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WEAK-OPEN COMPACT IMAGES OF METRIC SPACES

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Abstract. The main results of this paper are that

(1) a space X is g-developable if and only if it is a weak-open π image of a metric space, one consequence of the result being the correction of an error in the paper of Z. Li and S. Lin;

(2) characterizations of weak-open compact images of metric spaces, which is another answer to a question in in the paper of Y. Ikeda, C. liu and Y. Tanaka.

Keywords: g-developable, π-mapping, weak-open mapping, CWC-map, uniform weak base

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1. Introduction

A. Arhangel'skii [4] showed that a T_1-space X has a uniform base if and only if it is an open compact image of a metric space. As is well known, a space X has a uniform base if and only if it is a metacompact developable space. R. Heath [9] showed that a T_1-space X is developable if and only if it is an open π image of a metric space. In this paper characterizations of weak-open compact (π) images of metric spaces are given in terms of the concept of g-developable spaces, which are similar to Arhangel'skii's (Heath's) above result.

In this paper, all spaces are T_2 and all mappings are continuous and surjective. $\mathbb{N}$ denotes the set of all natural numbers.

2. Weak-open π images of metric spaces

Definition 2.1. A cover $\mathscr{P} = \bigcup_{x\in X} \mathscr{P}_x$ of a space X is called a weak base [5] for X if it has the following properties:

(1) $x \in \bigcap \mathscr{P}_x$ for each $x \in X$,

(2) if $P_1, P_2 \in \mathscr{P}_x$, then there exists $P \in \mathscr{P}_x$ with $P \subset P_1 \cap P_2$,

(3) for $U \subset X$, U is open in X if and only if for each $x \in U$ there exists $P \in \mathscr{P}_x$ with $P \subset U$.

$\mathscr{P}_x$ is called a weak neighborhood base of x in X. If each $\mathscr{P}_x$ is countable for $x \in X$, then X is called g-first countable.

For a space X, let g be a map defined on $\mathbb{N} \times X$ to the power-set of X such that $x \in g(n,x)$ and $g(n+1,x) \subset g(n,x)$ for each n and x; a subset U of X is open if for each $x \in U$ there is an n such that $g(n,x) \subset U$. We call such a map a CWC-map (countable weakly-open covering map).

As mentioned in [11], X is g-first countable if X has a CWC-map g such that if $x_n \in g(n,x)$ for each n, then the sequence $\{x_n\}$ converges to x.

Definition 2.2 ([11])**.** A space is g-developable if it has a CWC-map g with the following property: If $x, x_n \in g(n, y_n)$ for each n, then the sequence $\{x_n\}$ converges to x.

As stated in [19], if X has a g-first countable CWC-map g such that $\{\mathrm{st}(x,\mathscr{G}_n)\colon x \in X, n \in \mathbb{N}\}$ is a weak base for X, then X is g-developable (where $\mathscr{G}_n = \{g(n,x)\colon x \in X\}$). Conversely, let g be a g-developable CWC-map for a space X, then $\{\mathrm{st}(x,\mathscr{G}_n)\colon x \in X, n \in \mathbb{N}\}$ is a weak base for X. Thus, a g-developable space is symmetrizable.

Definition 2.3. Let $f\colon X \to Y$ be a mapping.

(1) f is called a π-mapping [1] if (X,d) is a metric space and $d(f^{-1}(y), X\setminus f^{-1}(U)) > 0$ for each $y \in Y$ and a neighborhood U of y in Y.

(2) f is weak-open [21] if there is a weak base $\mathscr{B} = \bigcup\{\mathscr{B}_y\colon y \in Y\}$ for Y and there is an $x(y) \in f^{-1}(y)$ for each $y \in Y$ such that for each open neighborhood U of $x(y)$ in X, $B_y \subset f(U)$ for some $B_y \in \mathscr{B}_y$.

(3) f is called a sequence-covering mapping [19] if each convergent sequence from Y is the image of some convergent sequence from X.

(4) f is called a 1-sequence-covering mapping [14] if for each $y \in Y$, there is an $x \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y, then there is a sequence $\{x_n\}$ converging to x in X with each $x_n \in f^{-1}(y_n)$.

(5) f is called a σ-locally finite mapping [16], if there exists a base $\mathscr{B}$ for X such that $f(\mathscr{B}) = \{f(B)\colon B \in \mathscr{B}\}$ is σ-locally finite in Y.

Obviously, every compact mapping on a metric space is a π-mapping, and each 1-sequence-covering mapping is a sequence-covering mapping. It is easy to check that a weak-open mapping is a quotient mapping.

Theorem 2.4. *A space X is g-developable if and only if it is a weak-open π image of a metric space.*

P r o o f. Necessity. Suppose that X has a CWC-map g satisfying Definition 2.2. Set $\mathscr{G}_n = \{g(n,x)\colon x \in X\} = \{G_\alpha\colon \alpha \in A_n\}$ for each $n \in \mathbb{N}$. Let A_n be the space with discrete topology. Put $M = \{\alpha = (\alpha_n) \in \prod\limits_{n\in\mathbb{N}} A_n\colon \{P_{\alpha_n}\colon n \in \mathbb{N}\}$ forms a network at some point $x(\alpha)$ in $X\}$, and equip M with the subspace topology induced from the usual product topology of the discrete spaces A_n. We define a distance function d on M as follows. For each $\alpha, \beta \in M$, if $\alpha = \beta$, then $d(\alpha,\beta) = 0$, if $\alpha \neq \beta$, then $d(\alpha,\beta) = 1/n$, where n is the smallest natural number such that $\alpha_n \neq \beta_n$. Hence d is a metric on M. We define $f\colon M \to X$ by $f(\alpha) = x(\alpha)$. It is easy to check that f is continuous and surjective.

(1) f is a weak-open mapping. For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = G_{\gamma_n}$, then $\gamma = (\gamma_n) \in M$ and $\gamma \in f^{-1}(x)$. Set $B_n = \{\beta \in M\colon$ the i-th coordinate of β is γ_i, for all $i \leqslant n\}$. It is easy to check that $\{B_n\colon n \in \mathbb{N}\}$ is a local decreasing base of $\gamma = (\gamma_i)$ in M, and $f(B_n) = \bigcap\limits_{i\leqslant n} G_{\gamma_i} = G_{\gamma_n}$. Let U be any open neighborhood of γ in M; then there exists $m \in \mathbb{N}$ such that $B_m \subset U$. Hence $f(B_n) \subset f(U)$. If we set $\mathscr{B}_x = \{g(n,x),\ n \in \mathbb{N}\}$ and $\mathscr{B} = \bigcup\limits_{x\in X} \mathscr{B}_x$, then $\mathscr{B}$ is a weak base for X which satisfies Definition 2.3 (2). Therefore f is a weak-open mapping.

(2) f is a π mapping. Suppose that $x \in X$ and U is an open neighborhood of x. It is not difficult to show that $\mathrm{st}(x,\mathscr{G}_n) \subset U$ for some $n \in \mathbb{N}$. For $\alpha \in f^{-1}(x)$, $\beta \in M$, if $d(\alpha,\beta) < 1/n$, then $\alpha_i = \beta_i$ for all $i \leqslant n$. Hence $f(\beta) = \bigcap\limits_{i\in\mathbb{N}} G_{\beta_i} = G_{\beta_n} \subset U$. Thus $d(f^{-1}(x), X \setminus f^{-1}(U)) \geqslant 1/n$. It follows that f is a π mapping.

Sufficiency. Suppose X is a weak-open π image of a metric space M under f. Let $\mathscr{P} = \bigcup\limits_{x\in X} \mathscr{P}_x$ be the weak base for X satisfying Definition 2.3 (2). For each $x \in X$, there is $\alpha(x) \in f^{-1}(x)$ satisfying Definition 2.3 (2). Set $g(n,x) = f(B(\alpha(x),1/n))$ for each $x \in X$ and $n \in \mathbb{N}$, where $B(\alpha,1/n) = \{\beta \in M\colon d(\alpha,\beta) < 1/n\}$.

(1) g is a g-first countable CWC-map for X. Suppose that U is a subset of X such that for each $x \in U$ there is n with $g(n,x) \subset U$. As f is weak-open and $B(\alpha(x),1/n)$ is an open neighborhood of $\alpha(x)$, there is $P \in \mathscr{P}_x$ such that $P \subset f(B(\alpha(x),1/n)) \subset U$. So U is open in X. On the other hand, suppose that U is an open subset of X. For each $x \in U$, we then have $B(\alpha(x),1/m) \subset f^{-1}(U)$ for some $m \in \mathbb{N}$. Hence $g(m,x) = f(B(\alpha(x),1/m)) \subset U$.

(2) g possesses the desired property: if $x, x_n \in g(n, y_n)$ for each n, then the sequence $\{x_n\}$ converges to x. Suppose that U is an open neighborhood of x in X. Then there is $n \in \mathbb{N}$ such that $d(f^{-1}(x), X \backslash f^{-1}(U)) > 1/n$. Taking a natural number $m \geqslant n$, if $x \in f(B(\alpha(z), 1/m)) = g(m, z)$, then $(f^{-1}(x) \cap B(\alpha(z), 1/m)) \neq \emptyset$. Suppose $B(\alpha(z), 1/m)) \subsetneq f^{-1}(U)$, then $d(f^{-1}(x), X \setminus f^{-1}(U)) < 2/m \leqslant 1/n$, a contradiction. Hence $B(\alpha(z), 1/m)) \subset f^{-1}(U)$, $f(B(\alpha(z), 1/m)) \subset U$. So $\mathrm{st}(x, \mathscr{G}_m) \subset U$, hence x_n converges to x. Therefore X is g-developable. □

Lemma 2.5 ([21]). *Let $f\colon X \to Y$ be a weak-open mapping. If X is first countable, then f is a 1-sequence-covering mapping.*

We recall that X is a Cauchy space [2] if X has a symmetric such that each convergent sequence is Cauchy. K. B. Lee [11] showed that a space X is g-develoable if and only if it is a Cauchy space. Y. Tanaka [20] characterized sequence-covering quotient π images of metric spaces by Cauchy spaces. Hence we have the following result.

Corollary 2.6. *For a space X, the following assertions are equivalent:*
(1) *X is a weak-open π image of a metric space.*
(2) *X is a 1-sequence-covering quotient π image of a metric space.*
(3) *X is a sequence-covering quotient π image of a metric space.*
(4) *X is a Cauchy space.*
(5) *X is a g-developable space.*

Example 2.7. There exists a g-metrizable space which is not a weak-open π image of a metric space.

L. Foged [6] showed that there is a g-metrizable space which is not g-developable. By Theorem 2.4, the g-metrizable space is not a weak-open π image of a metric space.

Remark 2.8. Example 2.7 shows that Theorem 2.2 [13] is wrong. And Lemma 2.1 [13] is not correct either, see Example 16 [16].

Examining the proof of Theorem 2.2 [13], we see that (1) and (3) of the theorem are equivalent. Thus we have the following result.

Theorem 2.9 ([12], [13], [16]). *For a regular space X, the following assertions are equivalent:*
(1) *X is a weak-open σ-image of a metric space.*
(2) *X is a 1-sequence-covering σ-image of a metric space.*
(3) *X is a compact-covering quotient π σ-image of a metric space.*
(4) *X is a g-metrizable space.*

Definition 3.1 ([8]). Let $\mathscr{P}$ be a cover of a space X. $\mathscr{P}$ is called a cs-network if whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is a neighborhood of x, then $\{x\} \cup \{x_n\colon n \geqslant m\} \subset P \subset U$ for some $m \in \mathbb{N}$ and some $P \in \mathscr{P}$.

Definition 3.2 ([3]). Let $\mathscr{P}$ be a cover of a space X. $\mathscr{P}$ is called a point-regular cover for X if $\{P \in (\mathscr{P})_x\colon P \subsetneq U\}$ is finite for each open neighborhood U of x.

$\mathscr{P}$ is called a point-regular cs-network (weak base) for X if $\mathscr{P}$ is called a point-regular cover and a cs-network (weak base) for X.

Definition 3.3 ([3]). Let $\mathscr{P}$ be a cover of a space X. $\mathscr{P}$ is called a uniform regular cover for X, if whenever $\mathscr{P}'$ is a countable infinite subset of $(\mathscr{P})_x$, then $\mathscr{P}'$ is a network for x in X.

Definition 3.4 ([7]). Let X be a space.

(1) Let $x \in P \subset X$. P is called a sequential neighborhood of x in X if whenever $\{x_n\}$ is a sequence converging to the point x, then $\{x_n\colon n \geqslant m\} \subset P$ for some $m \in \mathbb{N}$.
(2) Let $P \subset X$. P is called a sequential open subset in X if P is a sequential neighborhood of x for each $x \in P$.
(3) X is called a sequential space if each sequential open subset in X is open.

For a space X, set $W(X) = \{x \in X\colon \{x\}$ is a weak neighborhood of $x\}$ and $\mathscr{W}(X) = \{\{x\}\colon x \in W(X)\}$.

Proposition 3.5. *For a space X, the following assertions are equivalent:*

(1) *X has a point finite g-developable* CWC*-map g (i.e. $\mathscr{G}_n = \{g(n,x)\colon x \in X\}$ is point finite for each $n \in \mathbb{N}$).*
(2) *X has a uniform weak base.*

Proof. (1) $\Rightarrow$ (2). Set $\mathscr{G} = \bigcup\limits_{n\in\mathbb{N}} \mathscr{G}_n$. For each $x \in X$, let $\mathscr{G}'$ be a countable subset of $(\mathscr{G})_x$, and set $\mathscr{G}' = \{g(n_k, x_k)\colon k \in \mathbb{N}\}$. Then $\{n_k\colon k \in \mathbb{N}\}$ is an infinite set. Suppose that U is an open neighborhood of x, then there is $m \in \mathbb{N}$ such that $\operatorname{st}(x, \mathscr{G}_m) \subset U$. Taking $k \in \mathbb{N}$ with $n_k \geqslant m$, then $x \in g(n_k, x_k) \subset \operatorname{st}(x, \mathscr{G}_m) \subset U$. Hence $\mathscr{G}$ is a uniform weak base for X.

(2) $\Rightarrow$ (1) Let $\mathscr{P} = \bigcup\limits_{x\in X} \mathscr{P}_x$ be a uniform weak base for X.

(i) $\mathscr{P}$ is point countable. For $x \in X$, suppose that $(\mathscr{P})_x$ is not countable. For $y \neq x$, as $\mathscr{P}$ is a uniform weak base for X, hence $\{P \in (\mathscr{P})_x\colon y \in P\}$ is finite. So there are an infinite subset $\{P_n\colon n \in \mathbb{N}\}$ of $(\mathscr{P})_x$, $x_n \in P_n \setminus \{x\}$ and $k \in \mathbb{N}$ such that $\operatorname{ord}(x_n, (\mathscr{P})_x) = k$ for each n. Note that $\mathscr{P}$ is a uniform weak base for X, hence $x_n \to x$. Since a weak base for X is a cs-network for X, there are a subset $\{F_n\colon n \in \mathbb{N}\}$ of $(\mathscr{P})_x$ and a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_i}\colon i \geqslant j\} \subset F_j \subset X \setminus \{x_{n_i}\colon i < j\}$, $j \in \mathbb{N}$. So $\operatorname{ord}(x_{n_j}, (\mathscr{P})_x) \geqslant j$, a contradiction. Therefore $\mathscr{P}$ is point countable.

(ii) $\mathscr{P}$ has the property $(*)$: $\{R \in \mathscr{P}\colon P \subset R\}$ is finite for each $P \in \mathscr{P}$. Otherwise, there is an infinite subset $\{P_n\colon n \in \mathbb{N}\}$ of $\mathscr{P} \setminus \{P\}$ such that $P \subset P_n$ for each n. By (i), X is g-first countable, hence X is a sequential space. If P is a single point, then since a weak neighborhood is a sequential neighborhood, P is a sequential open subset of X, thus P is open in X and $P_n \subset P$ for some P_n, a contradiction. If P is not a single point, then we may suppose that $\{x, y\} \subset P$, thus there is $n \in \mathbb{N}$ such that $x \in P_n \subset X - \{y\}$, a contradiction.

(iii) Put $\mathscr{P}^m = \{H \in \mathscr{P}\colon \text{if } H \subset P \in \mathscr{P}, \text{ then } P = H\}$. Then $\mathscr{P}^m$ is a point finite cover for X. For $P \in \mathscr{P}$, by Property $(*)$, there is $H \in \mathscr{P}^m$ with $P \subset H$, thus $\mathscr{P}^m$ is a cover for X. For $x \in X$, if $(\mathscr{P}^m)_x$ is infinite, we denote it by $\{H_n\colon n \in \mathbb{N}\}$. Since $\mathscr{P}$ is a uniform weak base for X, there is a weak neighborhood $P \in \mathscr{P}$ of x and we may assume $P \subset H_1$. Taking $x_n \in (H_{n+1} \setminus H_1)$, then $x_n \to x$ and $\{x_n\} \cap P = \emptyset$, a contradiction. Therefore $\mathscr{P}^m$ is a point finite cover for X.

(iv) Put $\mathscr{P}' = (\mathscr{P} \setminus \mathscr{P}^m) \cup \mathscr{W}(X)$. Then $\mathscr{P}'$ is a uniform weak base for X. Let U be an open neighborhood of x and we may assume $x \notin \mathscr{W}(X)$. Then there are distinct elements P_1, P_2 and P of $\mathscr{P}_x$ with $x \in P \subset P_1 \cap P_2 \subset P_1 \cup P_2 \subset U$. Thus $P \in \mathscr{P}'$, $\mathscr{P}'$ is a weak base for X. Hence $\mathscr{P}'$ is a uniform weak base for X.

(v) X has a point finite g-developable CWC-map g. Put $\mathscr{P}_1 = \mathscr{P}^m$, $\mathscr{P}_{n+1} = [(\mathscr{P} \setminus \bigcup\{\mathscr{P}_j\colon j \leqslant n\}) \cup \mathscr{W}(X)]^m$, $n \in \mathbb{N}$. It is easy to see that $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$ for each $n \in \mathbb{N}$, and $\mathscr{P} = \bigcup\limits_{n\in\mathbb{N}} \mathscr{P}_n$ by the property $(*)$. For each $x \in X$, $n \in \mathbb{N}$, if $x \in W(X)$, then we define $g(n, x) = \{x\}$. If $x \notin W(X)$, since each $\mathscr{P}_n$ is point finite, we may assume that $\mathscr{P}_x = \{P_i\colon i \in \mathbb{N}\} = \{P_1, P_2, \ldots, P_{k_1}, P_{k_1+1}, \ldots, P_{k_2}, P_{k_2+1}, \ldots, P_{k_n} \ldots\}$ with $P_i \in \mathscr{P}_{m_n}$ when $k_{n-1} < i \leqslant k_n$ and $m_n < m_{n+1}$ (where $k_0 = 0$). We define $g(n, x) = \bigcap\limits_{j \leqslant k_n} P_j$. Since each $\mathscr{P}_n$ is point finite and $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$ for $n \in \mathbb{N}$, $\mathscr{G}_n = \{g(n, x)\colon x \in X\}$ is point finite for each $n \in \mathbb{N}$. Let U be a subset of X such that for each $x \in U$, $g(n, x) \subset U$ for some $g(n, x)$. Then there is $P \in \mathscr{P}_x$ with $P \subset g(n, x)$. Hence U is open in X, thus g is a g-first CWC-map for X. To complete the proof, we need only to show that $\{\operatorname{st}(x, \mathscr{G}_n)\colon x \in X,\ n \in \mathbb{N}\}$ is a weak base for X. Let U be an open neighborhood of x and suppose that $\operatorname{st}(x, \mathscr{G}_n) \subsetneq U$ for each $n \in \mathbb{N}$. Then there are $x_n, x \in g(n, y_n)$ with $x_n \notin U$ for each $n \in \mathbb{N}$. Note that if $P_n \in \mathscr{P}_n$ then $\{P_n\colon n \in \mathbb{N}\}$ are different from each other. Thus there is an infinite subset $\mathscr{P}''$ of $(\mathscr{P})_x$ such that each element of $\mathscr{P}''$ contains some $g(n, y_n)$. Since $\mathscr{P}$ is a uniform weak base for X, there is $P \in \mathscr{P}''$ such that $g(n, y_n) \subset P \subset U$ for some $g(n, y_n)$, a contradiction. Hence $\mathrm{st}(x, \mathscr{G}_n) \subset U$ for some $n \in \mathbb{N}$, and $\{\mathrm{st}(x, \mathscr{G}_n) \colon n \in \mathbb{N}\}$ is a weak base for X. $\square$

Theorem 3.6. *The following assertions are equivalent for a space X.*

(1) *X is a weak-open compact image of a metric space.*

(2) *X has a point finite g-developable CWC-map g.*

Proof. (1) $\Rightarrow$ (2) Let X be a weak-open compact image of a metric space M under f. By Theorem 1.3.3 [15], there is a sequence $\{\mathscr{P}_n\}$ of open covers of M such that $\{\mathrm{st}(K, \mathscr{P}_n) \colon n \in \mathbb{N}\}$ is a neighborhood base of K in M for each $K \in \mathscr{K}(M)$ (where $\mathscr{K}(M)$ is the collection of all compact subsets of M). By the paracompactness of M, we may assume that each $\mathscr{P}_n$ is local finite and $\mathscr{P}_{n+1}$ refines $\mathscr{P}_n$ for each $n \in \mathbb{N}$. Thus $f(\mathscr{P}_n)$ is a point finite cover of X for each $n \in \mathbb{N}$. Let $\mathscr{B} = \bigcup\limits_{x \in X} \mathscr{B}_x$ be a weak base for X and let $\alpha(x) \in f^{-1}(x)$ for $x \in X$ satisfying Definition 2.3 (2). For each $n \in \mathbb{N}$ and $x \in X$, there exists $P_n \in \mathscr{P}_n$ with $\alpha(x) \in P_n$. For this P_n there exists $B_n \in \mathscr{B}_x$ with $B_n \subset f(P_n)$. Put $g(n, x) = \bigcap\limits_{j \leqslant n} B_j$, then $\mathscr{G}_n = \{g(n, x) \colon x \in X\}$ is point finite for each $n \in \mathbb{N}$. Suppose that U is an open subset of X for $x \in U$, then $f^{-1}(x) \subset f^{-1}(U)$, so there exists $n \in \mathbb{N}$ such that $\mathrm{st}(f^{-1}(x), \mathscr{P}_n) \subset f^{-1}(U)$. Hence $g(n, x) \subset \mathrm{st}(x, \mathscr{G}_n) \subset \mathrm{st}(x, f(\mathscr{P}_n)) \subset U$. Conversely, let U be a subset of X such that for each $x \in U$ there exists $n \in \mathbb{N}$ with $g(n, x) \subset U$. By the definition of $g(n, x)$, we have $B \subset g(n, x) \subset U$ for some $B \in \mathscr{B}_x$. As $\mathscr{B}$ is a weak base for X, U is open in X. Therefore g is a g-first countable CWC-map for X. Similarly, as shown above, $\{\mathrm{st}(x, \mathscr{G}_n) \colon x \in X,\ n \in \mathbb{N}\}$ is a weak base for X, so g is a point finite g-developable CWC-map for X.

(2) $\Rightarrow$ (1) Put $\mathscr{G}_n = \{g(n, x), x \in X\}$, $n \in \mathbb{N}$, then each $\mathscr{G}_n$ is point finite. It is easy to check that $\{G_i \colon i \in \mathbb{N}\}$ is a network of x if $x \in G_i \in \mathscr{G}_i$ for each $i \in \mathbb{N}$. Set $\mathscr{G}_n = \{G_\alpha \colon \alpha \in A_n\}$, $n \in \mathbb{N}$, and $M = \bigl\{\alpha = (\alpha_n) \in \prod\limits_{n \in \mathbb{N}} A_n \colon \{G_{\alpha_n} \colon n \in \mathbb{N}\}$ forms a network at some point $x(\alpha)$ in $X\bigr\}$, and equip M with the subspace topology induced from the usual product topology of the discrete spaces A_n. Then M is a metric space. We define $f \colon M \to X$ by $f(\alpha) = x(\alpha)$. It is easy to see that f is a continuous and surjective mapping.

(i) f is a weak-open mapping. As in the proof of Theorem 2.4, f is a weak-open mapping

(ii) f is a compact mapping. For each $x \in X$, $i \in \mathbb{N}$, set

$$C_i = \{\alpha \in A_i \colon x \in G_\alpha \in \mathscr{G}_i\};$$

then $\prod\limits_{i \in \mathbb{N}} C_i$ is a compact subset of $\prod\limits_{i \in \mathbb{N}} A_i$. If $\alpha = (\alpha_i) \in \prod\limits_{i \in \mathbb{N}} C_i$, then $x \in \bigcap\limits_{i \in \mathbb{N}} G_{\alpha_i}$, $\alpha \in M$ and $f(\alpha) = x$. So $\prod\limits_{i \in \mathbb{N}} C_i \subset f^{-1}(x)$. On the other hand, if $\alpha = (\alpha_i) \in f^{-1}(x)$, then $x \in \bigcap_{i\in\mathbb{N}} G_{\alpha_i}$. Hence $\alpha \in \prod_{i\in\mathbb{N}} C_i$, $f^{-1}(x) \subset \prod_{i\in\mathbb{N}} C_i$. Thus $f^{-1}(x) = \prod_{i\in\mathbb{N}} C_i$, and f is a compact mapping. □

Corollary 3.7. *The following assertions are equivalent for a space X.*

(1) *X is a weak-open compact image of a metric space.*

(2) *X is a 1-sequence-covering quotient compact image of a metric space.*

(3) *X is a sequence-covering quotient compact image of a metric space.*

(4) *X has a point-regular weak base.*

(5) *X is a sequential space with a point-regular cs-network.*

(6) *X has a uniform weak base.*

(7) *X has a point finite g-developable* CWC*-map.*

(3) ⇔ (4) was shown in [10], (2) ⇔ (3) ⇔ (4) ⇔ (5) ⇔ (6) were shown by S. Lin and P. Yan [17], which answered a question from [10]: For a sequential space X with a point-regular cs-network, characterize X by means of a nice image of a metric space. Thus Corollary 3.7 is another answer to the question.

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