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## APPROXIMATION OF ENTIRE FUNCTIONS OF SLOW GROWTH ON COMPACT SETS

G. S. SRIVASTAVA AND SUSHEEL KUMAR

ABSTRACT. In the present paper, we study the polynomial approximation of entire functions of several complex variables. The characterizations of generalized order and generalized type of entire functions of slow growth have been obtained in terms of approximation and interpolation errors.

### 1. INTRODUCTION

The concept of generalized order and generalized type for entire transcendental functions was given by Seremeta [5] and Shah [6]. Hence, let  $L^0$  denote the class of functions  $h(x)$  satisfying the following conditions:

(i)  $h(x)$  is defined on  $[a, \infty)$  and is positive, strictly increasing, differentiable and tends to  $\infty$  as  $x \rightarrow \infty$ ,

(ii)  $\lim_{x \rightarrow \infty} \frac{h[\{1 + 1/\psi(x)\}x]}{h(x)} = 1$  for every function  $\psi(x)$  such that  $\psi(x) \rightarrow \infty$  as  $x \rightarrow \infty$ .

Let  $\Lambda$  denote the class of functions  $h(x)$  satisfying conditions (i) and

(iii)  $\lim_{x \rightarrow \infty} \frac{h(cx)}{h(x)} = 1$  for every  $c > 0$ , that is  $h(x)$  is slowly increasing.

For an entire transcendental function  $f(z) = \sum_{n=1}^{\infty} b_n z^n$ ,  $M(r) = \max_{|z|=r} |f(z)|$  and functions  $\alpha(x) \in \Lambda$ ,  $\beta(x) \in L^0$ , the generalized order is given by

$$\rho(\alpha, \beta, f) = \limsup_{r \rightarrow \infty} \frac{\alpha[\log M(r)]}{\beta(\log r)}.$$

Further, for  $\alpha(x)$ ,  $\beta^{-1}(x)$  and  $\gamma(x) \in L^0$ , generalized type of an entire transcendental function  $f(z)$  is given as

$$\sigma(\alpha, \beta, \rho, f) = \limsup_{r \rightarrow \infty} \frac{\alpha[\log M(r)]}{\beta[\{\gamma(r)\}^\rho]},$$

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where  $0 < \rho < \infty$  is a fixed number.

Let  $g: C^N \rightarrow C$ ,  $N \geq 1$ , be an entire transcendental function. For  $z = (z_1, z_2, \dots, z_N) \in C^N$ , we put

$$S(r, g) = \sup \{ |g(z)| : |z_1|^2 + |z_2|^2 + \dots + |z_N|^2 = r^2 \}, \quad r > 0.$$

Then we define the generalized order and generalized type of  $g(z)$  as

$$\rho(\alpha, \beta, g) = \limsup_{r \rightarrow \infty} \frac{\alpha[\log S(r, g)]}{\beta(\log r)}$$

and

$$\sigma(\alpha, \beta, \rho, g) = \limsup_{r \rightarrow \infty} \frac{\alpha[\log S(r, g)]}{\beta[\{\gamma(r)\}^\rho]}.$$

Let  $K$  be a compact set in  $C^N$  and let  $\|\cdot\|_K$  denote the sup norm on  $K$ . The function  $\Phi_K(z) = \sup (|p(z)|^{1/n} : p - \text{polynomial, } \deg p \leq n, \|p\|_K \leq 1, n \in N)$ ,  $z \in C^N$  is called the Siciak extremal function of the compact set  $K$  (see [2] and [3]). Given a function  $f$  defined and bounded on  $K$ , we put for  $n = 1, 2, \dots$

$$E_n^1(f, K) = \|f - t_n\|_K;$$

$$E_n^2(f, K) = \|f - l_n\|_K;$$

$$E_{n+1}^3(f, K) = \|l_{n+1} - l_n\|_K;$$

where  $t_n$  denotes the  $n^{\text{th}}$  Chebyshev polynomial of the best approximation to  $f$  on  $K$  and  $l_n$  denotes the  $n^{\text{th}}$  Lagrange interpolation polynomial for  $f$  with nodes at extremal points of  $K$  (see [2] and [3]).

The generalized order of an entire function of several complex variables has been characterized by Janik [3]. His characterization of order in terms of the above errors has been obtained under the condition

$$(1.1) \quad \left| \frac{d(\beta^{-1}[c\alpha(x)])}{d(\log x)} \right| \leq b; \quad x \geq a.$$

Clearly (1.1) is not satisfied for  $\alpha(x) = \beta(x)$ . Thus in this case, the corresponding result of Janik is not applicable. In the present paper we define generalized order and generalized type of entire functions of several complex variables in a new way. Our results apply satisfactorily to entire functions of slow growth and generalize many previous results.

Let  $\Omega$  be the class of functions  $h(x)$  satisfying conditions (i) and

(iv) there exist a function  $\delta(x) \in \Lambda$  and constants  $x_0, c_1$  and  $c_2$  such that

$$0 < c_1 \leq \frac{d\{h(x)\}}{d\{\delta(\log x)\}} \leq c_2 < \infty \quad \text{for all } x > x_0.$$

Let  $\bar{\Omega}$  be the class of functions  $h(x)$  satisfying (i) and

$$(v) \quad \lim_{x \rightarrow \infty} \frac{d\{h(x)\}}{d(\log x)} = c_3, \quad 0 < c_3 < \infty.$$

Kapoor and Nautiyal [4] showed that classes  $\Omega$  and  $\bar{\Omega}$  are contained in  $\Lambda$  and  $\Omega \cap \bar{\Omega} = \phi$ . They defined the generalized order  $\rho(\alpha, \alpha, f)$  for entire functions as

$$\rho(\alpha, \alpha, f) = \limsup_{r \rightarrow \infty} \frac{\alpha[\log M(r)]}{\alpha(\log r)},$$

where  $\alpha(x)$  either belongs to  $\Omega$  or to  $\bar{\Omega}$ . Ganti and Srivastava [1] defined the generalized type  $\sigma(\alpha, \alpha, \rho, f)$  of an entire function  $f(z)$  having finite generalized order  $\rho(\alpha, \alpha, f)$  as

$$\sigma(\alpha, \alpha, \rho, f) = \limsup_{r \rightarrow \infty} \frac{\alpha[\log M(r)]}{[\alpha(\log r)]^\rho}.$$

2. MAIN RESULTS

**Theorem 2.1.** *Let  $K$  be a compact set in  $C^N$ . If  $\alpha(x)$  either belongs to  $\Omega$  or to  $\bar{\Omega}$  then the function  $f$  defined and bounded on  $K$ , is a restriction to  $K$  of an entire function  $g$  of finite generalized order  $\rho(\alpha, \alpha, g)$  if and only if*

$$\rho(\alpha, \alpha, g) = \limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\alpha \left\{ -\frac{1}{n} \log E_n^s(f, K) \right\}}; \quad s = 1, 2, 3.$$

**Proof.** Let  $g$  be an entire transcendental function. Write  $\rho = \rho(\alpha, \alpha, g)$  and

$$\theta_s = \limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\alpha \left\{ -\frac{1}{n} \log E_n^s \right\}}; \quad s = 1, 2, 3.$$

Here  $E_n^s$  stands for  $E_n^s(g|_K, K)$ ,  $s = 1, 2, 3$ . We claim that  $\rho = \theta_s$ ,  $s = 1, 2, 3$ . It is known (see e.g. [7]) that

$$(2.1) \quad E_n^1 \leq E_n^2 \leq (n_* + 2)E_n^1, \quad n \geq 0,$$

$$(2.2) \quad E_n^3 \leq 2(n_* + 2)E_{n-1}^1, \quad n \geq 1,$$

where  $n_* = \binom{n+N}{n}$ . Using Stirling formula for the approximate value of

$$n! \approx e^{-n} n^{n+1/2} \sqrt{2\pi},$$

we get  $n_* \approx \frac{n^N}{N!}$  for all large values of  $n$ . Hence for all large values of  $n$ , we have

$$E_n^1 \leq E_n^2 \leq \frac{n^N}{N!} [1 + o(1)] E_n^1$$

and

$$E_n^3 \leq 2 \frac{n^N}{N!} [1 + o(1)] E_n^1.$$

Thus  $\theta_3 \leq \theta_2 = \theta_1$  and it suffices to prove that  $\theta_1 \leq \rho \leq \theta_3$ . First we prove that  $\theta_1 \leq \rho$ . Using the definition of generalized order, for  $\varepsilon > 0$  and  $r > r_0(\varepsilon)$ , we have

$$\log S(r, g) \leq \alpha^{-1} \{ \bar{\rho} \alpha(\log r) \},$$

where  $\bar{\rho} = \rho + \varepsilon$  provided  $r$  is sufficiently large. Without loss of generality, we may suppose that

$$K \subset B = \{z \in C^N : |z_1|^2 + |z_2|^2 + \dots + |z_N|^2 \leq 1\}.$$

Then

$$E_n^1 \leq E_n^1(g, B).$$

Now following Janik ([3, p.324]), we get

$$E_n^1(g, B) \leq r^{-n} S(r, g), \quad r \geq 2, \quad n \geq 0$$

or

$$\log E_n^1 \leq -n \log r + \alpha^{-1} \{ \bar{\rho} \alpha(\log r) \}.$$

Putting  $r = \exp [\alpha^{-1} \{ \alpha(n) / \bar{\rho} \}]$  in the above inequality, we obtain

$$\log E_n^1 \leq -n [\alpha^{-1} \{ \alpha(n) / \bar{\rho} \}] + n$$

or

$$\frac{\alpha(n)}{\alpha \{ 1 - \frac{1}{n} \log E_n^1 \}} \leq \bar{\rho}.$$

Taking limits as  $n \rightarrow \infty$ , we get

$$\limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\alpha \{ -\frac{1}{n} \log E_n^1 \}} \leq \bar{\rho}.$$

Since  $\varepsilon > 0$  is arbitrary small. Therefore finally we get

$$(2.3) \quad \theta_1 \leq \rho.$$

Now we will prove that  $\rho \leq \theta_3$ . If  $\theta_3 = \infty$ , then there is nothing to prove. So let us assume that  $0 \leq \theta_3 < \infty$ . Therefore for a given  $\varepsilon > 0$  there exist  $n_0 \in N$  such that for all  $n > n_0$ , we have

$$0 \leq \frac{\alpha(n)}{\alpha \{ -\frac{1}{n} \log E_n^3 \}} \leq \theta_3 + \varepsilon = \bar{\theta}_3$$

or

$$E_n^3 \leq \exp [ -n \alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} ].$$

Now from the property of maximum modulus, we have

$$\begin{aligned} S(r, g) &\leq \sum_{n=0}^{\infty} E_n^3 r^n, \\ S(r, g) &\leq \sum_{n=0}^{n_0} E_n^3 r^n + \sum_{n=n_0+1}^{\infty} r^n \exp [ -n \alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} ]. \end{aligned}$$

Now for  $r > 1$ , we have

$$(2.4) \quad S(r, g) \leq A_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n \exp [ -n \alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} ],$$

where  $A_1$  is a positive real constant. We take

$$(2.5) \quad N(r) = \alpha^{-1} ( \bar{\theta}_3 \alpha [ \log \{ (N+1)r \} ] ).$$

Now if  $r$  is sufficiently large, then from (2.4) and (2.5) we have

$$S(r, g) \leq A_1 r^{n_0} + r^{N(r)} \sum_{n_0+1 \leq n \leq N(r)} \exp \left[ -n\alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} \right] \\ + \sum_{n > N(r)} r^n \exp \left[ -n\alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} \right]$$

or

$$(2.6) \quad S(r, g) \leq A_1 r^{n_0} + r^{N(r)} \sum_{n=1}^{\infty} \exp \left[ -n\alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} \right] \\ + \sum_{n > N(r)} r^n \exp \left[ -n\alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} \right].$$

Now we have

$$\limsup_{n \rightarrow \infty} \left( \exp \left[ -n\alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} \right] \right)^{1/n} = 0.$$

Hence the first series in (2.6) converges to a positive real constant  $A_2$ . So from (2.6) we get

$$S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n > N(r)} r^n \exp \left[ -n\alpha^{-1} \{ \alpha(n) / \bar{\theta}_3 \} \right], \\ S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n > N(r)} r^n \exp \left[ -n \log \{ (N+1)r \} \right], \\ S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n > N(r)} \left( \frac{1}{N+1} \right)^n, \\ (2.7) \quad S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n=1}^{\infty} \left( \frac{1}{N+1} \right)^n.$$

It can be easily seen that the series in (2.7) converges to a positive real constant  $A_3$ . Therefore from (2.7), we get

$$S(r, g) \leq A_1 r^{n_0} + A_2 r^{N(r)} + A_3, \\ S(r, g) \leq A_2 r^{N(r)} [1 + o(1)], \\ \log S(r, g) \leq [1 + o(1)] N(r) \log r, \\ \log S(r, g) \leq [1 + o(1)] \alpha^{-1} (\bar{\theta}_3 \alpha [\log \{ (N+1)r \}]) \log r, \\ \log S(r, g) \leq [1 + o(1)] [\alpha^{-1} \{ (\bar{\theta}_3 + \delta) \alpha [\log \{ (N+1)r \} \}]],$$

where  $\delta > 0$  is suitably small.

Hence

$$\alpha [\log S(r, g)] \leq (\bar{\theta}_3 + \delta) \alpha [\log \{ (N+1)r \}] \\ \frac{\alpha [\log S(r, g)]}{\alpha [\log r]} \leq (\bar{\theta}_3 + \delta) [1 + o(1)].$$

Proceedings to limits as  $r \rightarrow \infty$ , since  $\delta$  is arbitrarily small, we get

$$\rho \leq \theta_3.$$

Now let  $f$  be a function defined and bounded on  $K$  and such that for  $s = 1, 2, 3$

$$\theta_s = \limsup_{n \rightarrow \infty} \frac{\alpha(n)}{\alpha \left\{ -\frac{1}{n} \log E_n^s \right\}}$$

is finite. We claim that the function

$$g = l_0 + \sum_{n=1}^{\infty} (l_n - l_{n-1})$$

is the required entire continuation of  $f$  and  $\rho(\alpha, \alpha, g) = \theta_s$ . Indeed, for every  $d_1 > \theta_s$

$$\frac{\alpha(n)}{\alpha \left\{ -\frac{1}{n} \log E_n^s \right\}} \leq d_1$$

provided  $n$  is sufficiently large. Hence

$$E_n^s \leq \exp \left[ -n \alpha^{-1} \left\{ \alpha(n) / d_1 \right\} \right].$$

Using the inequalities (2.1), (2.2) and converse part of theorem, we find that the function  $g$  is entire and  $\rho(\alpha, \alpha, g)$  is finite. So by (2.3), we have  $\rho(\alpha, \alpha, g) = \theta_s$ , as claimed. This completes the proof of Theorem 2.1.  $\square$

Next we prove

**Theorem 2.2.** *Let  $K$  be a compact set in  $C^N$  such that  $\Phi_K$  is locally bounded in  $C^N$ . Set  $G(x, \sigma, \rho) = \alpha^{-1} [\{\sigma \alpha(x)\}^{1/\rho}]$ , where  $\rho$  is a fixed number,  $1 < \rho < \infty$ . Let  $\alpha(x) \in \bar{\Omega}$  and  $\frac{dG(x, \sigma, \rho)}{d \log x} = O(1)$  as  $x \rightarrow \infty$  for all  $0 < \sigma < \infty$ . Then the function  $f$  defined and bounded on  $K$ , is a restriction to  $K$  of an entire function  $g$  of generalized type  $\sigma(\alpha, \alpha, \rho, g)$  if and only if*

$$\sigma(\alpha, \alpha, \rho, g) = \limsup_{n \rightarrow \infty} \frac{\alpha(n/\rho)}{\left\{ \alpha \left( \frac{\rho}{\rho-1} \log [E_n^s(f, K)]^{-1/n} \right) \right\}^{\rho-1}}, \quad s = 1, 2, 3.$$

Before proving the Theorem 2.2 we state and prove a lemma.

**Lemma 2.1.** *Let  $K$  be a compact set in  $C^N$  such that  $\Phi_K$  is locally bounded in  $C^N$ . Set  $G(x, \mu, \lambda) = \alpha^{-1} [\{\mu \alpha(x)\}^{1/\lambda}]$ , where  $\lambda$  is a fixed number,  $1 < \lambda < \infty$ . Let  $\alpha(x) \in \bar{\Omega}$  and  $\frac{dG(x, \mu, \lambda)}{d \log x} = O(1)$  as  $x \rightarrow \infty$  for all  $0 < \mu < \infty$ . Let  $(p_n)_{n \in N}$  be a sequence of polynomials in  $C^N$  such that*

- (i)  $\deg p_n \leq n, n \in N$ ;
- (ii) for a given  $\varepsilon > 0$  there exists  $n_0 \in N$  such that

$$\|p_n\|_K \leq \exp \left( -\frac{\lambda-1}{\lambda} n \alpha^{-1} \left[ \left\{ \frac{1}{\mu} \alpha(n/\lambda) \right\}^{1/(\lambda-1)} \right] \right).$$

Then  $\sum_{n=0}^{\infty} p_n$  is an entire function and  $\sigma(\alpha, \alpha, \lambda, \sum_{n=0}^{\infty} p_n) \leq \mu$  provided  $\sum_{n=0}^{\infty} p_n$  is not a polynomial.

**Proof.** By assumption, we have

$$\|p_n\|_K r^n \leq r^n \exp \left( - \frac{\lambda - 1}{\lambda} n \alpha^{-1} \left[ \left\{ \frac{1}{\mu} \alpha(n/\lambda) \right\}^{1/(\lambda-1)} \right] \right), \quad n \geq n_0, \quad r > 0.$$

If  $\alpha(x) \in \bar{\Omega}$ , then by assumptions of lemma, there exists a number  $b > 0$  such that for  $x > a$ , we have

$$\left| \frac{dG(x, \mu, \lambda)}{d \log x} \right| < b.$$

Let us consider the function

$$\phi(x) = r^x \exp \left( - \frac{\lambda - 1}{\lambda} x \alpha^{-1} \left[ \left\{ \frac{1}{\mu} \alpha(x/\lambda) \right\}^{1/(\lambda-1)} \right] \right).$$

Using the technique of Seremeta [5], it can be easily seen that the maximum value of  $\phi(x)$  is attained for a value of  $x$  given by

$$x^*(r) = \lambda \alpha^{-1} \{ \mu [\alpha \{ \log r - a(r) \}]^{\lambda-1} \},$$

where

$$a(r) = \frac{dG(x/\lambda, 1/\mu, \lambda - 1)}{d \log x}.$$

Thus

$$(2.8) \quad \|p_n\|_K r^n \leq \exp \{ b \lambda \alpha^{-1} [\mu \{ \alpha(\log r + b) \}]^{\lambda-1} \}, \quad n \geq n_0, \quad r > 0.$$

Let us write  $K_r = \{z \in C^N : \Phi_K(z) < r, r > 1\}$ , then for every polynomial  $p$  of degree  $\leq n$ , we have (see [3])

$$|p_n(z)| \leq \|p_n\|_K \Phi_K^n(z), \quad z \in C^N.$$

So the series  $\sum_{n=0}^\infty p_n$  is convergent in every  $K_r, r > 1$ , whence  $\sum_{n=0}^\infty p_n$  is an entire function. Put

$$M^*(r) = \sup \{ \|p_n\|_K r^n : n \in N, r > 0 \}.$$

On account of (2.8), for every  $r > 0$ , there exists a positive integer  $\nu(r)$  such that

$$M^*(r) = \|p_{\nu(r)}\|_K r^{\nu(r)}$$

and

$$M^*(r) > \|p_n\|_K r^n, \quad n > \nu(r).$$

It is evident that  $\nu(r)$  increases with  $r$ . First suppose that  $\nu(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . Then putting  $n = \nu(r)$  in (2.8), we get for sufficiently large  $r$

$$(2.9) \quad M^*(r) \leq \exp \{ b \lambda \alpha^{-1} [\mu \{ \alpha(\log r + b) \}]^{\lambda-1} \}.$$

Put

$$F_r = \{z \in C^N : \Phi_K(z) = r\}, \quad r > 1$$



and

$$M(r) = \sup \left\{ \left| \sum_{n=0}^{\infty} p_n(z) \right| : z \in F_r \right\}, \quad r > 1.$$

Now following Janik ([3] p.323), we have for some positive constant  $k$

$$(2.10) \quad S\left(r, \sum_{n=0}^{\infty} p_n\right) \leq M(kr) \leq 2M^*(2kr).$$

Combining (2.9) and (2.10), we get

$$S\left(r, \sum_{n=0}^{\infty} p_n\right) \leq 2 \exp \left\{ b\lambda\alpha^{-1} [\mu\{\alpha(\log r + b)\}^{\lambda-1}] \right\}$$

or

$$\frac{\alpha \left[ \frac{1}{b\lambda} \log \left\{ \frac{1}{2} S\left(r, \sum_{n=0}^{\infty} p_n\right) \right\} \right]}{[\alpha(\log 2kr + b)]^{\lambda-1}} \leq \mu.$$

Since  $\alpha(x) \in \bar{\Omega}$ , we get on using (v)

$$\limsup_{r \rightarrow \infty} \frac{\alpha \left[ \log S\left(r, \sum_{n=0}^{\infty} p_n\right) \right]}{[\alpha(\log r)]^{\lambda}} \leq \mu$$

or

$$\sigma(\alpha, \alpha, \lambda, \sum_{n=0}^{\infty} p_n) \leq \mu.$$

In the case when  $\nu(r)$  is bounded then  $M^*(r)$  is also bounded, whence  $\sum_{n=0}^{\infty} p_n$  reduces to a polynomial. Hence the Lemma 2.1 is proved.  $\square$

**Proof of Theorem 2.2.** Let  $g$  be an entire transcendental function. Write  $\sigma = \sigma(\alpha, \alpha, \rho, g)$  and

$$\eta_s = \limsup_{n \rightarrow \infty} \frac{\alpha(n/\rho)}{\left\{ \alpha \left( \frac{\rho}{\rho-1} \log [E_n^s]^{-1/n} \right) \right\}^{\rho-1}}, \quad s = 1, 2, 3.$$

Here  $E_n^s$  stands for  $E_n^s(g|_K, K)$ ,  $s = 1, 2, 3$ . We claim that  $\sigma = \eta_s$ ,  $s = 1, 2, 3$ . Now following Theorem 2.1, here we prove that  $\eta_1 \leq \sigma \leq \eta_3$ . First we prove that  $\eta_1 \leq \sigma$ . Using the definition of generalized type, for  $\varepsilon > 0$  and  $r > r_0(\varepsilon)$ , we have

$$S(r, g) \leq \exp \left( \alpha^{-1} [\bar{\sigma} \{\alpha(\log r)\}^\rho] \right),$$

where  $\bar{\sigma} = \sigma + \varepsilon$  provided  $r$  is sufficiently large. Thus following Theorem 2.1, here we have

$$(2.11) \quad \begin{aligned} E_n^1 &\leq r^{-n} \exp \left( \alpha^{-1} [\bar{\sigma} \{\alpha(\log r)\}^\rho] \right) \\ E_n^1 &\leq \exp \left[ -n \log r + \left( \alpha^{-1} [\bar{\sigma} \{\alpha(\log r)\}^\rho] \right) \right]. \end{aligned}$$

Let  $r = r(n)$  be the unique root of the equation

$$(2.12) \quad \alpha \left[ \frac{n \log r}{\rho} \right] = \bar{\sigma} \{\alpha(\log r)\}^\rho.$$

Then

$$(2.13) \quad \log r \simeq \alpha^{-1} \left\{ \frac{1}{\bar{\sigma}} \alpha(n/\rho) \right\}^{1/(\rho-1)} = G(n/\rho, 1/\bar{\sigma}, \rho - 1).$$

Using (2.12) and (2.13) in (2.11), we get

$$\begin{aligned} E_n^1 &\leq \exp \left[ -n G(n/\rho, 1/\bar{\sigma}, \rho - 1) + \frac{n}{\rho} G(n/\rho, 1/\bar{\sigma}, \rho - 1) \right], \\ \frac{\rho}{\rho - 1} \log[E_n^1]^{-1/n} &\geq \alpha^{-1} \left\{ \left( \frac{1}{\bar{\sigma}} \alpha(n/\rho) \right)^{1/(\rho-1)} \right\}, \\ \frac{\alpha(n/\rho)}{\left\{ \alpha \left( \frac{\rho}{\rho-1} \log[E_n^1]^{-1/n} \right) \right\}^{\rho-1}} &\leq \bar{\sigma}. \end{aligned}$$

Proceedings to limits, we get

$$\limsup_{n \rightarrow \infty} \frac{\alpha(n/\rho)}{\left\{ \alpha \left( \frac{\rho}{\rho-1} \log[E_n^1]^{-1/n} \right) \right\}^{\rho-1}} \leq \bar{\sigma}$$

or

$$\eta_1 \leq \bar{\sigma}.$$

Since  $\varepsilon > 0$  is arbitrarily small, we finally get

$$(2.14) \quad \eta_1 \leq \sigma.$$

Now we will prove that  $\sigma \leq \eta_3$ . Suppose that  $\eta_3 < \sigma$ , then for every  $\mu_1, \eta_3 < \mu_1 < \sigma$ , we have

$$\frac{\alpha(n/\rho)}{\left\{ \alpha \left( \frac{\rho}{\rho-1} \log[E_n^3]^{-1/n} \right) \right\}^{\rho-1}} \leq \mu_1$$

provided  $n$  is sufficiently large. Thus

$$E_n^3 \leq \exp \left( -\frac{\rho-1}{\rho} n \alpha^{-1} \left[ \left\{ \frac{1}{\mu_1} \alpha(n/\rho) \right\}^{1/(\rho-1)} \right] \right).$$

Also by previous lemma,  $\sigma \leq \mu_1$ . Since  $\mu_1$  has been chosen less than  $\sigma$ , we get a contradiction. Hence

$$\sigma \leq \eta_3.$$

Now let  $f$  be a function defined and bounded on  $K$  such that for  $s = 1, 2, 3$

$$\eta_s = \limsup_{n \rightarrow \infty} \frac{\alpha(n/\rho)}{\left\{ \alpha \left( \frac{\rho}{\rho-1} \log[E_n^s]^{-1/n} \right) \right\}^{\rho-1}}$$

is finite. We claim that the function

$$g = l_0 + \sum_{n=1}^{\infty} (l_n - l_{n-1})$$

is the required entire continuation of  $f$  and  $\sigma(\alpha, \alpha, \rho, g) = \eta_s$ . Indeed, for every  $d_2 > \eta_s$

$$\frac{\alpha(n/\rho)}{\left\{ \alpha \left( \frac{\rho}{\rho-1} \log[E_n^s]^{-1/n} \right) \right\}^{\rho-1}} \leq d_2$$

provided  $n$  is sufficiently large. Hence

$$E_n^s \leq \exp \left( - \frac{\rho-1}{\rho} n \alpha^{-1} \left[ \left\{ \frac{1}{d_2} \alpha(n/\rho) \right\}^{1/(\rho-1)} \right] \right).$$

Using the inequalities (2.1), (2.2) and previous lemma, we find that the function  $g$  is entire and  $\sigma(\alpha, \alpha, \rho, g)$  is finite. So by (2.14), we have  $\sigma(\alpha, \alpha, \rho, g) = \eta_s$ , as claimed. This completes the proof of the Theorem 2.2.  $\square$

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