Marko Švec

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OSCILLATORY PROPERTIES OF SOLUTIONS TO A DIFFERENTIAL INCLUSION OF ORDER n

MARKO ŠVEC, Bratislava

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The aim of this paper is to investigate the oscillatory as well as the nonoscillatory solutions and their asymptotic behaviour, of the differential inclusion

$$L_n x(t) \in F\big(t, x(\varphi(t))\big),\ n > 1 \tag{E}$$

where $L_n x(t)$ is the n-th quasiderivative of $x(t)$ with respect to the continuous functions $a_i(t)\colon J = [t_0, \infty) \to (0, \infty)$, $i = 0, 1, \ldots, n$, $L_0 x(t) = a_0(t) x(t)$, $L_i x(t) = a_i(t)\big(L_{i-1} x(t)\big)'$, $\int^{\infty} a_i^{-1}(t)\,\mathrm{d}t = \infty$; $F(t, x)\colon J \times \mathbf{R} \to \{$nonempty convex compact subsets of $\mathbf{R}\}$, $\mathbf{R} = (-\infty, \infty)$; $\varphi\colon J \to \mathbf{R}$ a continuous function such that $\lim \varphi(t) = \infty$ as $t \to \infty$.

Under a solution $x(t)$ of (E) we will understand a solution existing on some ray $[T_x, \infty)$ such that

$$\sup\big\{|x(t)|: t_1 \leqslant t < \infty\big\} > 0 \text{ for any } t_1 > T_x.$$

We will assume existence of such solutions.

Notation. $F(t, x)x > 0$ (< 0) means $yx > 0$ (< 0) for each $y \in F(t, x)$; if $h\colon J \times \mathbf{R} \to \mathbf{R}$, then $F(t, x) \geqslant (\leqslant)\ h(t, x)$ means: $y \geqslant (\leqslant)\ h(t, x)$ for each $y \in F(t, x)$. If $B \subset \mathbf{R}$ then $\|B\| = \inf\big\{|x|: x \in B\big\}$.

The following basic assumptions will be used:

1° $F(t, x)$ is upper semicontinuous on $J \times \mathbf{R}$;

2° $F(t, 0) = 0$ for each $t \in J$;

3° $F(t, x)x < 0$ for each $(t, x) \in J \times \mathbf{R}$, $x \neq 0$;

or 4° $F(t, x)x > 0$ for each $(t, x) \in J \times \mathbf{R}$, $x \neq 0$.

The notions of oscillatory and nonoscillatory solutions will be used in the usual sense.

Consider the inclusion (E) and assume that the assumptions 1°–4° are satisfied. Let $x(t)$ be a nonoscillatory solution of (E). Then from the assumption $\lim \varphi(t) = \infty$

as $t \to \infty$ it follows the existence of such $t_1 \geqslant t_0$ that $x(\varphi(t)) \neq 0$ on $[t_1, \infty)$. Taking into consideration the assumptions 1°–4° we get that $x(t)L_n x(t) \neq 0$ on $[t_1, \infty)$. Therefore, $x(t)L_n x(t) > 0$ if 1°, 2°, 4° are satisfied and $x(t)L_n x(t) < 0$ if 1°, 2°, 3° are satisfied on $[t_1, \infty)$. This implies that there exists $t_2 \geqslant t_1$ such that each $L_i x(t)$, $i = 0, 1, \ldots, n$, has a constant sign on $[t_2, \infty)$. Therefore, each $L_i x(t)$, $i = 0, 1, \ldots, n-1$, is monotone on $[t_2, \infty)$, and $\lim L_i x(t)$ as $t \to \infty$, $i = 0, 1, \ldots, n-1$, exists in the extended sense, i.e. $\lim |L_i x(t)|$ is finite or ∞ as $t \to \infty$ and $i = 0, 1, \ldots, n-1$. More detailed considerations [1] lead to the following result: For the nonoscillatory solutions of (E) the following two cases are posible:

a) $\lim\limits_{t\to\infty} |L_i x(t)| = \infty$ for $i = 0, 1, \ldots, n-1$;

b) there exists $k \in \{0, 1, \ldots, n-1\}$ such that $\lim\limits_{t\to\infty} L_k x(t)$ is finite,
$\lim\limits_{t\to\infty} L_i x(t) = \infty \cdot \operatorname{sgn} x(t)$, $i = 0, 1, \ldots, k-1$,
$\lim\limits_{t\to\infty} L_i x(t) = 0$, $i = k+1, \ldots, n-1$.

Remark 1. The case a) can occour only if the assumptions 1°, 2°, 4° are satisfied.

In fact, if the assumptions 1°, 2°, 3° are satisfied then $x(t)L_n x(t) < 0$. Therefore, if $x(t) > 0$ then $L_{n-1}x(t)$ descreases and must be ultimately positive. If $x(t) < 0$ then $L_{n-1}x(t)$ increases and must be ultimately negative. Thus $|\lim\limits_{t\to\infty} L_{n-1}x(t)| < \infty$.

These considerations show that the set of all nonoscillatory solutions of (E) can be divided into disjoint classes in the following way.

Definition 1. We will say that a nonoscillatory solution $x(t)$ of (E) belongs to the class V_n if the case a) occurs. We will say that a nonoscillatory solution $x(t)$ of (E) belongs to the class V_k, $k \in \{0, 1, \ldots, n-1\}$, if the case b) occurs.

In the sequel we will use the following notation and lemmas:
Let $t_0 \leqslant c < t < \infty$. Then

$$P_0(t,c) = 1,$$

$$P_i(t,c) = \int_c^t a_1^{-1}(s_1) \int_c^{s_1} a_2^{-1}(s_2) \ldots \int_c^{s_{i-1}} a_i^{-1}(s_i)\, \mathrm{d}s_i \ldots \mathrm{d}s_1,$$

$i = 1, 2, \ldots, n-1$;

$$Q_n(t,c) = 1,$$

$$Q_j(t,c) = \int_c^t a_{n-1}^{-1}(s_{n-1}) \int_c^{s_{n-1}} a_{n-2}^{-1}(s_{n-2}) \ldots \int_c^{s_{j+1}} a_j^{-1}(s_j)\, \mathrm{d}s_j \ldots \mathrm{d}s_{n-1},$$

$j = 1, 2, \ldots, n-1$.

It is easy to see that

$$\lim_{t\to\infty} P_i(t,c) = \infty, \ \lim_{t\to\infty} Q_i(t,c) = \infty, \ \text{for } i = 1,2,\dots,n-1,$$

and taking into account the properties of $a_i(t)$, by the l'Hospital rule we get

$$\lim_{t\to\infty} P_i(t,c)P_j^{-1}(t,c) = 0 \text{ for } 0 \leqslant i < j \leqslant n-1,$$
$$\lim_{t\to\infty} Q_j(t,c)Q_i^{-1}(t,c) = 0 \text{ for } 0 < i < j \leqslant n-1.$$

Lemma 1 ([1], Lemma 4). *Let $z(t)$ be such that $z(t) \neq 0$ on $[t_1,\infty)$ and $L_n z(t)$ exists on $[t_1,\infty)$. Let $z(t)L_n z(t) \leqslant 0$ on $[t_1,\infty)$, where the equality may eventually hold at isolated points. Let $k \in \{0,1,\dots,n-1\}$ be such that b) is fulfilled. Then there exists a $T_1 \geqslant t_1$ such that $\operatorname{sgn} z(t) = \operatorname{sgn} L_k z(t)$ for $t \geqslant T_1$.*

If $n+k$ is even then $|L_k z(t)|$ increases on $[T_1,\infty)$ and there exist two constants $0 < c_1 < c_2$ such that for $t > T_1$

$$0 < c_1 < |L_k z(t)| < c_2$$

and

$$0 < c_1 < \lim_{t\to\infty}|L_0 z(t)P_k^{-1}(t,c)| < c_2, \ \lim_{t\to\infty} L_0 z(t)P_{k+1}^{-1}(t,c) = 0.$$

If $n+k$ is odd then $|L_k z(t)|$ descreases on $[T_1,\infty)$ and there exists a constant $c > 0$ such that we have

$$0 < |L_k z(t)| < c \text{ for } t > T_1,$$
$$0 \leqslant \lim_{t\to\infty}|L_0 z(t)P_k^{-1}(t,c)| < c, \ \lim_{t\to\infty} L_0 z(t)P_{k+1}^{-1}(t,c) = 0.$$

Lemma 2 ([1], Lemma 6). *Let $z(t)$ be such that $z(t) \neq 0$ and $L_n z(t)$ exists, both on $[t_1,\infty)$. Let $z(t)L_n z(t) \geqslant 0$ for $t \geqslant t_1$, where the equality may hold at isolated points. Let $k \in \{0,1,\dots,n-1\}$ be such that b) is fulfilled. Then there exists a $T_1 \geqslant t_1$ such that $\operatorname{sgn} z(t) = \operatorname{sgn} L_k z(t)$ for $t > T_1$.*

If $n+k$ is odd then $|L_k z(t)|$ increases on $[T_1,\infty)$ and there exist two constants $0 < c_1 < c_2$ such that

$$0 < c_1 < |L_k z(t)| < c_2 \text{ for } t > T_1$$

and

$$0 < c_1 < \lim_{t\to\infty}|L_0 z(t)P_k^{-1}(t,c)| < c_2, \ \lim_{t\to\infty} L_0 z(t)P_{k+1}^{-1}(t,c) = 0.$$

If $n+k$ is even then $|L_k z(t)|$ descreases on $[T_1,\infty)$ and there exists a constant $c_3 > 0$ such that

$$0 < |L_k z(t)| < c_3 \text{ for } t > T_1,$$
$$0 \leqslant \lim_{t\to\infty}|L_0 z(t)P_k^{-1}(t,c)| < c_3, \ \lim_{t\to\infty} Lz(t)P_{k+1}^{-1}(t,c) = 0.$$

Lemma 3 ([2], Lemma 3). *Let $x(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$. Then*

$$\lim_{t\to\infty} L_0 x(t) P_k^{-1}(t,c) = \lim_{t\to\infty} L_k x(t) = c_k.$$

If $c_k \neq 0$ then there exist constants $\alpha_k > 0$, $\beta_k > 0$ and $T_k' > t_0$ such that

$$\alpha_k a_0^{-1}(t) P_k(t,c) \leqslant |x(t)| \leqslant \beta_k a_0^{-1}(t) P_k(t,c),\ t > T_k'. \tag{1}$$

We will consider two problems. The first problem is to find conditions which guarantee that $\lim L_k x(t) = 0$ as $t \to \infty$ for each $x(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$. The second problem is to state conditions which guarantee that the class V_k, $k \in \{0, 1, \dots, n-1\}$, is empty. These problems were discussed in [1], [2], [3] if instead of the inclusion (E) we have a differential equation.

Theorem 1. *Let the assumptions 1°-4° be satisfied. Let $G(t,u)\colon Jx[0,\infty) \to [0,\infty)$ be a continuous and nondecreasing function in u for each fixed $t \in J$, such that*

$$G(t,|x|) \leqslant \|F(t,x)\|,\ x \in \mathbf{R}.$$

Let $k \in \{0, 1, \dots, n-1\}$. Suppose that

$$\int_t^\infty a_n^{-1}(s) Q_{k+1}(s,t) G\big(s, \alpha a_0^{-1}(\varphi(s)) P_k(\varphi(s),c)\big)\, \mathrm{d}s = \infty \tag{2}$$

for all $t \geqslant T_k$ such that $\varphi(s) > c$ for $s > T_k \geqslant T_k'$, $c \geqslant t_0$, and for each $\alpha > 0$, or

$$\limsup_{t\to\infty} \int_t^\infty a_n^{-1}(s) Q_{k+1}(s,t) G\big(s, \alpha a_0^{-1}(\varphi(s)) P_k(\varphi(s),c)\big)\, \mathrm{d}s > 0 \tag{3}$$

for each $\alpha > 0$.

Then for each $x(t) \in V_k$ we have $\lim L_k x(t) = 0$ as $t \to \infty$.

Proof. Let $x(t) \in V_k$, $k \in \{0, 1, \dots, n-1\}$ and let $\lim L_k x(t) = c_k \neq 0$ as $t \to \infty$. Then

$$0 \leqslant G\big(t, |x(\varphi(t))|\big) \leqslant \big\|F\big(t, x(\varphi(t))\big)\big\|,\ t > T_k,$$

and

$$0 \leqslant G\big(t, |x(\varphi(t))|\big) \leqslant |L_n x(t)|,\ t > T_k. \tag{4}$$

Assume that T_k is such that for $t \geqslant T_k$, $x(t)$ has a constant sign, $\operatorname{sgn} x(t) = \operatorname{sgn} L_k x(t)$ for $t \geqslant T_k$ and (1) from Lemma 3 holds. Then the successive integrations on $[t, \infty)$,

$t > T_k$, of (4), by virtue of the fact that $\lim L_i x(t) = 0$ as $t \to \infty$, $i = k+1, \ldots, n-1$, give

$$0 \leqslant \int_t^\infty a_n^{-1}(s) Q_{k+1}(s,t) G\big(s, \big|x\big(\varphi(s)\big)\big|\big)\,\mathrm{d}s \leqslant \big|L_k x(t) - c_k\big|.$$

From Lemma 3 we have

$$\big|x\big(\varphi(t)\big)\big| \geqslant \alpha_k a_0^{-1}\big(\varphi(t)\big) P_k\big(\varphi(t), c\big).$$

Therefore, $G(t,u)$ being nondecreasing, we get

$$0 \leqslant \int_t^\infty a_n^{-1}(s) Q_{k+1}(s,t) G\big(s, \alpha a_0^{-1}\big(\varphi(s)\big) P_k\big(\varphi(s), c\big)\big)\,\mathrm{d}s \leqslant \big|L_k x(t) - c_k\big|.$$

The expression on the right-hand side is bounded. This leads to a contradiction with (2). If (3) is satisfied then we have once more a contradiction, because $\lim\big|L_k x(t) - c_k\big| = 0$ as $t \to \infty$. □

Theorem 2. *Let all assumptions of Theorem* 1 *be satisfied. Then, provided the assumptions* 1°, 2°, 3° *are satisfied, the sets* V_k *for* $n+k$ *even are empty. If the assumptions* 1°, 2°, 4° *are satisfied then the sets* V_k *for* $n+k$ *odd are empty.*

Proof follows from Theorem 1 and from Lemma 1 and 2, respectively. Denote

$$\gamma(t) = \sup\big\{s \geqslant t_0 : \varphi(s) \leqslant t\big\} \text{ for all } t \geqslant t_0$$

and

$$m(t) = \max\big\{\gamma(t), t\big\},\ t \geqslant t_0.$$

We see that $m(t) \geqslant t$. From the continuity of $\varphi(t)$ we get $\varphi(s) > t$ for $s > \gamma(t)$ and $\varphi(s) \geqslant t$ for $s \geqslant m(t)$, $t \geqslant t_0$. Evidently $\lim m(t) = \infty$ as $t \to \infty$.

Consider the class V_k, $k \in \{0, 1, \ldots, n-1\}$. Form the properties of the set V_k we get that $\lim L_{n-1} x(t)$ as $t \to \infty$ is finite for each $x(t) \in V_k$. Then by virtue of the assumptions of Theorem 1, (4) yields

$$0 \leqslant \int_t^\infty a_n^{-1}(s) G\big(s, \big|x\big(\varphi(s)\big)\big|\big)\,\mathrm{d}s \leqslant \big|L_{n-1} x(t)\big| < \infty. \tag{5}$$

Our forthcoming considerations are based on this fact. Successive integration of (5), together with the fact that $\lim L_i x(t) = 0$ as $t \to \infty$, $i = k, k+1, \ldots, n-1$, give

$$0 \leqslant \int_t^\infty a_n^{-1}(s) Q_{k+1}(s,t) G\big(s, \big|x\big(\varphi(s)\big)\big|\big)\,\mathrm{d}s \leqslant \big|L_k x(t)\big|. \tag{6}$$

a) Assume that $x(t) \in V_k$, $x(t) > 0$ for $t > t_3$, $k > 0$, where t_3 is such that $L_i x(t)$, $i = 0, 1, \ldots, n-1$, has a constant sign. Then $L_k x(t) > 0$ for $t > t_3$ and the integration of (6) between u and v, $t_3 \leqslant u < v$, and the application of Fubini's theorem yield

$$\begin{aligned} 0 &\leqslant \int_u^v a_n^{-1}(s) G\big(s, |x(\varphi(s))|\big) \int_u^s a_k^{-1}(t) Q_{k+1}(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ &+ \int_u^\infty a_n^{-1}(s) G\big(s, |x(\varphi(s))|\big) \int_u^v a_k^{-1}(t) Q_{k+1}(s,t)\,\mathrm{d}t\,\mathrm{d}s \\ &\leqslant L_{k-1}x(v) - L_{k-1}x(u) \leqslant L_{k-1}x(v) \end{aligned} \tag{7}$$

because $L_{k-1}x(t) > 0$ for $t > t_3$. It follows from the definiton of $Q_{k+1}(s,t)$ than for $t \leqslant v \leqslant s$

$$Q_{k+1}(s,t) \geqslant Q_{k+1}(v,t).$$

Therefore, from (7) we get

$$0 \leqslant \int_u^v a_k^{-1}(t) Q_{k+1}(v,t)\,\mathrm{d}t \int_v^\infty a_n^{-1}(s) G\big(s, |x(\varphi(s))|\big)\,\mathrm{d}s \leqslant L_{k-1}x(v). \tag{8}$$

Repeating this procedure $(k-1)$-times, we get

$$\begin{aligned} 0 \leqslant \int_u^v a_1^{-1}(t_1) \int_u^{t_1} a_2^{-1}(t_2) \ldots \int_u^{t_{k-1}} a_k^{-1}(t_k) Q_{k+1}(t_{k-1}, t_k)\mathrm{d}w_k \\ \cdot \int_v^\infty a_n^{-1}(s) G\big(s, |x(\varphi(s))|\big)\,\mathrm{d}s \leqslant L_0 x(v) \end{aligned} \tag{9}$$

for $t_3 \leqslant u < v$, where $\mathrm{d}w_k = \mathrm{d}t_k\,\mathrm{d}t_{k-1} \ldots \mathrm{d}t_1$. Denote

$$R_k(v,u) = \int_u^v a_1^{-1}(t_1) \int_u^{t_1} a_2^{-1}(t_2) \ldots \int_u^{t_{k-1}} a_k^{-1}(t_k) Q_{k+1}(t_{k-1}, t_k)\,\mathrm{d}w_k. \tag{10}$$

Then we have

$$0 \leqslant R_k(v,u) \int_v^\infty a_n^{-1}(s) G\big(s, |x(\varphi(s))|\big)\,\mathrm{d}s \leqslant L_0 x(v),\ t_3 \leqslant u < v. \tag{11}$$

The monotonicity of G and the properties of $m(t)$ yield

$$
\begin{aligned}
\left|L_0x(v)\right| &\geqslant R_k(v,u)\int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,\left|x(\varphi(s))\right|\big)\,\mathrm{d}s \\
&= R_k(v,u)\int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,a_0^{-1}(\varphi(s))\left|L_0x(\varphi(s))\right|\big)\,\mathrm{d}s.
\end{aligned} \tag{12}
$$

But $\left|L_0x(t)\right|$ is nondecreasing, $\varphi(s)\geqslant v$ for $s\geqslant m(v)$ and $G(t,z)$ is nondecreasing in z. Therefore, (12) implies

$$
\left|L_0x(v)\right| \geqslant R_k(v,u)\int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,a_0^{-1}(\varphi(s))\left|L_0x(v)\right|\big)\,\mathrm{d}s. \tag{13}
$$

for $t_3\leqslant u<v$. Once more by virtue of the monotonicity of $G(t,z)$ we get

$$
\begin{aligned}
&\int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,a_0^{-1}(\varphi(s))\left|L_0x(v)\right|\big)\,\mathrm{d}s \\
&\geqslant \int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\Big(s,a_0^{-1}(\varphi(s))\,R_k(v,u)\int\limits_{m(v)}^{\infty} a_n^{-1}(\tau)G\big(\tau,a_0^{-1}(\varphi(\tau))\left|L_0x(v)\right|\big)\,\mathrm{d}\tau\Big)\,\mathrm{d}s.
\end{aligned}
$$

Denote

$$
p(v)=\int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,a_0^{-1}(\varphi(s))\left|L_0x(v)\right|\big)\,\mathrm{d}s \tag{14}
$$

Then we have

$$
p(v)\geqslant \int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,a_0^{-1}(\varphi(s))\,R_k(v,u)p(v)\big)\,\mathrm{d}s. \tag{15}
$$

From (5) and (14) we obtain

$$
\begin{aligned}
\left|L_{n-1}x(m(v))\right| &\geqslant \int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,\left|x(\varphi(s))\right|\big)\,\mathrm{d}s \\
&\geqslant \int\limits_{m(v)}^{\infty} a_n^{-1}(s)G\big(s,a_0^{-1}(\varphi(s))\left|L_0x(v)\right|\big)\,\mathrm{d}s = p(v)\geqslant 0
\end{aligned}
$$

and

$$0 \leqslant \lim_{v\to\infty} p(v) \leqslant \lim_{v\to\infty} L_{n-1}x\big(m(v)\big) = 0.$$

Thus

$$\lim_{v\to\infty} p(v) = 0. \tag{16}$$

b) Let $x(t) \in V_k$, $x(t) < 0$ for $T \geqslant t_3$, $k > 0$. Then $\operatorname{sgn} L_k x(t) = \operatorname{sgn} x(t) = -1$ and from (6) we get

$$0 \leqslant \int_t^\infty a_n^{-1}(s)Q_{k+1}(s,t)G\big(s,\big|(\varphi(s))\big|\big)\,\mathrm{d}s \leqslant -L_k x(t),\ t \geqslant t_3.$$

Similar considerations as in the case a) lead to the inequalities (13), (15) and equality (16). □

Now we are able to prove the following theorems:

Theorem 3. *Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for every fixed $t \geqslant t_0$*

$$z^{-1}G(t,z) \text{ is nondecreasing for } z > 0, \tag{17}$$

and for $k \in \{1,2,\dots,n-1\}$,

$$\lim_{v\to\infty} \sup R_k(v,u) \int_{m(v)}^\infty a_n^{-1}(s)c^{-1}G\big(s, a_0^{-1}(\varphi(s))c\big)\,\mathrm{d}s > 1 \tag{18}$$

for some $c > 0$. Then the set V_k is empty.

Proof. Let $x(t) \in V_k$, $k \in \{1,2,\dots,n-1\}$. Then $\lim_{t\to\infty}\big|L_0x(t)\big| = \infty$. Therefore, for $c > 0$ there exists $v_1 > u \geqslant t_3$ such that $\big|L_0x(v)\big| > c$ for all $v > v_1$. Then from (13) and (17) we obtain

$$1 \geqslant R_k(v,u) \int_{m(v)}^\infty a_n^{-1}(s)a_0^{-1}\big(\varphi(s)\big)\frac{G\big(s,a_0^{-1}(\varphi(s))c\big)}{a_0^{-1}\big(\varphi(s)\big)c}\,\mathrm{d}s,$$

which contradicts (18). □

Theorem 4. *Let all assumptions of Theorem 1 be satisfied. Moreover, assume that for every fixed $t \geqslant t_0$*

$$z^{-1}G(t,z) \text{ is nonincreasing for } z > 0, \tag{19}$$

and for $k \in \{1, 2, \dots, n-1\}$

$$\limsup_{v\to\infty} \int_{m(v)}^{\infty} a_n^{-1}(s)c^{-1}G\big(s, R_k(v,u)a_0^{-1}(\varphi(s))c\big)\,\mathrm{d}s > 1 \tag{20}$$

for some $c > 0$. Then the set V_k is empty.

Proof. Let $K \in \{1, 2, \dots, n-1\}$ and $x(t) \in V_k$. Because $\lim p(v) = 0$ as $v \to \infty$ and $p(v) > 0$ for $v > u$, for $c > 0$ there exists $v_2 \geqslant u \geqslant t_3$ such that $c > p(v)$ for all $v > v_2$. Taking into account (15) and (19) we obtain

$$1 \geqslant \int_{m(v)}^{\infty} a_n^{-1}(s)a_0^{-1}\big(\varphi(s)\big)R_k(v,u)\frac{G\big(s, a_0^{-1}(\varphi(s))R_k(v,u)c\big)}{a_0^{-1}\big(\varphi(s)\big)R_k(v,u)c}\,\mathrm{d}s$$

for all $v > v_2$. This leads to a contradiction with (20). □

Definition 2. We will say that the inclusion (E) has property A if, provided n is even, all solutions of (E) are oscillatory and, provided n is odd, each solution $x(t)$ of (E) is either oscillatory or $\lim L_i x(t) = 0$ as $t \to \infty$ for $i = 0, 1, \dots, n-1$.

Definition 3. We will say that the inclusion (E) has property B if for n even each solution $x(t)$ of (E) is either oscillatory or $\lim L_i x(t) = 0$ as $t \to \infty$ for $i = 0, 1, \dots, n-1$ or it belongs to the class V_n, i.e. $\lim|L_i x(t)| = \infty$ as $t \to \infty$ for $i = 0, 1, \dots, n-1$, and for n odd each solution $x(t)$ of (E) either is oscillatory or belongs to the class V_n.

Now, from the Theorems 1–4 we obtain the final theorem:

Theorem 5. *Let all assumptions of Theorem 1 be satisfied.*

a) If the assumptions 1°, 2°, 3° are satisfied and if (17) and (18) (or (19) and (20)) hold for $k = 1, 2, \dots, n-1$, then the inclusion (E) has property A.

b) If the assumptions 1°, 2°, 4° are satisfied and if (17) and (18) (or (19) and (20)) hold for $k = 1, 2, \dots, n-1$, then the inclusion (E) has property B.

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Author's address: Matematicko-fyzikálna fakulta, Katedra matematickej analýzy, Mlynská dolina, 842 15 Bratislava, Czechoslovakia.