Ladislav Bican

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ON TORSIONFREE CLASSES WHICH ARE NOT PRECOVER CLASSES

Ladislav Bican, Praha

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Abstract. In the class of all exact torsion theories the torsionfree classes are cover (precover) classes if and only if the classes of torsionfree relatively injective modules or relatively exact modules are cover (precover) classes, and this happens exactly if and only if the torsion theory is of finite type. Using the transfinite induction in the second half of the paper a new construction of a torsionfree relatively injective cover of an arbitrary module with respect to Goldie's torsion theory of finite type is presented.

Keywords: hereditary torsion theory, exact, noetherian and perfect torsion theory, Goldie's torsion theory, precover class, cover class, precover and cover of a module

MSC 2000: 16S90, 16D50, 16D90, 18E40

In what follows, R stands for an associative ring with the identity element and R-mod denotes the category of all unitary left R-modules. The basic properties of rings and modules can be found in [1].

A class $\mathscr{G}$ of modules is called *abstract*, if it is closed under isomorphic copies. If $\mathscr{G}$ is an abstract class of modules, then a homomorphism $\varphi\colon G \to M$ with $G \in \mathscr{G}$ is called a *$\mathscr{G}$-precover* of the module M, if for each homomorphism $f\colon F \to M$ with $F \in \mathscr{G}$ there exists a homomorphism $g\colon F \to G$ such that $\varphi g = f$. A $\mathscr{G}$-precover φ of M is said to be a *$\mathscr{G}$-cover*, if every endomorphism f of G with $\varphi f = \varphi$ is an automorphism of the module G. An abstract class $\mathscr{G}$ of modules is called a *precover* (*cover*) *class*, if every module has a $\mathscr{G}$-precover ($\mathscr{G}$-cover). A more detailed study of precovers and covers can be found in [17].

Recall that a *hereditary torsion theory* $\tau = (\mathscr{T}, \mathscr{F})$ for the category R-mod consists of two abstract classes $\mathscr{T}$ and $\mathscr{F}$, the *τ-torsion class* and the *τ-torsionfree class*,

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respectively, such that $\mathrm{Hom}\,(T, F) = 0$ whenever $T \in \mathscr{T}$ and $F \in \mathscr{F}$, the class $\mathscr{T}$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathscr{F}$ is closed under submodules, extensions and arbitrary direct products and for each module M there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathscr{T}$ and $F \in \mathscr{F}$. It is easy to see that every module M contains the unique largest τ-torsion submodule (isomorphic to T), which is called the *τ-torsion part of the module M* and is usually denoted by $\tau(M)$. Note that a submodule K of a module $F \in \mathscr{F}$ is called *τ-pure* in F if $F/K \in \mathscr{F}$. For two hereditary torsion theories τ and τ' the symbol $\tau \leqslant \tau'$ means that $\mathscr{T}_\tau \subseteq \mathscr{T}_{\tau'}$ and consequently $\mathscr{F}_{\tau'} \subseteq \mathscr{F}_\tau$. With each hereditary torsion theory τ we associate the *Gabriel filter* $\mathscr{L}$ of left ideals of R consisting of all left ideals $I \leqslant R$ with $R/I \in \mathscr{T}$. Recall that τ is said to be of *finite type*, if $\mathscr{L}$ contains a cofinal subset of finitely generated left ideals. A module Q is called *τ-injective*, if it is injective with respect to all short exact sequences $0 \to A \to B \to C \to 0$, where $C \in \mathscr{T}$. Following [14] we say that a τ-torsionfree module is *τ-exact*, if any of its τ-torsionfree homomorphic images is τ-injective. The class of all τ-injective modules will be denoted by $\mathscr{I}_\tau$, while the class of all τ-exact modules will be denoted by $\mathscr{E}_\tau$. A τ-torsionfree module is called *τ-cocritical*, if any of its proper homomorphic images is τ-torsion. Further, a hereditary torsion theory τ is called *exact*, if $E(Q)/Q$ is τ-torsionfree τ-injective whenever Q is so. Here $E(Q)$ denotes as usual the injective hull of the module Q. Note that the hereditary torsion theory τ is exact if and only if $\mathscr{E}_\tau = \mathscr{F} \cap \mathscr{I}_\tau$. If τ is exact and of finite type then it is called *perfect*. A hereditary torsion theory τ is called *noetherian*, if $I_0 \subseteq I_1 \subseteq \ldots, I = \bigcup_{n<\omega} I_n$ and $I \in \mathscr{L}$ implies $I_n \in \mathscr{L}$ for some $n < \omega$, I_n being the left ideals of the ring R. Note that by [12; Proposition 41.1] τ is noetherian if and only if the class $\mathscr{F} \cap \mathscr{I}_\tau$ is closed under arbitrary direct sums.

For a module M, the *singular submodule* $Z(M)$ consists of all elements $a \in M$ the *annihilator left ideal* $(0\colon a) = \{r \in R;\ ra = 0\}$ of which is essential in R. *Goldie's torsion theory* for the category R-mod is the hereditary torsion theory $\sigma = (\mathscr{T}, \mathscr{F})$, where $\mathscr{T} = \{M \in R\text{-mod};\ Z(M/Z(M)) = M/Z(M)\}$ and $\mathscr{F} = \{M \in R\text{-mod};\ Z(M) = 0\}$. For more details on torsion theories we refer to [12] or [11].

In the first part of this note we will characterize the exact torsion theories for which the corresponding torsionfree class is a precover class. It is shown that this is just the case of torsion theories of finite type and that this is equivalent to the facts that $\mathscr{F}$ is a cover class, $\mathscr{E}_\tau$ is a precover class and $\mathscr{E}_\tau$ is a cover class. Moreover, these conditions are equivalent to that ensuring the existence of "large" $\mathscr{F}$-pure submodules in arbitrary "large" submodules of τ-torsionfree modules. In view of the fact that Goldie's torsion theory is exact (see e.g. [12; Corollary 44.3]) we see that Goldie's torsionfree class is a precover class if and only if this torsion theory is of finite type. On the other hand, Teply in [16; Theorem 2.1] proved that Goldie's torsion theory is of finite type if and only if the ring R contains no infinite direct sum of torsionfree left ideals. From this result we easily obtain that the torsionfree class of Goldie's torsion theory over the ring $Z^\omega = \prod_{n<\omega} Z_n$, $Z_n \cong Z$, is not a precover class. In the second half of the paper we will give a new construction of a σ-torsiofree σ-injective cover of an arbitrary module, where σ is Goldie's torsion theory for the category R-mod which is of finite type. This construction is based on the transfinite induction principle.

Lemma 1. *Let $\tau = (\mathscr{T}, \mathscr{F})$ be a hereditary torsion theory for the category R-mod. If $\mathscr{F} \cap \mathscr{I}_\tau$ is a precover class, then τ is noetherian.*

Proof. By virtue of [12; Proposition 41.1] it suffices to verify that the class $\mathscr{F} \cap \mathscr{I}_\tau$ is closed under arbitrary direct sums. So, let $\{F_\alpha;\ \alpha \in A\}$ be any collection of elements of $\mathscr{F} \cap \mathscr{I}_\tau$ and let $F = \bigoplus_{\alpha\in A} F_\alpha$ together with the canonical embeddings $i_\alpha\colon F_\alpha \to F$ be their direct sum. By the hypothesis there exists an $(\mathscr{F} \cap \mathscr{I}_\tau)$-precover $\varphi\colon G \to F$ of the module F. Now for each $\alpha \in A$ there is a homomorphism $f_\alpha\colon F_\alpha \to G$ with $\varphi f_\alpha = i_\alpha$ and consequently $\psi i_\alpha = f_\alpha$ for each $\alpha \in A$ and a (unique) homomorphism $\psi\colon F \to G$. Thus, for each $\alpha \in A$ we have $\varphi\psi i_\alpha = \varphi f_\alpha = 1_F i_\alpha$, hence $\varphi\psi = 1_F$ and F belongs to $\mathscr{F} \cap \mathscr{I}_\tau$ as a direct summand of the module $G \in \mathscr{F} \cap \mathscr{I}_\tau$. □

Now we can present the following characterization of exact torsion theories which are noetherian.

Theorem 2. *The following conditions are equivalent for an exact torsion theory $\tau = (\mathscr{T}, \mathscr{F})$ for the category R-mod:*

(i) *τ is noetherian;*
(ii) *τ is of finite type;*
(iii) *τ is perfect;*
(iv) *$\mathscr{F}$ is a precover class;*
(v) *$\mathscr{F} \cap \mathscr{I}_\tau$ is a precover class;*
(vi) *$\mathscr{E}_\tau$ is a precover class;*
(vii) *$\mathscr{F}$ is a cover class;*
(viii) *$\mathscr{F} \cap \mathscr{I}_\tau$ is a cover class;*
(ix) *$\mathscr{E}_\tau$ is a cover class;*
(x) *if λ is an infinite cardinal then there is a cardinal $\kappa > \lambda$ such that for each module $F \in \mathscr{F}$ with $|F| \geqslant \kappa$ and each submodule $L \leqslant F$ with $|F/L| \leqslant \lambda$ the submodule L contains a submodule K such that $F/K \in \mathscr{F}$ and $|F/K| < \kappa$.*

P r o o f. The conditions (i), (ii) and (iii) are equivalent by [12; Proposition 45.1] and the conditions (iv), (v) and (vi) are equivalent by [2; Theorem 7]. Further, the conditions (iii) and (vii) are equivalent by [3; Theorem 8] and (vii) is equivalent to (ix) by [10; Theorem 9]. Finally, (viii) implies (v) trivially, (v) implies (i) by Lemma 1, the conditions (viii) and (ix) are equivalent since $\mathscr{F} \cap \mathscr{I}_\tau = \mathscr{E}_\tau$ under the exactness by [12; Proposition 44.1] and (x) is equivalent to (iv) by [5; Theorem 9]. □

Corollary 3. *If σ is Goldie's torsion theory for the category R-mod, then for each hereditary torsion theory $\tau \geqslant \sigma$ the conditions* (i)–(x) *from Theorem* 2 *are equivalent.*

P r o o f. The torsion theory τ is exact by [12; Corollary 44.3]. □

Now we are ready to prove the existence of torsionfree classes which are not precover classes.

Theorem 4. *There are torsionfree classes which are not precover classes.*

P r o o f. In [16; Theorem 2.1] it was proved that Goldie's torsion theory is of finite type if and only if the ring R contains no infinite direct sum of σ-torsionfree left ideals. Taking $Z_n \cong Z$ (as rings) for each $n < \omega$, we see that the ring $R = \prod\limits_{n<\omega} Z_n$ contains the ideal $\bigoplus\limits_{n<\omega} Z_n$, where each Z_n is obviously a σ-torsionfree ideal of R and Corollary 3 applies. □

We are going now to prove some simple auxiliary results which will be useful later. From now on the symbol $\sigma = (\mathscr{T}, \mathscr{F})$ will always denote Goldie's torsion theory for the category R-mod.

Lemma 5. *Let $\mathscr{G}$ be any class of modules and let $M \in R$-mod be arbitrary. A homomorphism $\varphi\colon G \to M$ with $G \in \operatorname{Coprod}(\mathscr{G})$ is a $\operatorname{Coprod}(\mathscr{G})$-precover of the module M if and only if for each homomorphism $f\colon F \to M$ with $F \in \mathscr{G}$ there exists a homomorphism $g\colon F \to G$ such that $\varphi g = f$.*

P r o o f. Only the sufficiency requires verification. So, let $F = \bigoplus\limits_{\lambda\in\Lambda} F_\lambda$ together with the canonical injections $\iota_\lambda\colon F_\lambda \to F$ be an element of $\operatorname{Coprod}(\mathscr{G})$ and let $f\colon F \to M$ be an arbitrary homomorphism. Now we shall consider the commutative squares

$$\begin{array}{ccc} F_\lambda & \xrightarrow{\iota_\lambda} & F \\ {\scriptstyle g_\lambda}\downarrow & & \downarrow{\scriptstyle f} \\ G & \xrightarrow{\varphi} & M \end{array} \qquad\qquad \begin{array}{ccc} F_\lambda & \xrightarrow{\iota_\lambda} & F \\ {\scriptstyle g_\lambda}\downarrow & & \downarrow{\scriptstyle g} \\ G & = & G \end{array}$$

where $g_\lambda\colon F_\lambda \to G$ making the left square commutative exists by the hypothesis for each $\lambda \in \Lambda$ and the homomorphism $g\colon F \to G$ making the right square commutative for each $\lambda \in \Lambda$ exists by the universal property of direct sums. Now $\varphi g \iota_\lambda = \varphi g_\lambda = f \iota_\lambda$ for each $\lambda \in \Lambda$ and so $\varphi g = f$ by the same argument. □

Lemma 6. *Let $\sigma = (\mathscr{T}, \mathscr{F})$ be Goldie's torsion theory for the category R-*mod*. If σ is noetherian then every element of the class $\mathscr{F} \cap \mathscr{I}_\sigma$ is a direct sum of indecomposable injective modules.*

Proof. It is easy to see that the elements of $\mathscr{F} \cap \mathscr{I}_\sigma$ are just the σ-torsionfree injective modules (see e.g. [16; Lemma 1]. Thus it suffices to show that each non-zero element of $\mathscr{F} \cap \mathscr{I}_\sigma$ contains an indecomposable injective submodule, since this class is closed under arbitrary direct sums by the hypothesis that σ is noetherian. Proving indirectly, let $E \in \mathscr{F}$ be the injective hull of a cyclic module Ru, $E = E(Ru)$, which contains no indecomposable injective submodule. Thus we have $E = E_0 = Q_1 \oplus E_1$, $E_1 = Q_2 \oplus E_2, \ldots, E_n = Q_{n+1} \oplus E_{n+1}, \ldots$ So, for each $n < \omega$ we obtain $E = Q_1 \oplus \ldots \oplus Q_n \oplus E_n$. Setting $Q = \bigoplus\limits_{n<\omega} Q_n$, where $Q_0 = 0$, and taking $K \leqslant E$ maximal with respect to $K \cap Q = 0$, we easily get that $E = K \oplus Q$. So, $u \in K \oplus Q_1 \oplus \ldots \oplus Q_n$ for some $n < \omega$ and consequently $Ru \subseteq K \oplus Q_1 \oplus \ldots \oplus Q_n$. Hence $Ru \cap Q_{n+1} = 0$, which contradicts the fact that Ru is essential in E. From this the assertion now easily follows. □

Lemma 7. *Let $\sigma = (\mathscr{T}, \mathscr{F})$ be Goldie's torsion theory for the category R-*mod. *If $E \in \mathscr{F}$ is an indecomposable injective module, then E is σ-cocritical.*

Proof. It clearly suffices to verify that each non-zero submodule of E is essential in E. Proving indirectly, let U, V be non-zero submodules of E such that $U \cap V = 0$. Since $E(U)$ and $E(V)$ are submodules of E, it remains to verify that $E(U) \cap E(V) = 0$ in view of the obvious fact that $E(U) \oplus E(V)$ is a direct summand of E in this case. However, for $0 \neq w \in E(U) \cap E(V)$ there are elements $r, s \in R$ such that $0 \neq rw \in U$ and $0 \neq srw \in U \cap V$, which is a contradiction completing the proof. □

Remark 8. Let $\sigma = (\mathscr{T}, \mathscr{F})$ be Goldie's torsion theory for the category R-mod. If σ is noetherian, it is of finite type and so $\mathscr{F}$ and $\mathscr{F} \cap \mathscr{I}_\sigma$ are cover classes by Corollary 3 above. It should be noted that $\mathscr{F}$ is a cover class by [15; Theorem] and [7; Corollary 4.1] (for some other aspects see also [4]). The known construction of an $\mathscr{F} \cap \mathscr{I}_\sigma$-cover (see e.g. [7; Corollary 4.5]) starts with an arbitrary $(\mathscr{F} \cap \mathscr{I}_\sigma)$-precover, uses the inductivity of the corresponding pure submodules and then produces the $(\mathscr{F} \cap \mathscr{I}_\sigma)$-cover as a suitable factor-module of the given precover. Thus the construction uses the property of covers which asserts that the corresponding kernel contains no non-zero relatively pure submodule. In the rest of this paper we are going to give a direct construction of an $(\mathscr{F} \cap \mathscr{I}_\sigma)$-cover of an arbitrary module which uses the transfinite induction rather than Zorn's Lemma for relatively pure submodules.

Construction 9. Let $\{H_\lambda;\ \lambda \in \Lambda\}$ be any representative set of indecomposable σ-torsionfree (i.e. non-singular) injective modules. Now if M is an arbitrary module, then for each $\lambda \in \Lambda$ and each $0 \neq f \in \operatorname{Hom}(H_\lambda, M)$ we take an isomorphic copy $H_{\lambda f}$ of H_λ and consider the set $\mathscr{G}$ of all couples $(H_{\lambda f}, \tilde{f})$ with $\lambda \in \Lambda$ and $0 \neq f \in \operatorname{Hom}(H_\lambda, M)$, where $\tilde{f}\colon\ H_{\lambda f} \to M$ means the composition of f with the isomorphism of $H_{\lambda f}$ onto H_λ. We denote by ν the cardinality of this set if it is infinite and we put $\nu = |\mathscr{G}| + 1$ in the opposite case. Without loss of generality we can assume that $\mathscr{G}$ is well-ordered, i.e. that $\mathscr{G} = \{(H_\alpha, f_\alpha);\ \alpha < \nu\}$, where f_α is a non-zero element of $\operatorname{Hom}(H_\alpha, M)$. Now we are ready to start our construction.

We put $G_0 = 0$ and proceed by the transfinite induction. So, let $\alpha < \nu$ be an arbitrary ordinal and let G_β and $\varphi_\beta\colon\ G_\beta \to M$, $\beta < \alpha$, have been already defined in such a way that $G_\gamma \subseteq G_\beta$ and $\varphi_\gamma = \varphi_\beta | G_\gamma$ whenever $\gamma \leqslant \beta < \alpha$. If α is a limit ordinal then we simply put $G_\alpha = \bigcup\limits_{\beta<\alpha} G_\beta$ and $\varphi_\alpha = \bigcup\limits_{\beta<\alpha} \varphi_\beta$ in the natural way. Now let $\alpha = \beta + 1$ be a successor ordinal. If there is a monomorphism $g\colon\ H_\beta \to G_\beta$ such that $\varphi_\beta g = f_\beta$, then we put $G_\alpha = G_\beta$ and $\varphi_\alpha = \varphi_\beta$. In the opposite case we put $G_\alpha = G_\beta \oplus H_\beta$ and φ_α will be the corresponding natural evaluation map given by φ_β and f_β. Finally, we put $G = \bigcup\limits_{\alpha<\nu} G_\alpha$ and $\varphi = \bigcup\limits_{\alpha<\nu} \varphi_\alpha$. Now we are ready to prove that the homomorphism $\varphi\colon\ G \to M$ is an $(\mathscr{F} \cap \mathscr{I}_\sigma)$-cover of the module M.

Theorem 10. *Let $\sigma = (\mathscr{T}, \mathscr{F})$ be Goldie's torsion theory for the category R-*mod*, let $M \in R$-*mod *be arbitrary and let $\varphi\colon\ G \to M$ be as in Construction* 9. *If σ is noetherian, then φ is an $(\mathscr{F} \cap \mathscr{I}_\sigma)$-cover of the module M.*

P r o o f. It follows immediately from Construction 9 and Lemmas 5 and 6 that φ is an $(\mathscr{F} \cap \mathscr{I}_\sigma)$-precover of the module M, every σ-torsionfree σ-injective module being a direct sum of directly indecomposable injective modules. In order to show that φ is in fact an $(\mathscr{F} \cap \mathscr{I}_\sigma)$-cover of the module M let us consider the commutative square

$$(*) \qquad \begin{array}{ccc} G & \xrightarrow{\varphi} & M \\ {\scriptstyle f}\big\downarrow & & \big\| \\ F & \xrightarrow{\psi} & M \end{array}$$

where ψ is a $\mathscr{G}$-precover of M. Clearly, $\operatorname{Ker} f \subseteq \operatorname{Ker} \varphi$ and since $\operatorname{Im} f \cong G / \operatorname{Ker} f \in \mathscr{F}$, $\operatorname{Ker} f$ is σ-pure in G and consequently, $\operatorname{Ker} f$ is an injective module. Assuming $\operatorname{Ker} f \neq 0$ we see that $\operatorname{Ker} \varphi$ contains by Lemma 6 an indecomposable σ-torsionfree injective module H. Now we shall explore the construction of the module G. Taking $0 \neq v \in H$ arbitrarily, we have $v \in G_{\alpha_1} \oplus \ldots \oplus G_{\alpha_n}$ for suitable indices $\alpha_1 < \ldots < \alpha_n < \nu$ such that the element v has a non-zero component in each G_{α_i}, $i = 1, \ldots, n$. From this fact it immediately follows that $H \leqslant G_{\alpha_1} \oplus \ldots \oplus G_{\alpha_n}$ and so if $p_i \colon G \to G_{\alpha_i}$ are the canonical projections, then $\operatorname{Im}(p_i|H) \cong H/\operatorname{Ker}(p_i|H) \in \mathscr{T} \cap \mathscr{F} = 0$ by Lemma 7 whenever $\operatorname{Ker}(p_i|H) \neq 0$. Thus $\sigma_i = p_i|H \colon H \to G_{\alpha_i}$ is an isomorphism for each $i = 1, \ldots, n$. Assume first that $n > 1$. In this case the mapping $\sigma_n - 1_H \colon H \to G$ is a monomorphism and for each $u \in G_{\alpha_n}$ we have $\varphi(\sigma_n - 1_H)\sigma_n^{-1}(u) = \varphi(\sigma_n - 1_H)(v_1 + \ldots + v_n) = \varphi\Big(v_n - \sum\limits_{i=1}^{n} v_i\Big) = -\varphi\Big(\sum\limits_{i=1}^{n-1} v_i\Big) = -\sum\limits_{i=1}^{n-1} f_{\alpha_i}(v_i) = f_{\alpha_n}(v_n) = f_{\alpha_n}(\sigma_n\sigma_n^{-1}(u)) = f_{\alpha_n}(u)$, which contradicts Construction 9 in view of the fact that for each $v \in H$ we have $0 = \varphi(v) = \varphi(v_1 + \ldots + v_n) = f_{\alpha_1}(v_1) + \ldots f_{\alpha_n}(v_n)$. Now the case $n = 1$ remains. However, in this case we have $H = G_\alpha \leqslant G$ for some $\alpha < \nu$, hence $\varphi(H) = f_\alpha(G_\alpha) \neq 0$, which is a contradiction showing that f is a monomorphism. Since σ is of finite type, the class $\mathscr{F} \cap \mathscr{I}_\sigma$ is inductive by [12; Proposition 42.9] and it suffices to use [7; Theorem 3.1] stating that φ is an $(\mathscr{F} \cap \mathscr{I}_\sigma)$-cover of the module M if and only if in every commutative diagram $(*)$ where ψ is an $(\mathscr{F} \cap \mathscr{I}_\sigma)$-precover of M the homomorphism f is injective. $\square$

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Author's address: Ladislav Bican, KA MFF UK, Sokolovská 83, 186 00 Praha 8-Karlín, Czech Republic, e-mail: bican@karlin.mff.cuni.cz.