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SEQUENTIAL CONVERGENCES IN ℓ-GROUPS WITHOUT URYSOHN'S AXIOM

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The system $\operatorname{Conv} G$ of all sequential convergences on an ℓ-group G satisfying Urysohn's axiom was investigated in the papers [4–9], [11], [12].

All ℓ-groups which are considered in the present paper are assumed to be abelian. Let us denote by $\operatorname{conv} G$ the system of all sequential convergences on G which satisfy the usual conditions (as in the above mentioned papers) except Urysohn's axiom. (For a detailed definition cf. Section 1 below.)

One of the reasons for studying $\operatorname{conv} G$ is the fact that the o-convergence on G belongs to $\operatorname{conv} G$, but it need not belong in general to the system $\operatorname{Conv} G$. For example, the o-convergence on the vector lattice S does not satisfy Urysohn's axiom (cf. e.g., [13], Chap. III, § 9).

Both the systems $\operatorname{Conv} G$ and $\operatorname{conv} G$ are partially ordered by inclusion.

For each $\alpha \in \operatorname{conv} G$ there exists a uniquely determined element α^* of $\operatorname{Conv} G$ such that $\alpha \leqslant \alpha^*$ and whenever $\beta \in \operatorname{Conv} G$ with $\alpha \leqslant \beta$, then $\alpha^* \leqslant \beta$. Hence the intersection of the interval $[\alpha, \alpha^*]$ of $\operatorname{conv} G$ with the system $\operatorname{Conv} G$ is a one-element set.

Sample results:

For each cardinal m there exist an ℓ-group H and $\alpha \in \operatorname{conv} H$ such that $\operatorname{card}[\alpha, \alpha^*] > m$.

The following conditions are equivalent:

$$\text{(i)} \quad \operatorname{conv} G = \operatorname{Conv} G; \qquad \text{(ii)} \quad \operatorname{card} \operatorname{Conv} G = 1.$$

Let $\operatorname{conv} G \neq \operatorname{Conv} G$. Then the set $\operatorname{conv} G \setminus \operatorname{Conv} G$ is infinite. Moreover, if the breadth of G is infinite, then

$$\operatorname{card}(\operatorname{conv} G \setminus \operatorname{Conv} G) \geqslant 2^{\aleph_0}.$$

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A constructive description of atoms of $\operatorname{Conv} G$ was given in [7]. It will be proved below that there are no atoms in $\operatorname{conv} G$.

The system $\operatorname{conv} G$ is a lower semilattice, but it need not be a lattice. If α_o is the o-convergence on G and $\beta \in \operatorname{conv} G$, then the join $\alpha_o \vee \beta$ does exist in $\operatorname{conv} G$. If G is $(\aleph_0, 2)$-distributive, then $\operatorname{conv} G$ is a complete lattice.

The system $\operatorname{Conv} G$ is in a certain sense a closed subset of $\operatorname{conv} G$ (cf. 2.9). Each interval of $\operatorname{conv} G$ is a Brouwerian lattice. For the corresponding dual infinite distributive law the following negative result will be proved. Let the breadth of G be infinite and suppose that G is archimedean, orthogonally complete and divisible; then there are α_n $(n \in N)$ and β in $\operatorname{Conv} G$ such that both the elements $\beta \vee (\bigwedge_{n\in N} \alpha_n)$ and $\bigwedge_{n\in N} (\beta \vee \alpha_n)$ do exist in $\operatorname{conv} G$ (and in $\operatorname{Conv} G$), but these elements fail to be equal.

1. Preliminaries

Let G be an ℓ-group. Next, let $g \in G$ and $(g_n) \in G^N$. If $g_n = g$ for each $n \in N$, then we write $(g_n) = \operatorname{const} g$. For $(h_n) \in G^N$ we set $(h_n) \sim (g_n)$ if there is $m \in N$ such that $h_n = g_n$ for each $n \in N$ with $n \geqslant m$.

Let α be a subset of the semigroup $(G^N)^+$. Consider the following conditions for the set α:

(I) If $(g_n) \in \alpha$, then each subsequence of (g_n) belongs to α.

(II) Let $(g_n) \in (G^N)^+$. If each subsequence of (g_n) has a subsequence belonging to α, then (g_n) belongs to α.

(II′) Let $(g_n) \in \alpha$ and $(h_n) \in (G^N)^+$. If $(h_n) \sim (g_n)$, then $(h_n) \in \alpha$.

(III) Let $g \in G$. Then $\operatorname{const} g$ belongs to α if and only if $g = 0$.

The system of all convex semigroups α of $(G^N)^+$ which satisfy the conditions (I), (II) and (III) (or the conditions (I), (II') and (III)) will be denoted by $\operatorname{Conv} G$ (or $\operatorname{conv} G$, respectively). (Cf. e.g., [10], Section 1.) It is obvious that $\operatorname{Conv} G \subseteq \operatorname{conv} G$.

For $(g_n) \in G^N$, $g \in G$ and $\alpha \in \operatorname{conv} G$ we put $g_n \to_\alpha g$ if and only if $(|g_n - g|) \in \alpha$.

Let $\alpha(o)$ be the set of all sequences (g_n) in G^+ having the property that there is $(h_n) \in (G^N)^+$ such that (i) $h_{n+1} \geqslant h_n$ is valid for each $n \in N$; (ii) $\bigwedge_{n\in N} h_n = 0$; (iii) there is $m \in N$ such that $h_n \geqslant g_n$ for each $n \in N$ with $n \geqslant m$. (Then we clearly have $\alpha(o) \in \operatorname{conv} G$.) The set $\alpha(o)$ will be said to be the o-convergence in G.

As we have already remarked above, $\alpha(o)$ need not belong to $\operatorname{Conv} G$.

Both $\operatorname{Conv} G$ and $\operatorname{conv} G$ are partially ordered by inclusion.

For α_1 and α_2 in $\operatorname{conv} G$ with $\alpha_1 \leqslant \alpha_2$ we denote by $[\alpha_1, \alpha_2]$ the corresponding interval of $\operatorname{conv} G$. Let $\alpha(d)$ be the set of all $(g_n) \in (G^N)^+$ such that the set $\{n \in N : g_n \neq 0\}$ is finite. Then $\alpha(d)$ is the least element of both $\operatorname{Conv} G$ and $\operatorname{conv} G$.

Let $\alpha \in \operatorname{conv} G$. We denote by α^* the set of all elements (g_n) of $(G^N)^+$ such that each subsequence of (g_n) has a subsequence belonging to α. Clearly $\alpha \subseteq \alpha^*$.

Lemma 1.1. *Let $\alpha \in \operatorname{conv} G$. Then $\alpha^* \in \operatorname{Conv} G$. If $\beta \in \operatorname{Conv} G$ and $\beta \geqslant \alpha$, then $\beta \geqslant \alpha^*$.*

Proof. The first assertion is a consequence of [5], Theorem 2; the latter is obvious. □

Remark 1.2. In [12] the author studied several types of kernels in a convergence ℓ-group, where "convergence ℓ-group" denoted an ℓ-group with a fixed convergence belonging to $\operatorname{Conv} G$. Nevertheless, the condition (II) was not applied and thus the results and their proofs are valid also in the case when the convergence under consideration belongs to $\operatorname{conv} G$. [In the original version (which concerns convergences belonging to $\operatorname{Conv} G$), Lemma 4.1 is to be cancelled; namely in the proof of this lemma the notion of o-convergence was used. Lemma 4.1 was not applied in the proofs of further results of [12].]

2. The partially ordered system conv G

Again, let G be an ℓ-group. If $\{\alpha_i\}_{i\in I}$ is a nonempty system of elements of $\operatorname{conv} G$, then the set $\bigcap_{i\in I} \alpha_i$ is nonempty and satisfies the conditions (I), (II') and (III). Hence we have

Proposition 2.1. *Let $X \neq \emptyset$ be an upper-bounded subset of $\operatorname{conv} G$. Then X is a complete lattice. If $\{\alpha_i\}_{i\in I}$ is as above, then $\bigcap_{i\in I} \alpha_i = \bigwedge_{i\in I} \alpha_i$ is valid in $\operatorname{conv} G$.*

We recall the following notation (cf. [5], Section 2).

Let $\emptyset \neq A \subseteq (G^N)^+$. We denote δA — the set of all $(g_n) \in (G^N)^+$ such that (g_n) is a subsequence of some sequence belonging to A;

$\langle A\rangle$ — the set of all $(g_n) \in (G^N)^+$ having the property that there exist $k \in N$ and $(g_n^1), (g_n^2), \ldots, (g_n^k) \in A$ such that $g_n \leqslant g_n^1 + g_n^2 + \ldots + g_n^k$ holds for each $n \in N$;

$[A]$ — the set of all $(g_n) \in (G^N)^+$ having the property that there exists $(h_n) \in A$ such that $g_n \leqslant h_n$ is valid for each $n \in N$.

Now, let A° be the set of all $(g_n) \in (G^N)^+$ that there exists $(h_n) \in A$ with $(g_n) \sim (h_n)$.

Lemma 2.2. *Let $\emptyset \neq A \subseteq (G^N)^+$. Put $B = [\langle\delta A\rangle]$, $B_1 = B^\circ$. Then*
(i) $B = \delta B = \langle B\rangle$, and
(ii) $B_1 = B_1^\circ = \delta B_1 = \langle B_1\rangle$.

Proof. (i) is a consequence of 1.15 in [6]. It is obvious that $\delta(A^\circ) = (\delta A)^\circ$, $\langle A^\circ\rangle = \langle A\rangle^\circ$ and $[A^\circ] = [A]^\circ$. Hence (i) implies that (ii) holds. □

From the definition of $\operatorname{conv} G$ and from 2.2 we immediately obtain:

Proposition 2.3. *Let $\emptyset \neq A \subseteq (G^N)^+$. Put $B = [\langle \delta A \rangle]^\circ$. If there exists $0 \neq g \in G$ such that* $\operatorname{const} g \in B$, *then there is no $\alpha \in \operatorname{conv} G$ with $A \subseteq \alpha$. If there is no element $g \in G$ such that $g \neq 0$ and* $\operatorname{const} g \in B$, *then $B \in \operatorname{conv} G$; moreover, whenever $\alpha \in \operatorname{conv} G$ and $A \subseteq \alpha$, then $B \subseteq \alpha$.*

Next, 1.1 yields:

Lemma 2.4. *Let $\emptyset \neq A \subseteq (G^N)^+$. Then the following conditions are equivalent:*
(i) *There exists $\alpha \in \operatorname{Conv} G$ with $A \subseteq \alpha$.*
(ii) *There exists $\beta \in \operatorname{conv} G$ with $A \subseteq \beta$.*

Proposition 2.5. *There exists an ℓ-group G such that the partially ordered set* $\operatorname{conv} G$ *fails to be a lattice.*

Proof. In [6], Example 7.6, it was proved that there exists an ℓ-group G having the property that there are α_1 and α_2 in $\operatorname{Conv} G$ such that whenever $\alpha \in \operatorname{Conv} G$, then $\alpha_1 \cup \alpha_2$ fails to be a subset of α. Now from 2.4 we obtain that whenever $\beta \in \operatorname{conv} G$, then $\alpha_1 \cup \alpha_2$ fails to be a subset of β. Therefore the join $\alpha_1 \vee \alpha_2$ does not exist in $\operatorname{conv} G$. □

By appliyng 2.1, 2.4 and proceeding analogous by as in [5], Theorem 2.6 we obtain:

Proposition 2.6. *The following conditions are equivalent:*
(i) $\operatorname{conv} G$ *is a lattice.*
(ii) $\operatorname{conv} G$ *is a complete lattice.*
(iii) $\operatorname{conv} G$ *has a greatest element.*

Lemma 2.7. *Let $\{\alpha_i\}_{i \in I}$ be a nonempty subset of* $\operatorname{conv} G$. *Put $A = \bigcup_{i \in I} \alpha_i$ and $B = [\langle \delta A \rangle]$. Then the following conditions are equivalent:*
(i) *$B \in \operatorname{conv} G$.*
(ii) *$B = \bigvee_{i \in I} \alpha_i$.*

Proof. The implication (ii) ⇒ (i) is obvious. Clearly $A^\circ = A$. Thus, 2.2 and 2.3 yield that (i) ⇒ (ii) is valid. □

From 2.1, 2.4, 2.7 and [5], 2.1, 2.2 and 2.5 we obtain:

Proposition 2.8. *Let $\{\alpha_i\}_{i \in I}$ be a nonempty subset of* $\operatorname{Conv} G$.
(a) *The meet of the system $\{\alpha_i\}_{i \in I}$ in* $\operatorname{Conv} G$ *coincides with the meet of this system in* $\operatorname{conv} G$.
(b) *The join of the system $\{\alpha_i\}_{i \in I}$ in* $\operatorname{Conv} G$ *exists if and only if the join of this system in* $\operatorname{conv} G$ *exists, and in this case these joins coincide.*

If $\alpha \in \operatorname{conv} G$ and H is an ℓ-subgroup of G, then we put

$$\alpha[H] = \alpha \cap (H^N)^+.$$

It is obvious that $\alpha[H] \in \operatorname{conv} H$.

Example 2.9. Consider the vector lattice S (cf. [13], p. 79–80). Let m be a cardinal and let I be a set with $\operatorname{card} I > m$. Next, let $G_i = S$ for each $i \in I$. We denote by G the direct sum $\sum\limits_{i \in I} G_i$.

Let $i \in I$. For $g \in G$ let g_i be the component of g in G_i. We denote by α_i the set of all $(g_n) \in (G^N)^+$ having the property that there exists $m \in N$ such that $(g_{ni})_{n \geqslant m}$ belongs to the set $\alpha(o)[G_i]$, and for each $j \in I$ with $j \neq i$, the sequence $(g_{nj})_{n \geqslant m}$ belongs to the set $\alpha(d)[G_j]$. From 2.9 we infer $\alpha_i \in \operatorname{conv} G$ and that, whenever $i(1)$ and $i(2)$ are distinct elements of I, then $\alpha_{i(1)} \neq \alpha_{i(2)}$. Next, it is easy to verify that α_i^* consists of all $(h_n) \in (G^N)^+$ having the property that there exists $m \in N$ such that $(h_n)_{n \geqslant m}$ belongs to $\alpha(o)^*[G_i]$, and for each $j \in I$ with $j \neq i$, the sequence $(h_{nj})_{n \geqslant m}$ belongs to $\alpha(d)[G_j]$.

Now let α be the set of all $(x_n) \in (G^N)^+$ which satisfy the following condition: there exist $m \in N$ and a finite subset I_1 of I such that $(x_{ni})_{n \geqslant m} \in \alpha(o)[G_i]$ if $i \in I_1$, and $(x_{ni})_{n \geqslant m} \in \alpha(d)[G_i]$ otherwise. Then in view of 2.3 we have $\alpha \in \operatorname{conv} G$. Next, $\alpha_i < \alpha$ and $\alpha < \alpha \vee \alpha_i^*$ for each $i \in I$. Thus $\alpha \vee \alpha_i^* \leqslant \alpha^*$ for each $i \in I$. If $i(1)$ and $i(2)$ are distinct elements of I, then $\alpha \vee \alpha_{i(1)}^* \neq \alpha \vee \alpha_{i(2)}^*$. This yields that the power of the interval $[\alpha, \alpha^*]$ of $\operatorname{conv} G$ is greater or equal to $\operatorname{card} I > m$.

Lemma 2.10. *The following conditions are equivalent:*

(i) $\operatorname{Conv} G$ *has a greatest element;*

(ii) $\operatorname{conv} G$ *has a greatest element.*

Proof. This is an immediate consequence of 1.1. □

Proposition 2.11. *Assume that the ℓ-group G is $(\aleph_0, 2)$-distributive. Then* $\operatorname{conv} G$ *is a complete lattice.*

Proof. In view of [12], $\operatorname{Conv} G$ is a complete lattice. Hence according to 2.10, $\operatorname{conv} G$ has a greatest element. Now 2.6 implies that $\operatorname{conv} G$ is a complete lattice. □

3. Lattice ordered groups having finite breadth

A subset A of G^+ is said to be disjoint if $a_1 \wedge a_2 = 0$ whenever a_1 and a_2 are distinct elements of A. If G has an infinite disjoint subset, then we say that the breadth of G is infinite; otherwise G is said to have a finite breadth.

Lemma 3.1. *Let G be a linearly ordered group, $\alpha \in \operatorname{conv} G$, $\alpha \geqslant \alpha(o)$. Then $\alpha = \alpha(o)$.*

Proof. The case $\alpha = \alpha(d)$ being trivial we can suppose that $\alpha > \alpha(d)$, hence there exists $(g_n) \in \alpha$ such that $g_{n(1)} \neq g_{n(2)}$ whenever $n(1)$ and $n(2)$ are distinct elements of N. Let $0 < g \in G$. Proposition 2.3 yields that the set $\{n \in N : g_n \geqslant g\}$ is finite. Thus for each $n \in N$ there is $m(n) \in N$ such that

$$g_{m(n)} = \max\{g_t : t \in N \text{ and } t \geqslant n\}.$$

If $n(1)$ and $n(2)$ are positive integers with $n(1) < n(2)$, then $g_{m(n(1))} \geqslant g_{m(n(2))}$. By applying 2.3 again we get that $\bigwedge_{n\in N} g_{m(n)} = 0$. Thus $(g_n) \in \alpha(o)$ and hence $\alpha \leqslant \alpha(o)$. Therefore in view of the assumption we have $\alpha = \alpha(o)$. □

Lemma 3.2. *Let G_1 and G_2 be ℓ-groups, $\alpha_i \in \operatorname{conv} G_i$ $(i = 1,2)$ and $G = G_1 \times G_2$. For $g \in G$ let g^i $(i = 1,2)$ be the component of g in G_i. Let α be the set of all $(g_n) \in (G^N)^+$ having the property that there exists $m \in N$ such that $(g^i_n)_{n\geqslant m} \in \alpha_i$ $(i = 1,2)$. Then $\alpha \in \operatorname{conv} G$ and the mapping $(\alpha_1, \alpha_2) \to \alpha$ is an isomorphism of the partially ordered system $\operatorname{conv}(G_1 \times G_2)$ onto $\operatorname{conv} G$.*

Proof. This can be verified by using 2.3 and applying analogous steps as in [4]. Section 4. □

Similarly, from 2.3 and by applying the same procedure as in the proof of [4], Section 5, we obtain:

Lemma 3.3. *Let G and H be ℓ-groups such that G is a lexico extension of H. Let $\alpha \in \operatorname{Conv} H$. Next, let β be the set of all $(g_n) \in (G^N)^+$ having the property that there exists $m \in N$ such that $(g_{n+m})_{n\in N}$ belongs to α. Then $\beta \in \operatorname{conv} G$ and the mapping $\alpha \to \beta$ is an isomorphism of the partially ordered set $\operatorname{conv} H$ onto $\operatorname{conv} G$.*

Lemma 3.4. (a) *Let G_1, G_2 and G be as in 3.2. Let α_i be the o-convergence on G_i $(i = 1, 2)$. Next, let α be as in 3.2. Then α is the o-convergence on G.*

(b) *Let G and A be as in 3.3 and let α be the o-convergence on H. Next, let β be as in 3.3. Then β is the o-convergence on G.*

The proof is easy.

It is well-known that each ℓ-group having a finite breadth can be built up from a finite number of linearly ordered groups by forming direct products and lexico extensions (cf. [1], [2]). Next, if G is a linearly ordered group, then $\alpha(o) \in \operatorname{Conv} G$. Thus Lemmas 3.1–3.4 and [4], Theorem 3.9 yield:

Proposition 3.5. *Let G be an ℓ-group having a finite breadth. Then* $\operatorname{conv} G = \operatorname{Conv} G$.

Lemma 3.6. *Let G be an ℓ-group having a finite breadth. Then G is comletely distributive.*

Proof. This is an easy consequence of the fact that each interval $[u, v]$ of G with $u < v$ has a subinterval $[u_1, v_1]$ such that $u_1 < v_1$ and $[u_1, v_1]$ is linearly ordered. □

Propositon 3.7. *Let G be an ℓ-group having a finite breadth. Then* $\operatorname{conv} G$ *is a complete lattice.*

Proof. It suffices to apply 2.12 and 3.6. □

4. The system $\operatorname{conv} G \setminus \operatorname{Conv} G$

The main results of this section concern the case when the breadth of G is finite.

Let $(x_n) \in (G^N)^+$, $A = \{(x_n)\}$. If $\alpha = [\langle \delta A \rangle]^\circ$ and $\alpha \in \operatorname{conv} G$, then in view of 2.3, α is the least element of $\operatorname{conv} G$ which contains (x_n). In this case α will be said to be a principal convergence generated by the sequence (x_n).

The following assertion is obvious.

Lemma 4.1. *Let α be an atom of* $\operatorname{conv} G$. *Then α is a principal convergence generated by each sequence $(x_n) \in \alpha$ with $(x_n) \notin \alpha(d)(G)$.*

Lemma 4.2. *Let $(x_n), (y_n) \in (G^N)^+$, $x_n \geqslant x_{n+1}$ for each $n \in N$. Put $A = \{(x_n)\}$. Then the following conditions are equivalent:*

(i) *There are positive integers k_1 and m such that $y_{m+n} \leqslant k_1 x_n$ for each $n \in N$.*

(ii) $(y_n) \in [\langle \delta A \rangle]^\circ$.

Proof. The implication (i) ⇒ (ii) obviously holds. Assume that (ii) is valid. Then there exist subsequences (z_n^1), (z_n^2), …, (z_n^t) of (x_n) and positive integers k and m such that

$$y_{m+n} \leqslant k(z_n^1 + z_n^2 + \ldots + z_n^t)$$

is valid for each $n \in N$. Since $z_n^j \leqslant x_n$ for $j = 1, 2, \ldots, t$ we infer that (i) holds. □

Corollary 4.2.1. *Let α be a principal convergence in G which is generated by a strictly decreasing sequence (x_n). Let $(y_n) \in \alpha$. Assume that (y_n) is strictly decreasing. Let m and k_1 be as in 4.2. Then the set*

$$\{n \in N: y_{2n} < k_1 x_n\}$$

is infinite.

Proof. Let $n \in N$, $n > m$. Then $y_{2n} < y_{m+n}$, in view of 4.2 the relation $y_{2n} < k_1 x_n$ is valid. □

Lemma 4.3. *Assume that G is linearly ordered. Let $\alpha \in \operatorname{conv} G$, $\alpha \neq \alpha(d)$. Then there exists $(x_n) \in \alpha$ such that $x_n > x_{n+1}$ for each $n \in N$, and $\bigwedge_{n\in N} x_n = 0$.*

Proof. Since $\alpha \neq \alpha(d)$ there is $(y_n) \in \alpha$ such that $y_n \neq 0$ for each $n \in N$. Denote $z_n = y_1 \wedge y_2 \wedge \ldots \wedge y_n$. Hence $0 < z_n \leqslant y_n$ for each $n \in N$; thus $(z_n) \in \alpha$. According to 2.3, for each $n \in N$ there is $m \in N$ with $m > n$ such that $z_m < z_n$. Thus there is a subsequence (x_n) of (z_n) such that $x_n > x_{n+1}$ for each $n \in N$. Clearly $(x_n) \in \alpha$. Hence $\bigwedge_{n\in N} x_n = 0$. □

Lemma 4.4. *Let $\alpha \in \operatorname{conv} G$ and let (x_n) be a strictly decreasing sequence belonging to α. Then α fails to be an atom in $\operatorname{conv} G$.*

Proof. By way of contradiction, suppose that α is an atom of $\operatorname{conv} G$. From $(x_n) \in \alpha$ we infer that $\bigwedge_{n\in N} x_n = 0$. Next, α is a principal convergence generated by (x_n). We construct by induction a subsequence (t_n) of (x_n) as follows.

We put $t_1 = x_1$. Suppose that $t_1, t_2, \ldots, t_m$ are already defined. From $\bigwedge_{n\in N} x_n = 0$ we obtain $\bigwedge_{n\in N} (m+1)x_n = 0$. Hence there is $n(1) \in N$ such that $x_{n(1)} < t_m$ and $(m+1)x_{n(1)} \ngtr x_{2(m+1)}$. We put $t_{m+1} = x_{n(1)}$.

The relation $x_{2(m+1)} \nless (m+1)t_{m+1}$ is valid for each $m \in N$. Hence if $k_1 \in N$, $m \in N$ and $m+1 > k_1$, then

$$x_{2(m+1)} \nless k_1 t_{m+1}.$$

In view of 2.3 there exists a principal convergence β which is generated by (t_n). Clearly $(t_n) \in \alpha$ and hence $\beta \leqslant \alpha$. Next, according to 4.2.1 the sequence (x_n) does not belong to β. Hence $\beta < \alpha$, which is a contradiction. □

Lemma 4.5. *Let G be a linearly ordered. Then $\operatorname{conv} G$ has no atom.*

Proof. This is a consequence of 4.3 and 4.4.

In the remaining part of the present section we assume that G is a nonzero ℓ-group having a finite breadth. Thus (cf. [1] or [2]) there are nonzero linearly ordered convex ℓ-subgroups $G_1, G_2, \ldots, G_m$ of G such that

(i) if $H \neq \{0\}$ is a convex linearly ordered subgroup of G, then there is a uniquely determined $i \in \{1, 2, \ldots, m\}$ such that $H \subseteq G_i$;

(ii) if $0 < g_i \in G_i$ for each $i \in \{1, 2, \ldots, m\}$, then $\{g_1, g_2, \ldots, g_m\}$ is a maximal disjoint subset of G.

Let $\{g_1, g_2, \ldots, g_m\}$ be a fixed subset of G with the property as in (ii). □

Lemma 4.6. *Let α be a principal element of* conv G *which is genetated by (x_n), $\alpha \neq \alpha(d)$. Then there are $i \in \{1, 2, \ldots, m\}$ and $(y_n) \in \alpha$ such that $0 < y_n \in G_i$ for each $n \in N$.*

Proof. Let $i \in \{1, 2, \ldots, m\}$. Denote $x_n^i = x_n \wedge g_i$. In view of the condition (ii) above we conclude that for some i, the set $\{n \in N : x_n^i \neq 0\}$ is infinite. Hence for this i, there is a subsequence (y_n) of (x_n^i) having the desired properties.

For $i \in I = \{1, 2, \ldots, m\}$ and $\beta \in \operatorname{conv} G_i$ we denote by $f(\beta)$ the set of all sequences (v_n) in G^+ which have the following property: there exists $m \in N$ (depending on (v_n)) such that the sequence $(v_n \wedge g_i)_{n \geqslant m}$ belongs to β, and $v_n \wedge g_j = 0$ whenever $n \leqslant m$ and $j \in I \setminus \{i\}$. □

The following lemma is an obvious consequence of 2.3.

Lemma 4.7. *Let $i \in I$; next, let β_1 and β_2 be the elements of* conv G_i. *Then $f(\beta_i) \in$ conv G. If $\beta_1 < \beta_2$, then $f(\beta_1) < f(\beta_2)$.*

Lemma 4.8. *Let G be an ℓ-group of a finite breadth. Then* conv G *has no atom.*

Proof. By way of contradiction, suppose that α is an atom of conv G. Hence α is principal. Let (x_n) and (y_n) be as in 4.6. Then there is a principal element β of conv G_i which is generated by (y_n). Hence $f(\beta) \leqslant \alpha$ and $\alpha(d) \neq f(\beta)$. Since α is an atom we infer that $\alpha = f(\beta)$. Also, β fails to be the least element of conv G_i. Thus according to 4.5 there is $\beta_1 \in$ conv G_i with $\beta_1 < \beta$. In view of 4.7 we obtain that $f(\beta_1) \in$ conv G and $f(\beta_2) < \alpha$, which is a contradiction.

Let us remark that the above lemma will be sharpened in Section 5 below. □

Corollary 4.9. *Let G be an ℓ-group having a finite breadth. Assume that* card Conv $G > 1$. *Then the set* conv $G \setminus$ Conv G *is infinite.*

Proof. According to [4], Theorem 6.5, the set Conv G is finite. Next, in view of card Conv $G > 1$ there is $\alpha \in$ Conv G with $\alpha \neq \alpha(d)$. Hence the assertion follows from 4.8. □

Lemma 4.10. *Let G be an ℓ-group having a finite breadth. Then the following conditions are equivalent:*

(i) $\operatorname{card} \operatorname{Conv} G = 1$.

(ii) *The set* $\operatorname{conv} G \setminus \operatorname{Conv} G$ *is finite.*

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. Let (i) be valid. By way of contradiction, suppose that $\operatorname{card} \operatorname{conv} G > 1$. Hence there is $\alpha \in \operatorname{conv} G$ with $\alpha \neq \alpha(d)$. Without loss of generality we can assume that α is principal. Let (x_n) and (y_n) be as in 4.6. There exists $\beta \subset \operatorname{conv} G_i$ such that $(y_n) \in \beta$. Hence according to 4.3 there is $(t_n) \in (G_i^N)^+$ such that $t_n > t_{n+1}$ for each $n \in N$ and $\bigwedge_{n\in N} t_n = 0$. By applying [5], Theorem 2.2 we get the $\operatorname{Conv} G \neq \{\alpha(d)\}$, which is a contradiction. □

Lemma 4.11. *Let G be an ℓ-group of finite breadth. Then the following conditions are equivalent:*

(i) $\operatorname{card} \operatorname{Conv} G > 1$.

(ii) *The set* $\operatorname{conv} G \setminus \operatorname{Conv} G$ *is infinite.*

Proof. This is a consequence of 4.9 and 4.10. □

5. The case of ℓ-groups having infinite breadth

We denote by D the system of all sequences $(x_n) \in (G^N)^+$ which satisfy the following conditions:

(i) $x_n > 0$ for each $n \in N$;

(ii) $x_n \wedge x_m = 0$ whenever n and m are distinct positive integers.

Hence $D \neq \emptyset$ if and only if the breadth of G is infinite.

From [4], Theorem 7.3 we obtain

Lemma 5.1. *Let $(x_n) \in D$. Then there exists $\alpha \in \operatorname{Conv} G$ such that $(x_n) \in \alpha$.*

Lemma 5.2. *Let $(x_n) \in D$, $A = \{(x_n)\}$. Then $[\langle \delta A \rangle]^{\circ} \in \operatorname{conv} G$.*

Proof. This is a consequence of 5.1 and 2.3. □

Let $(x_n) \in D$. Denote

$$y_1 = x_1;$$
$$y_2 = y_3 = x_2;$$
$$y_4 = y_5 = y_6 = x_3;$$
$$\vdots$$

Lemma 5.3. *Let (x_n), (y_n) and A be as above. Then (y_n) does not belong to $[\langle\delta A\rangle]^\circ$.*

Proof. By way of contradiction, assume that (y_n) belongs to $[\langle\delta A\rangle]^\circ$. Hence there exist subsequences (x_n^1), (x_n^2), ..., (x_n^m) of (x_n) and positive integers k, k_1, m such that

$$y_n \leqslant k_1(x_n^1 + x_n^2 + \ldots + x_n^k)$$

is valid for each $n \in N$ with $n > m$.

Let $n \in N$ be such that $n > m$ and $n > 2$. Then $y_n = x_{n(1)}$ for some $n(1) < n$. Hence $y_n \wedge x_n^j = 0$ for each $j \in \{1, 2, \ldots, k\}$ and therefore $y_n \not\leqslant k_1(x_n^1 + x_n^2 + \ldots + x_n^k)$, which is a contradiction. □

Lemma 5.4. *Let (x_n) and A be as above. Then $[\langle\delta A\rangle]^\circ$ does not belong to* Conv G.

Proof. Let (y_n) be as above. Each subsequence (z_n) of (y_n) has a subsequence (t_n) such that (t_n) is a subsequence of (x_n), whence $(t_n) \in [\langle\delta A\rangle]^\circ$. On the other hand, in view of 5.3 the sequence (y_n) does not belong to $[\langle\delta A\rangle]^\circ$. Hence $[\langle\delta A\rangle]^\circ$ fails to be an element of Conv G. □

Lemma 5.5. *Let (x_n) and (x'_n) be sequences belonging to D. Assume that $x_n \wedge x'_m = 0$ whenever n and m are positive integers. Let $A = \{(x_n)\}$, $A' = \{(x'_n)\}$. Then $[\langle\delta A\rangle]^\circ \neq [\langle\delta A'\rangle]^\circ$.*

Proof. By an obvious verification. □

If G has infinite breadth, then there exist (x_n^m) in D ($m = 1, 2, \ldots$) such that $x_{n(1)}^{m(1)} \wedge x_{n(2)}^{m(2)} = 0$ whenever $m(1)$ and $m(2)$ are distinct positive integers and $n(1)$, $n(2)$ are arbitrary positive integers. Hence 5.4 and 5.5 yield

Proposition 5.6. *Let G be an ℓ-group with infinite breadth. Then the set* conv $G \setminus$ Conv G *is infinite.*

This result can be slightly sharpened if we apply the following argument. Let (x_n^m) be as above. For $\emptyset \neq M \subseteq N$ let α_M be the convergence which is generated by the $S(M) = \{(x_n^m)\}_{m\in M}$, i.e., $\alpha_M = [\langle\delta S(M)\rangle]^\circ$. (From 2.3 we infer that, in fact, $\alpha_M \in$ conv G.) Next, if M_1 and M_2 are nonempty subsets of N with $M_1 \neq M_2$, then $\alpha_{M_1} \neq \alpha_{M_2}$. Moreover, analogously as in 5.5 we have $\alpha_M \notin$ Conv G.

Thus we obtain

Theorem 5.6.1. *Let G be an ℓ-group with infinite breadth. Then* card(conv $G \setminus$ Conv G) $\geqslant 2^{\aleph_0}$.

From 5.6 and 4.11 we obtain

Corollary 5.7. *Let G be an ℓ-group. Then either* (i) $\operatorname{conv} G = \operatorname{Conv} G$, *or* (ii) *the set* $\operatorname{conv} G \setminus \operatorname{Conv} G$ *is infinite. The condition (i) is valid if and only if* $\operatorname{Conv} G$ *is a one-element set.*

The following assertion is easy to verify.

Lemma 5.8. *An ℓ-group G has a finite breadth if and only if there exists a finite set $M = \{a_1, a_2, \dots, a_m\}$ in G such that M is a maximal disjoint subset of G and each interval $[0, a_i]$ $(i \in \{1, 2, \dots, m\})$ is a chain.*

Let us denote by $S(G)$ the set of all $x \in G^+$ such that the interval $[0, x]$ of G is a chain, and whenever (x_n) is a sequence of elements in $[0, x]$ with $x_n > x_{n+1}$ for each $n \in N$, then the relation $\bigwedge_{n \in N} x_n = 0$ fails to hold.

Proposition 5.9. *Let G be an ℓ-group. Then* $\operatorname{conv} G = \operatorname{Conv} G$ *if and only if there exists a finite subset S_1 of $S(G)$ such that S_1 is a maximal disjoint subset of G.*

P r o o f. The case $G = \{0\}$ is trivial; suppose that $G \neq \{0\}$.

Assume that $\operatorname{conv} G = \operatorname{Conv} G$. Hence in view of 5.6, the breadth of G is finite. Thus there exists a set M with the properties as in 5.8. Put $S_1 = M$. Let $i \in I$ and suppose that a_i does not belong to $S(G)$. Then there exists a strictly decreasing sequence (x_n) in $[0, a_i]$ such that $\bigwedge_{n \in N} x_n = 0$. There exists $\alpha \in \operatorname{Conv} G$ with $(x_n) \in \alpha$. Clearly $(x_n) \notin \alpha(d)$, whence $\alpha \neq \alpha(d)$, which contradicts 5.7. Therefore $S_1 \subseteq S(G)$.

Conversely, assume that there is a finite subset $S_1 = \{g_1, \dots, g_m\}$ of $S(G)$ such that S_1 is a maximal disjoint subset of G. By 5.8 the breadth of G is finite. By virtue of 3.5 the relation $\operatorname{conv} G = \operatorname{Conv} G$ is valid. □

Again, let $(x_n) \in D$. Put $z_n = x_{2n}$ for each $n \in N$. We denote by α and β the elements of $\operatorname{conv} G$ generated by (x_n) and (z_n), respectively. Using this notation we have the following lemma.

Lemma 5.10. $\beta < \alpha$.

P r o o f. Since (z_n) is a subsequence of (x_n), the relation $(z_n) \in \alpha$ holds. Thus $\beta \leqslant \alpha$. By way of contradiction, suppose that $\beta = a$. Then there are $k, k_1, m \in N$ and subsequences (z_n^1), (z_n^2), …, (z_n^k) of (z_n) such that

$$x_n \leqslant k_1(z_n^1 + z_n^2 + \dots + z_n^k)$$

is valid for each $n \in N$ with $n > m$. But the relation $(x_n) \in D$ implies that if n is odd, then this relation cannot hold. □

Corollary 5.11. *Let $\alpha \in \operatorname{conv} G$. Assume that α contains a sequence belonging to D. Then α fails to be an atom of* $\operatorname{conv} G$.

Lemma 5.12. *Let $\alpha \in \operatorname{conv} G$, $\alpha \neq \alpha(d)$. Then at least one of the following conditions is valid:*

(i) *α contains a strictly decreasing sequence.*

(ii) *α contains a sequence belonging to D.*

Proof. Assume that (i) does not hold. We have to verify that (ii) is valid. Since $\alpha \neq \alpha(d)$ there exists $(x_n) \in \alpha$ such that $x_n > 0$ for each $n \in N$. We construct a sequence (y_n) as follows.

When defining y_1 we distinquish two cases.

(a) First, suppose that the set $\{n \in N: x_1 \wedge x_n = 0\}$ is infinite. Then we put $y_1 = x_1$. Further, for constructing y_2, y_3, ... we apply the subsequence of (x_n) consisting of those x_n which satisfy the condition $x_1 \wedge x_n = 0$.

(b1) Suppose that the set $\{n \in N: x_1 \wedge x_n = 0\}$ is finite and that the interval $[0, x_1]$ is a chain. Then by the same argument as in the proof 4.3 we can verify that there exists a strictly decreasing subsequence of the sequence $(x_1 \wedge x_n)$ such that all elements of this subsequence belong to the interval $[0, x_1]$. This subsequence obviously belongs to α, which is a contradiction.

(b2) Assume that the set $\{n \in N: x_1 \wedge x_n = 0\}$ is finite and that the interval $[0, x_1]$ fails to be a chain. Hence there are elements x_{11} and x_{12} such that $0 < x_{1i} < x_1$ is valid for $i = 1, 2$ and $x_{11} \wedge x_{12} = 0$.

If the set $\{n \in N: x_{11} \wedge x_n = 0\}$ is finite, then we put $y_1 = x_{12}$ and for constructing $y_2, y_3, \ldots$ we apply the sequence consisting of those $x_{11} \wedge x_n$ which are distinct from 0.

If the set $\{n \in N: x_{11} \wedge x_n = 0\}$ is infinite, then we put $y_1 = x_{11}$ and for constructing y_2, y_3, ... we apply the sequence consisting of those x_n which satisfy the condition $x_{11} \wedge x_n = 0$.

The next induction step is obvious. In this way we arrive at a sequence which belongs to $\alpha \cap D$. □

Theorem 5.13. *The partially ordered set* $\operatorname{conv} G$ *has no atom.*

Proof. This is a consequence of 5.12, 5.11 and 4.4. □

In this section we shall investigate the question whether the infinite distributive laws must be valid in $\operatorname{conv} G$.

Let $\alpha_i (i \in I)$ and β be elements of $\operatorname{conv} G$.

Lemma 6.1. *Assume that* $\bigvee_{i\in I} \alpha_i$ *does exist in* $\operatorname{conv} G$. *Then both* $\beta \wedge (\bigvee_{i\in I} \alpha_i)$ *and* $\bigvee_{i\in I} (\beta \wedge \alpha_i)$ *exist in* $\operatorname{conv} G$ *and*

$$(1) \qquad \beta \wedge \Big(\bigvee_{i\in I} \alpha_i\Big) = \bigvee_{i\in I} (\beta \wedge \alpha_i).$$

Proof. In view of 2.1, the element $\gamma = \beta \wedge (\bigvee_{i\in I} \alpha_i)$ exists in $\operatorname{conv} G$. Clearly $\beta \wedge \alpha_i \leqslant \gamma$ for each $i \in I$. Hence $\bigvee_{i\in I} (\beta \wedge \alpha_i)$ exists in $\operatorname{conv} G$ and $\bigvee_{i\in I} (\beta \wedge \alpha_i) \leqslant \gamma$. Let $(x_n) \in \gamma$. Thus $(x_n) \in \beta$ and in view of 2.7 there are $i(1), i(2), \ldots, i(m)$ in I, $(y_n^1) \in \alpha_{i(1)}, \ldots, (y_n^m) \in \alpha_{i(m)}$ such that $x_n \leqslant y_n^1 + y_n^2 + \ldots + y_n^m$ is valid for each $n \in N$. Hence there are elements x_n^j in G with $0 \leqslant x_n^j \leqslant y_n^j$ ($j = 1, 2, \ldots, m$; $n = 1, 2, \ldots$) such that $x_n = x_n^1 + x_n^2 + \ldots + x_n^m$ for each $n \in N$. Then $(x_n^j) \in \beta$ for $j = 1, 2, \ldots, m$ and hence $(x_n) \in \bigvee_{i\in I} (\beta \wedge \alpha_i)$. Thus the relation (1) holds. □

In view of 2.8 we obtain

Corollary 6.2. *Let* α_i $(i \in I)$ *and* β *be elements of* $\operatorname{Conv} G$ *such that* $\bigvee_{i\in I} \alpha_i$ *does exist in* $\operatorname{Conv} G$. *Then the relation (1) is valid in* $\operatorname{Conv} G$.

Corollary 6.3. *Each interval of* $\operatorname{conv} G$ *is a Brouwerian lattice.*

Corollary 6.4. (Cf. [5], Theorem 2.5.) *Each interval of* $\operatorname{Conv} G$ *is a Brouwerian lattice.*

Proposition 6.5. *Let* G *be a lattice ordered group of infinite breadth. Assume that* G *is orthogonally complete and divisible. Then there are* β *and* α_n $(n \in N)$ *in* $\operatorname{Conv} G$ *such that both* $\beta \vee (\bigwedge_{n\in N} \alpha_n)$ *and* $\bigwedge_{n\in N} (\beta \vee \alpha_n)$ *do exist in* $\operatorname{Conv} G$, *but these elements fail to be equal.*

For proving this we need some auxiliary results.

For a nonempty subset A of $(G^N)^+$ we denote by A^* the system of all $(x_n) \in (G^N)^+$ such that for each subsequence (y_n) of (x_n) there exists a subsequence (z_n) of (y_n) with $(z_n) \in A$.

We shall apply the following (slightly modified) version of 2.3. (Cf. also [4].)

Proposition 6.6. *Let A be a nonempty subset of $(G^N)^+$.*

(i) *If there is $0 \neq g \in G^+$ such that* const $g \in [\langle \delta A \rangle]^*$, *then there is no $\alpha \in$* Conv G *with $A \subseteq \alpha$.*

(ii) *If there is no element $g \in G$ with $g \neq 0$ such that* const $g \in [\langle \delta A \rangle]^*$, *then $[\langle \delta A \rangle]^* \in$* Conv G *and whenever $\alpha \in$* Conv G *with $A \subseteq \alpha$, then $\alpha \supseteq [\langle \delta A \rangle]^*$.*

If the condition from (ii) is satisfied, then A is said to be regular and the system $[\langle \delta A \rangle]^*$ is called the convergence in Conv G which is generated by A. If, moreover, $A = \{(x_n)\}$ is a one-element set, then (x_n) is said to be regular.

Now assume that G has an infinite breadth and that G is orthogonally complete, divisible and archimedean.

There exists (x_n) in $(G^N)^+$ such that $(x_n) \in D$. Next, because G is orthogonally complete, for each $t \in N$ there exists $y_t = \vee x_n (n \in N,\ n > t)$.

For each fixed $t \in N$ we consider the sequence $\left(\frac{1}{n} y_n\right)$.

Lemma 6.7. *Let $t \in N$. Then the sequence $\left(\frac{1}{n} y_t\right)$ is regular.*

Proof. This is an immediate consequence of 6.6 and of the fact that G is archimedean.

In view of 6.7 there exists $\alpha_t \in$ Conv G such that α_t is generated by the sequence $\left(\frac{1}{n} y_t\right)$ in Conv G. □

The above Lemmas 6.8 – 6.11 are also consequences of 6.6.

Lemma 6.8. *Let $t \in N$. Next, let $0 < a \in G$, $a \wedge y_t = 0$ and $(u_n) \in \alpha_t$. Then there is $m \in N$ such that $a \wedge u_n = 0$ for each $n \in N$ with $n \geqslant m$.*

Corollary 6.8.1. $\bigwedge_{n \in N} \alpha_n = \alpha(d)$.

For $t \in N$ we put $z_t = x_1 \vee x_2 \vee \ldots \vee x_t$. Let A be the system of all sequences $\left(\frac{1}{n} z_t\right)_{n \in N}$, where t runs over N.

Lemma 6.9. *The set A is regular.*

According to 6.9 there exists $\beta \in$ Conv G which is generated by A in Conv G.

Lemma 6.10. *Let $(v_n) \in \beta$. Then there are $m(1)$ and $m(2) \in N$ such that, whenever $n \in N$, $n \geqslant m(1)$ and $0 < a \in G$, $a \wedge x_m = 0$ for each $m < m(2)$, then $v_n \wedge a = 0$.*

Put $x = \bigvee_{n \in N} x_n$. From 6.10 we infer

Corollary 6.10.1. *The sequence $\left(\frac{1}{n} x\right)$ does not belong to β.*

Lemma 6.11. *Let $t \in N$. Then the set $A \cup \alpha_t$ is regular.*

Corollary 6.12. *Let $t \in N$. Then the join $\beta \vee \alpha_t$ does exist in* Conv G.

Lemma 6.13. *Let $t \in N$. Then $\left(\frac{1}{n}x\right) \in \beta \vee \alpha_t$.*

Proof. We have $x = z_t + y_t$. Next, $\left(\frac{1}{n}z_t\right) \in \beta$ and $\left(\frac{1}{n}y_t\right) \in \alpha_t$. Therefore $\left(\frac{1}{n}x_n\right) \in \beta \vee \alpha_t$. □

Proof of 6.5. In viev of 6.13 the relation $\left(\frac{1}{n}x\right) \in \bigwedge_{i \in N} (\beta \vee \alpha_t)$ is valid. Next, 6.8.1 yields that $\beta \vee (\bigwedge_{t \in N} \alpha_t) = \beta$. Thus according to 6.10.1 the sequence $\left(\frac{1}{n}x_n\right)$ does not belong to $\beta \vee (\bigwedge_{t \in N} \alpha_t)$, which completes the proof. □

Finally, 6.5 and 2.8 yield:

Corollary 6.14. *In 6.5, the set* Conv G *can be replaced by* conv G.

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