p(g(x))$, since $g \in Q(f)$. Hence $g(x) \in C$. □

2.6. Lemma. *Let $y \in \operatorname{dom} f_B - \operatorname{dom} g$ and $f(y) \in \operatorname{dom} g$. Suppose that either $f(y) \notin \operatorname{dom} f$ or $f^2(y) \neq f(y)$. Then $g(f(y)) = y$.*

Proof. Suppose that $g(f(y)) \neq y$. Let us define $\varphi \in F(A)$ such that $\varphi = \{[y, f(y)], [f(y), f(y)]\}$. If $g(f(y)) = f(y)$, then $\varphi(g(f(y))) = \varphi(f(y)) = f(y) = g(\varphi(f(y)))$ and hence $\varphi \in Q(g)$. Otherwise φ is a trivial element of $Q(g)$. Further $y \in \operatorname{dom} f$ and $y, f(y) \in \operatorname{dom} \varphi$, but either $\varphi(y) = f(y) \notin \operatorname{dom} f$ or $f(\varphi(y)) = f^2(y) \neq f(y) = \varphi(f(y))$. Thus $\varphi \notin Q(f)$ and the proof is complete. □

2.7. Lemma. *If $y \in \operatorname{dom} f_B$, $f(y) \neq y$ and $f(y) \in \operatorname{dom} f$, then $f(y) \in \operatorname{dom} g$.*

Proof. If $y \in \operatorname{dom} g$ and $g(y) \in \operatorname{dom} f$, then $f(y) \in \operatorname{dom} g$ according to 2.2. Let either $y \notin \operatorname{dom} g$ or $g(y) \notin \operatorname{dom} f$.

Suppose that $f(y) \notin \operatorname{dom} g$. We define $\varphi \in F(A)$ such that $\varphi = \{[y, y], [f(y), y]\}$. if $y \in \operatorname{dom} g$, then $g(y) \notin \{y, f(y)\}$. We have that φ is a trivial element of $Q(g)$. Further $\varphi(f(y)) = y$, $f(\varphi(y)) = f(y)$ and $f(y) \neq y$. Consequently $\varphi \notin Q(f)$, a contradiction. □

2.8. Lemma. *Let (B, f_B) have a cycle and suppose that it is not of type τ. Then $B \cap \operatorname{dom} g = B$.*

Proof. Because (B, f_B) is a connected algebra with a cycle C, the relation $\operatorname{dom} f_B = B$ is valid. We have $C \subset \operatorname{dom} g$ by 2.5.

Assume that $B - \operatorname{dom} g \neq \emptyset$. Then we can choose $z \in B - \operatorname{dom} g$ with $\{f^i(z) : i \in \mathcal{N}\} \subset \operatorname{dom} g$.

First let $f^2(z) \neq f(z)$ and consider $k \in \mathcal{N}$ such that $f^k(z) \in C, f^{k-1}(z) \notin C$. Since $g \in Q(f)$, the lemma 2.6 implies $f^{k-1}(z) = f^{k-1}(g(f(z))) = g(f^k(z))$. Further, $g(f^k(z)) \in C$ according to 2.5, which is a contradiction.

Now let $f^2(z) = f(z)$. Put $y = f(z)$. The algebra (B, f_B) is not of type τ and $\|B\| > 1$, therefore there exists $x \in B$ with $f^2(x) \neq f(x)$. Let us define $\varphi \in F(A)$ such that $\varphi = \{[z, x], [y, y]\}$. We get $\varphi \in Q(g)$, but $\varphi \notin Q(f)$, because $f(z) = y$, $\varphi(z) = x$, $\varphi(f(z)) = y$ and $f(\varphi(z)) = f(x) \neq y$. □

2.9. Lemma. *Suppose that (B, f_B) has no cycle and that (B, f_B) is not a chain. Then $\operatorname{dom} f_B \subset \operatorname{dom} g$.*

Proof. Assume that there exists $y_0 \in B$ such that $y_0 \in \operatorname{dom} f - \operatorname{dom} g$. The algebra (B, g_B) is a component of (A, g) by 1.7, hence $\|B - \operatorname{dom} g\| \leqslant 1$ and therefore $f(y_0) \in \operatorname{dom} g$. Then 2.6 yields that $g(f(y_0)) = y_0$.

Let $y \in B$ be such that $f(y) = y_0$. Then $y \in \operatorname{dom} g$ and $g(y) \neq y_0$ according to 2.2. We can use the partial mapping φ from the proof of lemma 2.7 and we conclude a contradictoin with $Q(f) = Q(g)$. Thus $y_0 \notin \operatorname{rng} f$.

For $k \in \mathcal{N}$ such that $f^{k-1}(y_0) \in \operatorname{dom} f$ let us put $y_k = f^k(y_0)$. We get $g(y_1) = y_0$ and, by induction, $g(y_k) = g(f(y_{k-1})) = f(g(y_{k-1})) = f(y_{k-2}) = y_{k-1}$.

Since (B, f_B) is not a chain, we can choose $a \in B$ such that $f(a) = y_m$ for some $m \in \mathcal{N}$ and $a \neq y_{m-1}$. Let us define $\varphi \in F(A)$ as $\varphi = \{[y_0, a], [y_1, y_m]\}$. We have $\varphi(f(y_0)) = \varphi(y_1) = y_m = f(a) = f(\varphi(y_0))$, thus $\varphi \in Q(f)$. Further $g(\varphi(y_1)) = g(y_m) = y_{m-1}$ and $\varphi(g(y_1)) = \varphi(y_0) = a$, hence $\varphi \notin Q(g)$, a contradiction. □

2.10. Lemma. *Let $(B, f_B) \notin \mathcal{U}_c$ and let $g_B = g \restriction B$. If $\operatorname{dom} f_B = \operatorname{dom} g_B$, then $f_B = g_B$.*

Proof. According to the assumption there exists $y_0 \notin \operatorname{dom} f_B$. Denote $S = \{x \in \operatorname{dom} f_B : g(x) \neq f(x)\}$. Assume that $S \neq \emptyset$. Choose $y' \in S$. In view of the connectivity of (A, f) there exists a positive integer r that $f^r(y') = y_0$. Consider $t = \max\,\{\in \{0, 1, \dots, r\} : f^p(y') \in S\}$. Since $y_0 \notin S$, the relation $t < r$ is valid. Put $y = f^t(y')$.

First let us show that there exists no $m \in \mathcal{N}$ such that $g(y) = f^m(y), m \in \mathcal{N}$. Then $m > 1$ because $y \in S$. Further we obtain $f^m(y) = f(f^{m-1}(y)) = g(f^{m-1}(y)) = f^{m-1}(g(y)) = f^{m-1}(f^m(y)) = f^{2m-1}(y)$. Hence (B, f_B) possesses a cycle, which is a contradiction with $(B, f_B) \notin \mathcal{U}_c$.

Put $k = r - t$. Then $k \in \mathcal{N}$ and $f^k(y) = y_0$. Let us define $\varphi = \{[y, y_0], [f(y), f(y)], \dots, [f^k(y), f^k(y)]\}$.

Let $x \in \operatorname{dom} g$ and $x, g(x) \in \operatorname{dom} \varphi$. Since $g(y) \neq y$ (in the opposite case we obtain $f(y) = y$ by 2.3 with replacement f and g) and $g(y) \notin \{f(y), f^2(y), \dots, f^k(y)\}$, we have $g(y) \notin \operatorname{dom} \varphi$. It means, that $x \neq y$. Thus $x = f^n(y) = \varphi(x)$ for some $n \in \mathcal{N}$, $n < k$. Therefore $g(x) = f(x)$. Further $\varphi(g(x)) = \varphi(f^{n+1}(y)) = f^{n+1}(y) = f(f^n(y)) = f(x) = g(x) = g(f^n(y)) = g(\varphi(f^n(y))) = g(\varphi(x))$, and $\varphi \in Q(g)$.

But $\varphi \notin Q(f)$, because $\varphi(y) = y_0$, $\varphi(f(y)) = f(y)$ and $\varphi(y) \notin \operatorname{dom} f$, which is a contradiction.

We get $S = \emptyset$ and it means that $f_B = g_B$. □

2.11. Lemma. *Let (B, f_B) be an algebra of type τ and let $g_B = g \restriction B$. Then $g_B = f_B$ or (B, g_B) is of type π and $\operatorname{rng} g_B = \operatorname{rng} f_B$.*

Proof. Assume that $g_B \neq f_B$ and that $f(x) = a$ for each $x \in B$. We know that $g(a) = a$ by 2.3 and if $y \in \operatorname{dom} g_B$, then $g(y) = f(y) = a$ according to 2.4. Let us show that (B, g_B) is an algebra of type π.

Suppose that (B, g_B) is not of type π. Hence (B, f_B) contains a subalgebra (C, f_C) such that $C = \{a, b, c\}, b \in \operatorname{dom} g$ and $c \notin \operatorname{dom} g$. To agrue the contrapositive let us define $\varphi \in F(A), \varphi = \{[a, a], [b, c]\}$. We have $\varphi \in Q(f) - Q(g)$, because $\varphi(g(b)) = \varphi(a) = a$ and $\varphi(b) \notin \operatorname{dom} g$. □

3. The system $EQ(f)$ for a connected partial monounary algebra (A, f)

Let us suppose that $(A, f) \in \mathcal{U}_p$ is connected and $\|A\| > 1$. We shall describe the partial mappings g of A into A which have the property $Q(f) = Q(g)$.

3.1. Lemma. *Let (A, f) be a chain. Then $EQ(f) = \{f, h\}$, where* $\operatorname{dom} h = \operatorname{rng} f$ *and $h(f(a)) = a$ for each $a \in \operatorname{dom} f$.*

Proof. First we shall show that $Q(f) = Q(h)$. Let $\varphi \in Q(f)$ and suppose that $y \in \operatorname{dom} h$ and $y, h(y) \in \operatorname{dom} \varphi$ for some $y \in A$. We obtain that the relations $\varphi(h(y)) \in \operatorname{dom} f$ and $\varphi(y) = f(\varphi(y))$ are valid, because $\varphi \in Q(f)$, $h(y) \in \operatorname{rng} h = \operatorname{dom} f$ and $h(y), f(h(y)) \in \operatorname{dom} \varphi$. Hence $Q(f) \subset Q(h)$. The opposite inclusion can be proved analogously. Thus $\{f, h\} \subset EQ(f)$.

Suppose that $g \in EQ(f)$. We want to prove that $g = f$ or $g = h$. To complete the proof we shall show the following assertions:

a) If $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$, then $g = h$.

b) If $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ and there exists $x_0 \in A$ with $g(x_0) = f(x_0)$, then $g = f$.

c) If $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ and $f(x) \neq g(x)$ for all $x \in \operatorname{dom} f \cap \operatorname{dom} g$, then $g = h$.

a) Assume that $\operatorname{dom} f \cap \operatorname{dom} g = \emptyset$. Then $\|A\| = 2$ and thus $\operatorname{dom} g = \operatorname{rng} f$. Let $\operatorname{dom} f = \{y\}$. If $g(f(y)) = f(y)$, then $f(y) \in \operatorname{dom} f$ and $f^2(y) = f(y)$ by 2.3, a contradiction. We get $g(f(y)) = y$, i.e., $g = h$.

b) Assume that $f(x_0) = g(x_0)$. Let us define x_k and x_{-k} for $k \in \mathcal{N}$ by induction. If $x_{k-1} \in \operatorname{dom} f$, then put $x_k = f(x_{k-1})$. If there exists $x \in \operatorname{dom} f$ such that $f(x) = x_{-k+1}$, then put $x_{-k} = x$. Since (A, f) is a chain, all elements of A are signed.

It is easy to see, by induction, that if $x_k \in \operatorname{dom} f$, then $x_k \in \operatorname{dom} g$ and $f(x_k) = g(x_k)$ for $k \in \mathcal{N}$.

Further let us show, by induction on k, that if $x_{-k} \in \operatorname{dom} f$, then $x_{-k} \in \operatorname{dom} g$ and $f(x_{-k}) = g(x_{-k})$. Suppose that the assertion holds for $k - 1$. We have $f(x_{-k}) \in \operatorname{dom} f$ according to the facts that (A, f) is a chain and $x_0 \in \operatorname{dom} f$. If $f(x_{-k}) \in \operatorname{dom} g$, then $x_{-k} \in \operatorname{dom} g$. Namely, if $x_{-k} \notin \operatorname{dom} g$, we can define $\varphi \in F(A), \varphi = \{[x_{-k}, x_{-k}], [f(x_{-k}), x_{-k}]\}$. Then $\varphi \in Q(g) - Q(f)$, a contradiction with $Q(f) = Q(g)$. Further the relation $Q(f) = Q(g)$ and the induction assumption yield $f(g(x_{-k})) = g(f(x_{-k})) = g(x_{-k+1}) = f(x_{-k+1}) = f(f(x_{-k}))$. Since (A, f) is a chain this implies $g(x_{-k}) = f(x_{-k})$.

We have proved that $\operatorname{dom} f \subset \operatorname{dom} g$ and that $f(x) = g(x)$ for each $x \in \operatorname{dom} f$.

The relation $\operatorname{dom} f \neq \operatorname{dom} g$ implies that $\operatorname{dom} g = A$ and that there exists $y \in A$ such that $\operatorname{dom} f = A - \{y\}$. Then $(A, f) \in \mathcal{U}_c$ and $g(y) \neq y$. Namely if $g(y) = y$, then in view of 2.3 we would have $y \in \operatorname{dom} f$, which is a contradiction. Put $y' = g(y)$. There exists $k \in \mathcal{N}$ such that $f^k(y') = y$. Then $g^{k+1}(y) = g^k(g(y)) = g^k(y') = f^k(y') = y$. Hence (A, g) has a cycle. Since the assumptions $g \in Q(f)$ and $Q(f) = Q(g)$ imply $f \in Q(g)$ we can interchange f and g in the assertion 2.8 and conclude that $\operatorname{dom} f = \operatorname{dom} f \cap A = A$, a contradiction. Thus $\operatorname{dom} f = \operatorname{dom} g$, as desired.

c) Let $\operatorname{dom} f \cap \operatorname{dom} g \neq \emptyset$ and $g(x) \neq f(x)$ for each $x \in \operatorname{dom} f \cap \operatorname{dom} g$.

Suppose that $\|A\| = 2$. Then $\operatorname{dom} f = \{z\}$ for some $z \in A$. According to the assumption $z \in \operatorname{dom} g$ and $g(z) \neq f(z)$. Consequently $g(z) = z$ and 2.1 implies $f(z) = z$, a contradiction. Therefore $\|A\| > 2$.

We want to prove that $\operatorname{dom} g = \operatorname{rng} f$ and $g(f(a)) = a$ for each $a \in \operatorname{dom} f$. We shall proceed as follows: First we show that $\operatorname{rng} f \subset \operatorname{dom} g$. In the second step we prove there is no $y \in \operatorname{dom} f$ having the property that $g(f(y)) \neq y$. Finally (in the third step) we show that $\operatorname{rng} f = \operatorname{dom} g$.

(1) Assume that $z \in \operatorname{dom} f$ and $f(z) \in \operatorname{dom} g$. Define $\zeta \in F(A)$, $\zeta = \{[z,z], [f(z),z]\}$. Since $g(z) \neq z$ and $f(z) \notin \operatorname{dom} g$, the mapping ζ is a trivial element of $Q(g)$. It is obvious that $\zeta \notin Q(f)$. We arrived at a contradiction. Consequently $\operatorname{rng} f \subset \operatorname{dom} g$.

(2) Let $y \in \operatorname{dom} f$ and $g(f(y)) \neq y$. In view of the assumptions of c) we have either $y \notin \operatorname{dom} g$ or $g(y) \neq f(y)$. If we replace the element z by the element y in the definition of ζ, then we obtain $\zeta \in Q(g) - Q(f)$.

(3) Suppose that $\operatorname{rng} f \neq \operatorname{dom} g$. Then $\operatorname{dom} g = A$ and there exists $u \in A$ with $A - \operatorname{rng} f = \{u\}$. Since $\|A\| > 2$, the relation $f(u) \in \operatorname{dom} f$ is valid. According to the relation $g \in Q(f)$ we get $f(g(f(u))) = g(f(f(u))) = f(u)$. Since f is injective, this implies $g(f(u)) = u$. Next $g \in Q(f)$ and $u \in \operatorname{dom} f, u, f(u) \in \operatorname{dom} g$, which yield that $g(u) \in \operatorname{dom} f$ and $f(g(u)) = g(f(u)) = u$. Therefore $u \in \operatorname{rng} f$, a contradiction. □

3.2. Lemma. *Let $(A,f) \in \mathcal{U}_c$ be neither a chain nor an algebra with a cycle. Let $g \in Q(f)$. If $f(y) \in \operatorname{dom} g$ for each $y \in A, A - \operatorname{dom} g = \{y_0\}$ and $g(f(y_0)) = y_0$, then $Q(f) \neq Q(g)$.*

Proof. Put $y_k = f^k(y_0)$ for each $k \in \mathcal{N}$. Then $g(y_1) = y_0$. Let $k > 1$. We have $y_{k-1} \in \operatorname{dom} f$ and $y_{k-1}, y_k \in \operatorname{dom} g$. Inasmuch as $g \in Q(f)$ we obtain $g(y_k) = g(f(y_{k-1})) = f(g(y_{k-1})) = f(y_{k-2}) = y_{k-1}$ by induction. There exist $z \notin \{y_k, k \in \mathcal{N}\}$ and $m \in \mathcal{N}$ such that $f(z) = y_m$ according to the assumption. Let us define $\varphi \in F(A), \varphi = \{[y_0, z], [y_1, y_m]\}$. It is obvious that $\varphi \in Q(f)$. Further $\varphi(g(y_1)) = \varphi(y_0) = z$ and $g(\varphi(y_1)) = g(y_m) = y_{m-1}$, hence $\varphi \notin Q(g)$. □

3.3. Lemma. *Suppose that (A,f) is a connected monounary algebra beeing not a chain, which is not of type τ.*

Then $EQ(f) \cap (Q(f) - H(f)) = \emptyset$.

Proof. It is necessary to show that $g \in Q(f)$ and $\operatorname{dom} g \neq A$ imply $Q(f) \neq Q(g)$. Assume that $g \in Q(f)$ and $\operatorname{dom} g \neq A$. Since (A,g) is connected in view of 1.7, there is $y_0 \in A$ with $A - \operatorname{dom} g = \{y_0\}$. If (A,f) possesses a cycle, then $Q(f) \neq Q(g)$ by 2.8. Let (A,f) contain no cycle. Then 2.6 implies $g(f(y_0)) = y_0$ and 3.2 yields $Q(f) \neq Q(g)$. □

3.4. Lemma. *Let (A, f) be of type τ, $\operatorname{rng} f = \{a\}$. Then $EQ(f) = \{f, h\}$, where (A, h) is an algebra of type π, $\operatorname{dom} h = \{a\}$.*

P r o o f. From 2.11 it follows that $\{f, h\} \supset EQ(f)$. It suffices to prove that $Q(f) = Q(h)$. Let $\varphi \in Q(h)$. If $a \notin \operatorname{dom}\varphi$, then φ is a trivial quasi-endomorphism of (A, f). If $a \in \operatorname{dom}\varphi$, then $\varphi(a) = a$. Let $x \in A$ and $x, f(x) \in \operatorname{dom}\varphi$. We get $\varphi(f(x)) = \varphi(a) = a = f(\varphi(x))$. Thus $\varphi \in Q(f)$ and $Q(h) \subset Q(f)$.

Conversely suppose that $\varphi \in Q(f)$. Let $a \in \operatorname{dom}\varphi$. Then $\varphi(a) = a$. If $x \in A$ is such that $x \in \operatorname{dom} h$, then $x = a$ and $\varphi(h(a)) = \varphi(a) = a = h(\varphi(a))$. Therefore $\varphi \in Q(h)$. □

3.5. Lemma. *Suppose that (A, f) is a connected monounary algebra beeing not a chain and having no cycle. Then $EQ(f) = \{f\}$.*

P r o o f. We have $EQ(f) \subset EH_0(f)$ according to 3.3 and 1.4. Consider the greatest chain (R, f_R), which is a subalgebra of (A, f).

If there exists $x \in A$ with $f(x) \notin R$ or if there exists $x' \in R$ with $x' \notin \operatorname{rng} f$, then Thm.3 of the paper [2] implies $EH_0(f) = \{f\}$.

Let $R = \operatorname{rng} f$. Since (A, f) is not a chain, let us choose $a \in A - R$. Further there are $y, y' \in R$ such that $f(y) = f(a)$, $f(y') = y$. We have $EH_0(f) = \{f, g\}$, where $g(y) = g(a) = y'$ and $g(f(a)) = y$ according to Thm.1 of the paper [2]. Let us define $\varphi \in F(A)$ such that $\varphi = \{[a, f(a)], [y', f(y)]\}$. Then φ is a trivial element of $Q(f)$, but $\varphi \notin Q(g)$, because $g(\varphi(a)) = g(f(a)) = y$ and $\varphi(g(a)) = \varphi(y') = f(y)$. Thus $Q(f) \neq Q(g)$ and $EQ(f) = \{f\}$. □

3.6. Lemma. *Suppose that (A, f) is a connected monounary algebra having a cycle C and beeing not of type τ.*

a) If there is $x \in A - C$, then $EQ(f) = \{f\}$.

b) If $A = C$, then $EQ(f) = \{f, f^{p-1}\}$, where $p = \|C\|$.

P r o o f. We have $EQ(f) \subset EH_0(f)$ by 3.3 and 1.4. If there exists $x \in A$ with $f(x) \notin C$, then Thm.3 of the paper [2] implies $EH_0(f) = \{f\}$.

Assume that $f(x) \in C$ for each $x \in A$. Then $\|C\| > 1$, because (A, f) is not of type τ. Further $EH_0(f) = \{f^k : 1 \leqslant k < p, k \in \mathcal{N}, k \text{ and } p \text{ are relatively prime}\}$ according to Thm.2 of the paper [2]. The assertion is obvious for $p = 2$. Let $1 < k < p - 1$ and choose $z \in C$. Define $\varphi \in F(A)$ such that $\varphi = \{[z, z], [f^k(z), f(z)]\}$. The mapping φ is a trivial element of $Q(f)$. We obtain $\varphi(f^k(z)) = f(z)$ and $f^k(\varphi(z)) = f^k(z)$, thus $\varphi \notin Q(f^k)$, therefore $Q(f^k) \neq Q(f)$.

Further let $k = p - 1$ and $a \in A - C$. Then $f(a) = f(b)$ for some $b \in C$. Let us define $\psi = \{[a, f(a)], [f^{p-1}(b), f^{p-1}(b)]\}$. Since $p > 2$, ψ is a trivial element of $Q(f)$. The relations $\psi(f^{p-1}(a)) = \psi(f^{p-1}(b)) = f^{p-1}(b)$ and $f^{p-1}(\psi(a)) = f^p(a) = b$ yield that $\psi \notin Q(f^{p-1})$. The proof of the first assertion is complete.

Now assume that $A = C$. Let $\zeta \in Q(f)$. If $x, f^{p-1}(x) \in \operatorname{dom}\zeta$ then $\zeta(f^{p-1}(x)) = f^p(\zeta(f^{p-1}(x))) = f^{p-1}(\zeta(f^p(x))) = f^{p-1}(\zeta(x))$. Thus $Q(f) \subset Q(f^{p-1})$. Similarly $Q(f^{p-1}) \subset Q(f)$. □

3.7. Lemma. *Let $(A, f) \notin \mathcal{U}_c$ and let (A, f) be not a chain. Then $EQ(f) = \{f\}$.*

Proof. Let $g \in Q(f)$ be such that $Q(f) = Q(g)$. Then 2.9 implies $\operatorname{dom} f \subset \operatorname{dom} g$. Since (A, f) is not a chain, (A, g) is not a chain as well in view of 3.1. Further (A, f) contains no cycle, hence (A, g) has no cycle by 3.6. We obtain $\operatorname{dom} g \subset \operatorname{dom} f$ using 2.9. Therefore $\operatorname{dom} f = \operatorname{dom} g$ and 2.10 implies $g = f$. □

3.8. Theorem. *Let $(A, f) \in \mathcal{U}_p$ be connected.*

1° *If $\|A\| = 1$, then $EQ(f) = \{g_1, g_2\}$, where $\operatorname{dom} g_1 = A$, $\operatorname{dom} g_2 = \emptyset$.*

2° *If (A, f) is a chain, then $EQ(f) = \{f, h\}$, where $\operatorname{dom} h = \operatorname{rng} f$ and $h(f(y)) = y$ for each $y \in \operatorname{dom} f$.*

3° *If (A, f) is of type τ with a value a, then $EQ(f) = \{f, g\}$, where (A, g) is of type π with a value a.*

4° *If (A, f) is a cycle, $\|A\| = p > 2$, then $EQ(f) = \{f, f^{p-1}\}$.*

5° *Otherwise $EQ(f) = \{f\}$.*

Proof. If $\|A\| = 1$, then $f = g_1$ or $f = g_2$ and $\{g_1, g_2\} = Q(g_1) = Q(g_2) = EQ(f)$.

The second assertion is proved in 3.1, the third one in 3.4 and the fourth one in 3.6.

Suppose that (A, f) fails to satisfy the assumptions of $1° - 4°$. If $(A, f) \notin \mathcal{U}_c$, then (A, f) is not a chain and 3.1 implies $EQ(f) = \{f\}$. If $(A, f) \in \mathcal{U}_c$ and (A, f) contains a cycle, then $EQ(f) = \{f\}$ by 3.6. Finally, if $(A, f) \in \mathcal{U}_c$ and (A, f) possesses no cycle, then $EQ(f) = \{f\}$ in view of 3.5. □

4. Algebras with common quasi-endomorphisms

In this section the characterization of the set $EQ(f)$ of an arbitrary partial mono-unary algebra (A, f) will be given.

4.1. Lemma. *Suppose that $(A, f) \in \mathcal{U}_p, (B, f_B)$ is a component of (A, f), $\|B\| > 1$ and $g \in EQ(f)$. If $g \restriction B = f_B$, then $g = f$.*

Proof. We can assume that (A, f) contains more then one component. Choose $z \in \operatorname{dom} f_B$ such that $f(z) \neq z$.

First we shall show that $\operatorname{dom} f = \operatorname{dom} g$. Suppose that $x \in \operatorname{dom} g$. Define $\psi \in F(A)$ as $\psi = \{[z, x], [g(z), g(x)]\}$. We obtain $\psi \in Q(g)$. Let $x \in \operatorname{dom} g - \operatorname{dom} f$. Then $\psi \notin Q(f)$, because $\psi(f(z)) = \psi(g(z)) = g(x)$ and $\psi(z) = x \notin \operatorname{dom} f$. The proof for $x \in \operatorname{dom} f - \operatorname{dom} g$ is analogous.

Consider $x \in \operatorname{dom} f$. Therefore we get $g(x) = \psi(g(z)) = \psi(f(z)) = f(\psi(z)) = f(x)$ for each $x \in \operatorname{dom} f = \operatorname{dom} g$. □

4.2. Lemma. *Suppose that $(A, f) \in \mathcal{U}_p, (B, f_B)$ is a component of (A, f) and $B \neq A$. If (B, f_B) is an algebra of type τ, then $EQ(f) = \{f\}$.*

Proof. Assume that $g \in Q(f)$ is such that $Q(f) = Q(g)$. Further let $\operatorname{rng} f_B = \{a\}$. If $g \restriction B = f_B$, then $g = f$ according to 4.1.

Let $g_B \neq f_B$, where $g_B = g \restriction B$. Then $Q(g_B) = Q(f_B)$ is valid in view of 1.8 and $\operatorname{dom} g_B = \{a\}, g(a) = a$ in view of 3.8. Choose $x, z \in A$ as follows: $x \notin B$ and $z \in B$ such that $f(z) \neq z$. Put $\varphi = \{[z, x], [a, a]\}$. The mapping φ belongs to $Q(g)$, because $z \notin \operatorname{dom} g$ and $g(f(z)) = g(a) = a$, $\varphi(g(a)) = \varphi(a) = a = g(\varphi(a))$. Since $\varphi(f(z)) = \varphi(a) = a \in B$ and $\varphi(z) \notin \operatorname{dom} f$ or $f(\varphi(z)) = f(x) \notin B$, we have $\varphi \notin Q(f)$, a contradiction. □

4.3. Lemma. *Let $(A, f) \in \mathcal{U}_p, \|A\| > 1, K_d \neq A$ and let $g \in EQ(f)$. If $a \in K_d$, then $a \in \operatorname{dom} g$ and $g(a) = f(a)$.*

Proof. It suffices to show that $a \in \operatorname{dom} g$ in view of 2.1.

Suppose that $a \notin \operatorname{dom} g$. The assumptions $\|A\| > 1$ and $K_d \neq A$ allow us to choose $x \in A$ such that either $x \notin \operatorname{dom} f$ or $f(x) \neq x$. Now we define $\varphi \in F(A), \varphi = \{[a, x]\}$. Then $\varphi \in Q(g) - Q(f)$. □

4.4. Lemma. *Let $(A, f) \in \mathcal{U}_p$ be of type δ. Then $EQ(f) = \{f, g\}$, where $\operatorname{dom} g = \emptyset$.*

Proof. Since f is the identity on A, we conclude $Q(f) = F(A)$. It is easy to see that $Q(g) = F(A)$. Thus $\{f, g\} \subset EQ(f)\}$.

Assume that $h \in Q(f)$ is such that $h \neq g, h \neq f$ and $Q(h) = Q(f)$. Then $\operatorname{dom} h \neq \emptyset$. Further $\operatorname{dom} h \neq A$, because $h(z) = z$ for each $z \in \operatorname{dom} h$ according to 2.1. Thus we can to choose $a \in \operatorname{dom} h$ and $b \notin \operatorname{dom} h$. Consider $\varphi \in F(A)$, $\varphi = \{[a, b]\}$. We have $\varphi \in Q(f) - Q(h)$, which is a contradiction. □

4.5. Corollary. *Let $(A, f) \in \mathcal{U}_p$ be of type γ. Then $EQ(f) = \{f, g\}$, where g is the identity on A.*

Proof. Analogously as the proof of the last assertion. □

4.6. Lemma. *Suppose that $(A, f) \in \mathcal{U}_p, \|A\| > 1, (A, f)$ is neither of type π nor of type γ and $g \in EQ(f)$. If $a \in K_n$, then $a \notin \operatorname{dom} g$ and $(\{a\}, g_a)$, where $g_a = g \restriction \{a\}$, is a component of (A, g).*

Proof. Let $a \in K_n$. Assume that there exists a component (B, g_B) of (A, g) such that $\|B\| > 1$ and $a \in B$. By virtue of 1.8 we get $Q(g_B) = Q(f_B)$, where

$f_B = f \restriction B$, and thus (B, g_B) is of type τ by 3.8. That means $B = A$ according to 4.2 and consequently, (A, f) is of type π. Thus $(\{a\}, g_a)$ is a component of (A, g).

Since (A, f) is not of type γ, the algebra (A, g) is not of type δ by 4.5. Let us choose $x \in A$ such that either $x \notin \operatorname{dom} g$ or $g(x) \neq x$.

Consider $a \in \operatorname{dom} g$. We obtain $g(a) = a$, because $(\{a\}, g_a)$ is a component of (A, g). Take $\varphi = \{[a, x]\}$. We have $\varphi \in Q(f) - Q(g)$, a contradiction with $Q(f) = Q(g)$. This gives the desired conclusion that $a \notin \operatorname{dom} g$. □

4.7. Corollary. *Let $(A, f) \in \mathcal{U}_p$ and $g \in EQ(f)$.*

1) If $K \neq A$, then $f \restriction K = g \restriction K$.

2) If $K = A, K_n \neq \emptyset$ and $\|K_d\| > 1$, then $f = g$.

P r o o f. Let $K \neq A$. Then the relation $f \restriction K_d = g \restriction K_d$ follows from 4.3 and the relation $f \restriction K_n = g \restriction K_n$ follows from 4.6.

Let the assumptions of the second assertion be satisfied in the algebra (A, f). Then as well as the assumptions of 4.3 and 4.6 are satisfied. We get $f = f \restriction (K_d \cup K_n) = g \restriction (K_d \cup K_n) = g$. □

4.8. Lemma. *Let $(A, f) \in \mathcal{U}_p$ and let (A, f) be of type α. Then $EQ(f) = \{f, g\}$, where $\operatorname{dom} g = \operatorname{rng} f$ and $g(f(a)) = a$ for each $a \in \operatorname{dom} f$.*

P r o o f. First we will show that $Q(f) = Q(g)$. Suppose that $\varphi \in Q(f)$. Further let $x \in \operatorname{dom} g$ and $x, g(x) \in \operatorname{dom} \varphi$. We can choose $y \in \operatorname{dom} f$ such that $f(y) = x$ and $y = g(f(y)) = g(x) \in \operatorname{dom} \varphi$. We obtain $g(\varphi(x)) = g(\varphi(f(y))) = g(f(\varphi(y))) = \varphi(y)$ and $\varphi(g(x)) = \varphi\big(g(f(y))\big) = \varphi(y)$, because $y \in \operatorname{dom} f$ and $y, f(y) \in \operatorname{dom} \varphi$. Therefore $\varphi \in Q(g)$.

Using $\operatorname{dom} f = \operatorname{rng} g$ and $f(g(a)) = a$ for each $a \in \operatorname{dom} g$, the inclusion $Q(g) \subset Q(f)$ can be proved in the same way.

Assume that $h \in EQ(f), h \neq f$. To complete the proof, let us show that $h_B = g_B$ for a set B such that (B, f_B) is a component of (A, f), where $h_B = h \restriction B, g_B = g \restriction B$.

If $\|B\| = 1$, then $h_B = f_B = g_B$ follows from 4.7 and from the definition of algebras of type α.

Now let $\|B\| > 1$. We get $Q(f_B) = Q(h_B)$ by 1.8. The algebra (B, f_B) is either a chain or a cycle and consequently $h_B = g_B$ in view of 3.8. □

4.9. Lemma. *Suppose that $(A, f) \in \mathcal{U}_p, K \neq A$ and that (A, f) is neither of type τ nor of type α. Then $EQ(f) = \{f\}$.*

P r o o f. If (A, f) is connected, then $EQ(f) = \{f\}$ according to 3.8. Assume that (A, f) is not connected. Then there exists a component (B, f_B) of (A, f) such that $\|B\| > 1$ and (B, f_B) is neither a cycle nor a chain. According to 4.2 we have $EQ(f) = \{f\}$ for (B, f_B) of type τ.

Let (B, f_B) be not of type τ and $h \in EQ(f)$. We conclude $Q(f_B) = Q(h_B)$ and $h_B = f_B$ by 1.8 and 3.8. That means $h = f$ in view of 4.1. □

4.10. Theorem. *Let* $(A, f) \in \mathcal{U}_p$.

1° *If* (A, f) *is of type* α, *then* $EQ(f) = \{f, g\}$, *where* $\operatorname{dom} g = \operatorname{rng} f$ *and* $g(f(a)) = a$ *for each* $a \in \operatorname{dom} f$.

2° *If* (A, f) *is of type* τ *with a value* a, *then* $EQ(f) = \{f, g\}$, *where* (A, g) *is of type* π *with a value* a.

3° *If* (A, f) *is of type* π *with a value* a, *then* $EQ(f) = \{f, g\}$, *where* (A, f) *is of type* τ *with a value* a.

4° *If* (A, f) *is of type* δ, *then* $EQ(f) = \{f, g\}$, *where* (A, g) *is of type* γ.

5° *If* (A, f) *is of type* γ, *then* $EQ(f) = \{f, g\}$, *where* (A, g) *is of type* δ.

6° *Otherwise* $EQ(f) = \{f\}$.

Proof. The assertion is the consequence of 3.8, 4.4, 4.5, 4.7, 4.8 and 4.9. □

4.11. Corollary. *The relation* $\|EQ(f)\| \leqslant 2$ *is valid for each* $(A, f) \in \mathcal{U}_p$.

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