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STRONG SEPARATIVITY OVER EXCHANGE RINGS

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Abstract. An exchange ring R is strongly separative provided that for all finitely generated projective right R-modules A and B, $A \oplus A \cong A \oplus B \Rightarrow A \cong B$. We prove that an exchange ring R is strongly separative if and only if for any corner S of R, $aS + bS = S$ implies that there exist $u, v \in S$ such that $au = bv$ and $Su + Sv = S$ if and only if for any corner S of R, $aS + bS = S$ implies that there exists a right invertible matrix $\begin{pmatrix} a & b \\ * & * \end{pmatrix} \in M_2(S)$. The dual assertions are also proved.

Keywords: strong separativity, exchange ring, regular ring

MSC 2000: 16E50, 19E99

1. Introduction

A ring R is an exchange ring provided that for every right R-module M and two decompositions $M = A \oplus B = \bigoplus\limits_{i\in I} A_i$, where $A_R \cong R$ and the index set I is finite, there exist submodules $A_i' \subseteq A_i$ such that $M = A \oplus \Big(\bigoplus\limits_{i\in I} A_i'\Big)$ (see [3]). An exchange ring R is strongly separative provided that for all finitely generated projective right R-modules A and B, $A \oplus A \cong A \oplus B \Rightarrow A \cong B$. Many authors studied strong separativity for exchange rings (cf. [3]–[6], [8] and [11]–[14]). In [4, Proposition 4.2], Ara et al. proved that a regular ring R is strongly separative if and only if for any corner S of R, $(a, b) \in M_{1\times 2}(S)$ implies that there exists an invertible matrix $U \in M_2(S)$ such that $(a, b)U = (*, 0)$, where S is a corner of a ring R provided that $S = eRe$ for an idempotent $e \in R$. This inspires us to develop new elementwise characterizations of strongly separative exchange rings. In this paper we prove that an exchange ring R is strongly separative if and only if for any corner S of R, $aS + bS = S$ implies that there exist $u, v \in S$ such that $au = bv$ and $Su + Sv = S$

if and only if for any corner S of R, $aS + bS = S$ implies that there exists a right invertible matrix $\begin{pmatrix} a & b \\ * & * \end{pmatrix} \in M_2(S)$. Furthermore, we prove that a regular ring R is strongly separative if and only if for any corner S of R, $aS + bS = S$ implies that there exist $u, v \in S$ such that $au = bv$ and $r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v) = 0$. The dual assertions are also proved.

Throughout this paper, all rings are associative with an identity and all modules are unital right R-modules. Let A and B be right R-modules. $A \lesssim^{\oplus} B$ means that A is isomorphic to a direct summand of B and $A \lesssim B$ means that A is isomorphic to a submodule of B. We write $r(a)(l(a))$ for the right (left) annihilator of an element $a \in R$. $\mathbb{N}$ stands for the set of all natural numbers.

2. Unimodular rows

We say that a 1×2 matrix (a, b) over a ring R is a unimodular row provided that it is right invertible, i.e., $aR + bR = R$. It is well known that a regular ring R is unit-regular if and only if for any unimodular row (a, b), there exists a $y \in R$ such that $a + by \in R$ is invertible (see [7, Proposition 4.12]). In this section, we characterize strong separativity for exchange rings by virtue of unimodular rows.

Lemma 2.1. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R is strongly separative.*

(2) *For any right R-modules A and B, $A \oplus A \cong A \oplus B \lesssim^{\oplus} R$ implies that $A \lesssim^{\oplus} B$.*

Proof. (1) $\Rightarrow$ (2) is trivial.

(2) $\Rightarrow$ (1) Let A and B be finitely generated projective right R-modules such that $A \oplus A \cong A \oplus B$. Then we can find $n \in \mathbb{N}$ such that $A \lesssim^{\oplus} nR$. Since R is an exchange ring, it follows by [3, Proposition 1.2] that there exist $A_1, \ldots, A_n \lesssim^{\oplus} R$ such that $A \cong A_1 \oplus \ldots \oplus A_n$. Hence $A_1 \oplus (A_2 \oplus \ldots \oplus A_n \oplus A) \cong A_1 \oplus (A_2 \oplus \ldots \oplus A_n \oplus B)$. Let $C_1 = A_2 \oplus \ldots \oplus A_n \oplus A$ and $B_1 = A_2 \oplus \ldots \oplus A_n \oplus B$. Then $A_1 \oplus C_1 \cong A_1 \oplus B_1$ with $A_1 \lesssim^{\oplus} C_1$. Analogously to [3, Lemma 2.7], we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} A_1 & C_1 \end{array} \\ \begin{array}{c} A_1 \\ B_1 \end{array} & \begin{pmatrix} A_1' & C_1' \\ B_1' & D_1 \end{pmatrix} \end{array},$$

where $A_1' \lesssim^{\oplus} C_1'$.

Clearly, $A_1' \oplus C_1' \cong A_1' \oplus B_1' \cong A_1 \lesssim^{\oplus} R$ with $A_1' \lesssim^{\oplus} C_1'$. Thus, we have a refinement matrix

$$\begin{array}{cc} & \begin{array}{cc} A_1' & C_1' \end{array} \\ \begin{array}{c} A_1' \\ B_1' \end{array} & \begin{pmatrix} A_1'' & C_1'' \\ B_1'' & D_1' \end{pmatrix}, \end{array}$$

where $A_1'' \lesssim C_1''$. It follows from $A_1'' \oplus C_1'' \cong A_1'' \oplus B_1'' \cong A_1'$ with $A_1'' \lesssim C_1''$ that $C_1'' \oplus C_1'' \cong C_1'' \oplus B_1'' \lesssim A_1' \oplus C_1' \cong A_1 \lesssim^{\oplus} R$. By assumption we have $C_1'' \lesssim^{\oplus} B_1''$, and so $B_1'' \cong C_1'' \oplus E$ for a right R-module E. As a result, we get $A_1' \cong A_1'' \oplus B_1'' \cong A_1'' \oplus C_1'' \oplus E \cong A_1' \oplus E$. Since $A_1' \lesssim^{\oplus} C_1'$, we see that $C_1' \cong C_1' \oplus E$, whence $C_1' \cong C_1'' \oplus D_1' \oplus E \cong B_1'' \oplus D_1' \cong B_1'$, and so $C_1 \cong B_1$. This means that $A_2 \oplus (A_3 \oplus \ldots \oplus A_n \oplus A) = A_2 \oplus (A_3 \oplus \ldots \oplus A_n \oplus B)$. By iterating this process, we prove that $A \cong B$. Therefore, R is strongly separative, which concludes the proof.

□

Lemma 2.2. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *$aR+bR=R$ implies that there exist $u,v \in R$ such that $au=bv$ and $Ru+Rv=R$.*

(2) *For any right R-module A, $R \oplus R \cong R \oplus A$ implies that $R \lesssim^{\oplus} A$.*

Proof. (1) ⇒ (2) Given $R \oplus R \cong R \oplus A$, we have a split exact sequence

$$0 \to A \xrightarrow{i} R \oplus R \xrightarrow{f} R \to 0.$$

Thus, there exists a right R-morphism $g\colon R \to R \oplus R$ such that $fg = 1_R$. Assume that $f(1,0)=a$, $f(0,1)=b$ and $g(1)=(x,y)$. Then $1 = fg(1) = f(x,y) = ax+by$; hence, $aR+bR=R$. By assumption, there exist $u,v \in R$ such that $au=bv$ and $Ru+Rv=R$. Construct a map $\varphi\colon R \to R \oplus R$ given by $\varphi(r)=(ur,-vr)$ for any $r \in R$. Since $f\varphi = 0$, there exists a right R-morphism $\omega\colon R \to A$ such that $i\omega = \varphi$. Clearly, $su+tv=1$ for some $s,t \in R$. Construct a map $\psi\colon R \oplus R \to R$ given by $\psi(r_1,r_2) = sr_1 - tr_2$ for any $(r_1,r_2) \in R \oplus R$. It is easy to verify that $\psi\varphi = 1_R$; hence, $\varphi\colon R \to R \oplus R$ is an R-monomorphism. This implies that $\omega\colon R \to A$ is an R-monomorphism, and so $R \lesssim^{\oplus} A$.

(2) ⇒ (1) Suppose that $aR+bR=R$ with $a,b \in R$. Then we have $x,y \in R$ such that $ax+by=1$. Define $f\colon R \oplus R \to R$ by $f(r_1,r_2) = ar_1+br_2$ for any $(r_1,r_2) \in R \oplus R$ and $g\colon R \to R \oplus R$ by $g(r)=(xr,yr)$ for any $r \in R$. As $fg=1_R$, we have a split exact sequence

$$0 \to \operatorname{Ker} f \xhookrightarrow{i} R \oplus R \xrightarrow{f} R \to 0,$$

where i is the inclusion map. This implies that $\operatorname{Ker} f \oplus R \cong R \oplus R$. By assumption, we get $R \lesssim^{\oplus} \operatorname{Ker} f$, and so we have two R-morphisms $\varphi\colon R \to \operatorname{Ker} f$ and $\omega\colon \operatorname{Ker} f \to R$ such that $\omega\varphi = 1_R$. Thus, $i\varphi\colon R \to R \oplus R$. Assume that $i\varphi(1) = (u, v') \in R \oplus R$. Then $0 = fi\varphi(1) = f(u, v') = au + bv'$; hence, $au = -bv'$. Let $v = -v'$. Then $au = bv$. Clearly, there exists a $j\colon R \oplus R \to \operatorname{Ker} f$ such that $ji = 1_{\operatorname{Ker} f}$. Hence $\omega j\colon R \oplus R \to R$. Assume that $\omega j(1, 0) = s$ and $\omega j(0, 1) = t$. Then $su - tv = \omega j(u, 0) + \omega j(0, v') = \omega j(u, v') = \omega j i\varphi(1_R) = 1_R$. This shows that $Ru + Rv = R$, as asserted. □

Theorem 2.3. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R is strongly separative.*

(2) *For any corner S of R, $aS + bS = S$ implies that there exist $u, v \in S$ such that $au = bv$ and $Su + Sv = S$.*

P r o o f. (2) ⇒ (1) Suppose that $A \oplus A \cong A \oplus B \lesssim^{\oplus} R$. Then we can find an idempotent $e \in R$ such that $A \cong eR$. Hence $eR \oplus eR \cong eR \oplus B$, and so $eR \bigotimes_R Re \oplus eR \bigotimes_R Re \cong eR \bigotimes_R Re \oplus B \bigotimes_R Re$. This implies that $eRe \oplus eRe \cong eRe \oplus B \bigotimes_R Re$ as right eRe-modules. By Lemma 2.2 we get $eRe \lesssim^{\oplus} B \bigotimes_R Re$. As a result, we get

$$A \cong eR \cong eRe \bigotimes_{eRe} eR \lesssim^{\oplus} \Big(B \bigotimes_R Re\Big) \bigotimes_{eRe} eR.$$

Since $B \lesssim^{\oplus} eR \oplus eR$, we see that $(B \bigotimes_R Re) \bigotimes_{eRe} eR \cong Be \bigotimes_{eRe} eR \cong B$ and so $A \lesssim^{\oplus} B$. According to Lemma 2.1, R is strongly separative.

(1) ⇒ (2) Let $e \in R$ be an idempotent. Given $eRe \oplus eRe \cong eRe \oplus A$ with a right eRe-module A, then $eR \oplus eR \cong eR \oplus A \bigotimes_{eRe} eR$. Since R is strongly separative, we have $eR \cong A \bigotimes_{eRe} eR$, and then $eR \bigotimes_R Re \cong (A \bigotimes_{eRe} eR) \bigotimes_R Re$ as right eRe-modules. Thus, $eRe \cong A$ as right eRe-modules. According to Lemma 2.2, we obtain the result. □

Following Ara et al. (see [4]), we say that a regular ring R has cancellation of small projectives provided it is strongly separative. Theorem 2.3 shows that a regular ring R having cancellation of small projectives can be characterized by an analogue of stable rank one.

Corollary 2.4. *Let R be an exchange ring. Then the following are equivalent:*

(1) *R is strongly separative.*

(2) *For any corner S of R and any $n \geqslant 2$, $a_1S + a_2S + \ldots + a_nS = S$ implies that there exist $u_1, u_2, \ldots u_n \in S$ such that $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$ and $Su_1 + Su_2 + \ldots + Su_n = S$.*

P r o o f. (2) $\Rightarrow$ (1) is trivial by Theorem 2.3.

(1) $\Rightarrow$ (2) Let S be a corner of R and $n \geqslant 2$. Suppose that $a_1S+a_2S+\ldots+a_nS = S$. Since R is strongly separative, so is S by [3, Lemma 1.5]. In view of [3, Theorem 3.3], the stable rank of S is less than 2. Thus, we have some $y_3, \ldots, y_n, z_3, \ldots, z_n \in S$ such that

$$(a_1 + a_3y_3 + \ldots + a_ny_n)S + (a_2 + a_3z_3 + \ldots + a_nz_n)S = S.$$

By virtue of Theorem 2.3, there exist $u, v \in S$ such that

$$(a_1 + a_3y_3 + \ldots + a_ny_n)u + (a_2 + a_3z_3 + \ldots + a_nz_n)v = 0$$

and $Su + Sv = S$. This implies that

$$a_1u + a_2v + a_3(y_3u + z_3v) + \ldots + a_n(y_nu + z_nv) = 0.$$

Furthermore, we see that

$$Ru + Rv + R(y_3u + z_3v) + \ldots + R(y_nu + z_nv) = R,$$

and therefore we complete the proof. $\square$

3. The symmetry

We say that a 2×1 matrix $\begin{pmatrix} a \\ b \end{pmatrix}$ over a ring R is a unimodular column provided that it is left invertible, i.e., $Ra + Rb = R$. In this section we characterize strongly separative exchange rings by using unimodular columns and give the dual of Theorem 2.3. As an application, we prove that strong separativity for exchange rings is symmetric.

Lemma 3.1. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *$Ra+Rb = R$ implies that there exist $u, v \in R$ such that $ua = vb$ and $uR+vR = R$.*

(2) *For any right R-module A, $R \oplus R \cong R \oplus A$ implies that $R \lesssim^{\oplus} A$.*

P r o o f. (1) $\Rightarrow$ (2) Given $R \oplus R \cong R \oplus A$, then we have a split exact sequence

$$0 \to R \xrightarrow{i} R \oplus R \xrightarrow{f} A \to 0.$$

So we can find a right R-morphism $j\colon R \oplus R \to R$ such that $ji = 1_R$. Assume that $j(1,0) = x$, $j(0,1) = y$ and $i(1) = (a,b)$. Then $1 = ji(1) = j(a,b) = xa + yb$ and so $Ra + Rb = R$. By assumption, there exist $u, v \in R$ such that $ua = vb$ and $uR + vR = R$. Construct a map $\varphi\colon R \oplus R \to R$ given by $\varphi(r_1, r_2) = ur_1 - vr_2$ for any $(r_1, r_2) \in R \oplus R$. For any $r \in R$, $\varphi i(r) = \varphi(ar, br) = uar - vbr = 0$, so $\varphi i = 0$. Thus, we can find a right R-morphism $\omega\colon A \to R$ such that $\omega f = \varphi$. Obviously, we have $s, t \in R$ such that $us + vt = 1$; hence, $\varphi(sr, -tr) = usr + vtr = r$ for any $r \in R$. This means that $\varphi\colon R \oplus R \to R$ is an R-epimorphism. Hence $\omega\colon A \to R$ is an R-epimorphism. As a result, we get $R \lesssim^{\oplus} A$.

(2) $\Rightarrow$ (1) Suppose that $Ra + Rb = R$ with $a, b \in R$. Then we have $x, y \in R$ such that $xa + yb = 1$. Define $j\colon R \oplus R \to R$ by $j(r_1, r_2) = xr_1 + yr_2$ for any $r_1, r_2 \in R$ and $i\colon R \to R \oplus R$ by $i(r) = (ar, br)$ for any $r \in R$. Then $ji = 1_R$, and so the exact sequence

$$0 \to R \xrightarrow{i} R \oplus R \xrightarrow{\sigma} \operatorname{Coker} i \to 0$$

splits, where σ is the natural R-epimorphism. Thus we have $R \oplus R \cong R \oplus \operatorname{Coker} i$. By hypothesis, we have $R \lesssim^{\oplus} \operatorname{Coker} f$. So there exists an R-epimorphism φ: $\operatorname{Coker} f \to R$, and then $\varphi\sigma\colon R \oplus R \to R$. Assume that $\varphi\sigma(1,0) = u$ and $\varphi\sigma(0,1) = -v$. Then $0 = \varphi\sigma i(1) = \varphi\sigma(a,b) = ua - vb$, i.e., $ua = vb$. Clearly, $\varphi\sigma$ is a split R-epimorphism. Hence we can find $\tau\colon R \to R \oplus R$ such that $\varphi\sigma\tau = 1_R$. Let $\tau(1) = (s,t) \in R \oplus R$. Then $1 = \varphi\sigma\tau(1) = \varphi\sigma(s,t) = us - vt$. Therefore $uR + vR = R$, as asserted. $\square$

Theorem 3.2. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R is strongly separative.*

(2) *For any corner S of R, $Sa + Sb = S$ implies that there exist $u, v \in S$ such that $ua = vb$ and $uS + vS = S$.*

P r o o f. (2) $\Rightarrow$ (1) Let S be a corner of R. Then S is an exchange ring by [1, Proposition 1.3]. For any right S-module A, it follows by Lemma 3.1 that $S \oplus S \cong S \oplus A$ implies that $S \lesssim^{\oplus} A$. According to Lemma 2.2 and Theorem 2.3, R is strongly separative.

(1) $\Rightarrow$ (2) Let S be a corner of R. Then S is an exchange ring. For any right S-module A, it follows by Theorem 2.3 and Lemma 2.2 that $S \oplus S \cong S \oplus A$ implies that $S \lesssim^{\oplus} A$. Therefore we complete the proof by Lemma 3.1. $\square$

Let $U_r(S)$ and $U_c(S)$ denote the sets of all unimodular rows and unimodular columns over S, respectively.

Corollary 3.3. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R is strongly separative.*

(2) *For any corner S of R and any $x \in U_c(S)$, there exists $y \in U_r(S)$ such that $yx = 0$.*

(3) *For any corner S of R and any $x \in U_r(S)$, there exists $y \in U_c(S)$ such that $xy = 0$.*

Proof. (1) $\Rightarrow$ (2) For any corner S of R and any $x = \begin{pmatrix} a \\ b \end{pmatrix} \in U_c(S)$, it follows by Theorem 3.2 that there exist $u, v \in S$ such that $ua = vb$ and $uS + vS = S$. Let $y = (u, -v)$. Then $y \in U_r(S)$. In addition, we have $yx = (u, -v)\begin{pmatrix} a \\ b \end{pmatrix} = 0$, as required.

(2) $\Rightarrow$ (1) For any corner S of R, $Sa + Sb = S$ implies that there exist $u, v' \in S$ such that $(u, v')\begin{pmatrix} a \\ b \end{pmatrix} = 0$ and $uS + v'S = S$. Let $u = -u'$. Then $ua = vb$ and $uS + vS = S$. In view of Theorem 3.2, R is strongly separative.

(1) $\Leftrightarrow$ (3) is proved in the same manner. □

We use R^o to denote the opposite ring of a ring R. As a consequence of Theorem 3.2, we now derive

Theorem 3.4. *An exchange ring R is strongly separative if and only if so is the opposite ring R^o.*

Proof. (1) $\Rightarrow$ (2) Let R be a strongly separative exchange ring. In view of [10, Proposition], R^o is an exchange ring. For any $S^o = e^o R^o e^o$ with idempotent $e^o \in R^o$, $S^o a^o + S^o b^o = S^o$ with $a^o, b^o \in S^o$ implies that $aS + bS = S$. Clearly, $e \in R$ is an idempotent. In view of Theorem 2.3, we can find $u, v \in R$ such that $au = bu$ and $Su + Sv = S$. Thus, $u^o a^o = v^o b^o$ and $u^o S^o + v^o S^o = S^o$. It follows by Theorem 3.2 that R^o is strongly separative.

(2) $\Rightarrow$ (1) is symmetric. □

Corollary 3.5. *An exchange ring R is strongly separative if and only if for all finitely generated projective left R-modules A and B, $A \oplus A \cong A \oplus B$ implies that $A \cong B$.*

Proof. Let R be an exchange ring. Clearly, the opposite ring R^o is strongly separative if and only if for all finitely generated projective left R-modules A and B, $A \oplus A \cong A \oplus B$ implies that $A \cong B$. Therefore we complete the proof by Theorem 3.4. □

By Theorem 3.4 and [6, Corollary 5] we prove that an exchange ring R is strongly separative if and only if for any finitely generated projective left R-module C, $2C \oplus A \cong C \oplus B$ implies that $C \oplus A \cong B$ for any left R-modules.

4. Completion of matrices

In [6, Theorem 5], the author proved that an exchange ring R is strongly separative if and only if every regular matrix over any corner of R admits a diagonal reduction. In this section we characterize strongly separative exchange rings by virtue of completion of matrices over their corners.

Theorem 4.1. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R is strongly separative.*

(2) *For any corner S of R, $aS + bS = S$ implies that there exists a right invertible matrix $\begin{pmatrix} a & b \\ * & * \end{pmatrix} \in M_2(S)$.*

Proof. (1) $\Rightarrow$ (2) Let S be a corner of R. Then S is a strongly separative exchange ring by [3, Lemma 1.5]. Suppose that $aS + bS = S$ with $a, b \in S$. Then we can find $a', b' \in S$ such that $aa' + bb' = 1_S$. Let $\alpha = (a, b)$, $\beta = \begin{pmatrix} a' \\ b' \end{pmatrix}$. Then $\alpha\beta = 1_S$. Let $\{\varepsilon\}$ and $\{\eta_1, \eta_2\}$ be bases of S and $S \oplus S$, respectively. Construct two right S-morphisms

$$\varphi\colon\ S \to S \oplus S, \quad \varphi(\varepsilon) = (\eta_1, \eta_2)\beta;$$
$$\omega\colon\ S \oplus S \to S, \quad \omega(\eta_1, \eta_2) = \varepsilon\alpha.$$

Obviously, $\omega\varphi = 1_S$. Thus, we have a split exact sequence

$$0 \longrightarrow \operatorname{Ker}\omega \longrightarrow S \oplus S \xrightarrow{\omega} S \longrightarrow 0.$$

This implies that $S \oplus S = \operatorname{Ker}\omega \oplus C$ for a right S-module C. In addition, $C \cong S$. As a result, $\operatorname{Ker}\omega \cong S$. Let $\{\delta_1\}$ and $\{\delta_2\}$ be bases of $\operatorname{Ker}\omega$ and C, respectively. Then $\{\delta_1, \delta_2\}$ and $\{\eta_1, \eta_2\}$ are both bases of $S \oplus S$ and so there exists an invertible matrix $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ such that

$$(\eta_1, \eta_2) = (\delta_1, \delta_2) \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}.$$

One easily checks that

$$\operatorname{Ker}\omega = \left\{(\eta_1,\eta_2)\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} : \varepsilon\alpha\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0,\ \ r_1, r_2 \in S\right\}$$
$$= \left\{(\delta_1,\delta_2)\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} : \varepsilon\alpha\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = 0,\ \ r_1, r_2 \in S\right\}.$$

Let

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}.$$

Then

$$\operatorname{Ker}\omega = \left\{(\delta_1,\delta_2)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} : \varepsilon\alpha\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^{-1}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0,\ \ s_1, s_2 \in S\right\}$$
$$= \left\{(\delta_1,\delta_2)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} : \alpha\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^{-1}\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} = 0,\ \ s_1, s_2 \in S\right\}.$$

Let

$$\alpha\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^{-1} = (t_1, t_2).$$

Then

$$\operatorname{Ker}\omega = \left\{(\delta_1,\delta_2)\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} : t_1s_1 + t_2s_2 = 0,\ \ s_1, s_2 \in S\right\} = \delta_1 S.$$

Clearly, $\delta_1 \in \operatorname{Ker}\omega$, and so

$$(\delta_1,\delta_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \operatorname{Ker}\omega.$$

This infers that $t_1 = 0$. As a result,

$$\alpha\beta = (0, t_2)\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\beta = t_2(c_{21}, c_{22})\beta.$$

As $\alpha\beta = 1_S$, we see that $t_2k = 1_S$ for a $k \in S$. This shows that $t_2(1_S - kt_2) = 0$, and so $\delta_2(1_S - kt_2) \in \operatorname{Ker}\omega$. Thus, we get $1_S - kt_2 = 0$, i.e., $t_2 \in S$ is invertible. Furthermore, $\alpha = (t_2c_{21}, t_2c_{22})$ is the first row of an invertible matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},$$

as required.

(2) $\Rightarrow$ (1) Let S be a corner of R. Suppose that $aS + bS = S$ with $a, b \in S$. By assumption, there exists a right invertible matrix $\begin{pmatrix} a & b \\ * & * \end{pmatrix} \in M_2(S)$. So we have $s, t \in S$ and $\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in M_2(S)$ such that

$$\begin{pmatrix} a & b \\ s & t \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}.$$

Thus $ac_{12} + bc_{22} = 0$ and $sc_{12} + tc_{22} = 1$. Let $u = c_{12}$ and $v = -c_{22}$. Then $au = bv$ and $Su + Sv = S$. In view of Theorem 2.3, we conclude that R is strongly separative. $\square$

Corollary 4.2. *Let R be an exchange ring. Then the following conditions are equivalent:*

(1) *R is strongly separative.*

(2) *For any corner S of R, $Sa + Sb = S$ implies that there exists a left invertible matrix $\begin{pmatrix} a & * \\ b & * \end{pmatrix} \in M_2(S)$.*

Proof. (1) $\Rightarrow$ (2) Let S be a corner of R. Suppose that $Sa + Sb = S$ with $a, b \in S$. Then S^o is a corner of R^o and $a^oS^o + b^oS^o = S^o$. Since R is strongly separative, so is S^o from Theorem 3.4. According to Theorem 4.1, we can find a right invertible matrix $\begin{pmatrix} a^o & b^o \\ *^o & *^o \end{pmatrix} \in M_2(S^o)$. This means that $\begin{pmatrix} a & * \\ b & * \end{pmatrix} \in M_2(S)$ is left invertible, as required.

(2) $\Rightarrow$ (1) Let S^o be a corner of R^o. Suppose that $a^oS^o + b^oS^o = S^o$ with $a^o, b^o \in S^o$. Then $Sa + Sb = S$. By assumption, there exists a left invertible matrix $\begin{pmatrix} a & * \\ b & * \end{pmatrix} \in M_2(S)$. This means that $\begin{pmatrix} a^o & b^o \\ *^o & *^o \end{pmatrix} \in M_2(S^o)$ is right invertible. By virtue of Theorem 4.1, R^o is strongly separative. It follows by Theorem 3.4 that R is strongly separative. $\square$

Analogously, we prove that an exchange ring R is strongly separative if and only if for any corner S of R, $aS + bS = S$ implies that there exists an invertible matrix $\begin{pmatrix} a & b \\ * & * \end{pmatrix} \in M_2(S)$ if and only if for any corner S of R, $Sa + Sb = S$ implies that there exists an invertible matrix $\begin{pmatrix} a & * \\ b & * \end{pmatrix} \in M_2(S)$.

A ring R is regular provided that for every $a \in R$ there exists $x \in R$ such that $a = axa$. Clearly, every regular ring is an exchange ring. In this section we characterize strongly separative regular rings in terms of annihilators.

Lemma 5.1. *Let R be a regular ring. Then the following conditions are equivalent:*

(1) *$aR + bR = R$ implies that there exist $u, v \in R$ such that $au = bv$ and $r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v) = 0$.*

(2) *For any right R-module A, $R \oplus R \cong R \oplus A$ implies that $R \lesssim A$.*

Proof. (1) $\Rightarrow$ (2) Given $R \oplus R \cong R \oplus A$, then we have a split exact sequence

$$0 \to A \xrightarrow{i} R \oplus R \xrightarrow{f} R \to 0.$$

So there is a right R-morphism $g\colon R \to R \oplus R$ such that $fg = 1_R$. Assume that $f(1,0) = a$, $f(0,1) = b$ and $g(1) = (x,y)$. Then $aR + bR = R$. By assumption, there exist $u, v \in R$ such that $au = bv$ and $r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v) = 0$. Construct a map $\varphi\colon R \to R \oplus R$ given by $\varphi(r) = (ur, -vr)$ for any $r \in R$. For any $r \in R$, $f\varphi(r) = f(ur, -vr) = aur - bvr = 0$. Hence $f\varphi = 0$. As a result, there exists a right R-morphism $\omega\colon R \to A$ such that $i\omega = \varphi$. If $\omega(r) = 0$, then $\varphi(r) = (0,0)$. Hence $(ur, -vr) = (0,0)$, and so $ur = -vr = 0$. This implies that $r \in r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v) = 0$. Thus $\omega\colon R \to A$ is an R-monomorphism, and so $R \lesssim A$.

(2) $\Rightarrow$ (1) Suppose that $aR + bR = R$ with $a, b \in R$. Then we have $x, y \in R$ such that $ax + by = 1$. Define $f\colon R \oplus R \to R$ by $f(r_1, r_2) = ar_1 + br_2$ for any $(r_1, r_2) \in R \oplus R$ and $g\colon R \to R \oplus R$ by $g(r) = (xr, yr)$ for any $r \in R$. Analogously to Lemma 2.2, we get $R \oplus R \cong R \oplus \operatorname{Ker} f$. By hypothesis, we have $R \lesssim \operatorname{Ker} f$. So there exists an R-monomorphism $\varphi\colon R \to \operatorname{Ker} f$, and so $i\varphi\colon R \to R \oplus R$. Assume that $i\varphi(1) = (u, v') \in R \oplus R$. Then $0 = fi\varphi(1) = f(u, v') = au + bv'$. Let $v = -v'$. Then $au = bv$. Given any $z \in r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v)$, we have $uz = -v'z = 0$. As a result, $i\varphi(z) = i\varphi(1)z = (u, v')z = (0,0)$. Since both i and φ are R-monomorphisms, we deduce that $z = 0$; hence, $r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v) = 0$. The proof is completed. □

Theorem 5.2. *A regular ring R is strongly separative if and only if for any corner S of R, $aS + bS = S$ implies that there exist $u, v \in S$ such that $au = bv$ and $r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v) = 0$.*

Proof. Let S be a corner of R. Suppose that $aS + bS = S$ with $a, b \in S$. In view of Theorem 2.3, there exist $u', v' \in S$ such that $au' = bv'$ and $Su' + Sv' = S$. For any right S-module A, it follows by Lemma 2.2 that $S \oplus S \cong S \oplus A$ implies that $S \lesssim^{\oplus} A$. Therefore there exist $u, v \in S$ such that $au = bv$ and $r \cdot \mathrm{ann}(u) \cap r \cdot \mathrm{ann}(v) = 0$ from Lemma 5.1. Conversely, we prove that R is strongly separative by Lemma 5.1, Lemma 2.2 and Theorem 2.3. □

Corollary 5.3. *A regular ring R is strongly separative if and only if for any corner S of R, $Sa + Sb = S$ implies that there exist $u, v \in S$ such that $ua = vb$ and $l \cdot \mathrm{ann}(u) \cap l \cdot \mathrm{ann}(v) = 0$.*

P r o o f. Let S be a corner of R. Assume that $Sa + Sb = S$ with $a, b \in S$. Then $a^o S^o + b^o S^o = S^o$. In view of Theorem 3.4 and Theorem 2.3, we can find $u, v \in S$ such that $a^o u^o = b^o v^o$ and $r \cdot \mathrm{ann}(u^o) \cap r \cdot \mathrm{ann}(v^o) = 0$. As a result, $ua = vb$ and $l \cdot \mathrm{ann}(u) \cap l \cdot \mathrm{ann}(v) = 0$.

Conversely, assume that $a^o S^o + b^o S^o = S^o$ with $a^o, b^o \in S^o$. Then $Sa + Sb = S$. By assumption, there exist $u, v \in S$ such that $ua = vb$ and $l \cdot \mathrm{ann}(u) \cap l \cdot \mathrm{ann}(v) = 0$. Thus, $a^o u^o = b^o v^o$ and $r \cdot \mathrm{ann}(u^o) \cap r \cdot \mathrm{ann}(v^o) = 0$. In view of Theorem 2.3, R^o is strongly separative. Therefore we complete the proof by Theorem 3.4. □

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