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# CLASSIFICATION OF 4-DIMENSIONAL HOMOGENEOUS D'ATRI SPACES

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*Abstract.* The property of being a D'Atri space (i.e., a space with volume-preserving symmetries) is equivalent to the infinite number of curvature identities called the odd Ledger conditions. In particular, a Riemannian manifold $(M, g)$ satisfying the first odd Ledger condition is said to be of type $\mathcal{A}$. The classification of all 3-dimensional D'Atri spaces is well-known. All of them are locally naturally reductive. The first attempts to classify all 4-dimensional homogeneous D'Atri spaces were done in the papers by Podesta-Spiro and Bueken-Vanhecke (which are mutually complementary). The authors started with the corresponding classification of all spaces of type $\mathcal{A}$, but this classification was incomplete. Here we present the complete classification of all homogeneous spaces of type $\mathcal{A}$ in a simple and explicit form and, as a consequence, we prove correctly that all homogeneous 4-dimensional D'Atri spaces are locally naturally reductive.

*Keywords*: Riemannian manifold, naturally reductive Riemannian homogeneous space, D'Atri space

*MSC 2000*: 53C21, 53B21, 53C25, 53C30

## 1. Introduction and preliminaries

A D'Atri space is defined as a Riemannian manifold $(M, g)$ whose local geodesic symmetries are volume-preserving. D'Atri and Nickerson (see [6]) proved that every naturally reductive Riemannian manifold has this property. See [10] for a survey of the whole topic. The second author in [9] classified all 3-dimensional D'Atri spaces by showing that they are all locally isometric to naturally reductive homogeneous spaces

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(including the trivial cases of locally symmetric spaces). Hence all these spaces are *locally homogeneous.* A similar result is not known in dimension 4. Then a natural problem occurs to classify all four-dimensional homogeneous D'Atri spaces. The first attempts in this direction were made in two subsequent papers [13] and [4]. As we shall see later, the final classification announced in [4] is correct but not well-founded and its proof needs to be completed. This is the main purpose of the present paper.

Let us recall that the property of being a D'Atri space is equivalent to the infinite number of curvature identities called the *odd Ledger conditions* $L_{2k+1}$, $k \geqslant 1$ (see [5] and [15]). In particular, the first two non-trivial Ledger conditions are

$$L_3\colon\ (\nabla_X \varrho)(X,X) = 0 \quad \text{and} \quad L_5\colon \sum_{a,b=1}^{n} \mathcal{R}_{XE_aXE_b}(\nabla_X \mathcal{R})_{XE_aXE_b} = 0$$

where $X$ is any tangent vector at any point $m \in M$ and $\{E_1, \dots, E_n\}$ is any orthonormal basis of $T_mM$. Here $\mathcal{R}$ denotes the curvature tensor and $\varrho$ the Ricci tensor of $(M, g)$, and $n = \dim M$.

Thus, it is natural to start with the investigation of all homogeneous Riemannian 4-manifolds satisfying the simplest Ledger condition $L_3$, which is the first approximation of the D'Atri property. This condition is called in [13] the "class $\mathcal{A}$ condition". More explicitly, we have

**Definition 1.** A Riemannian manifold $M$ is said to belong to class $\mathcal{A}$, or to be of type $\mathcal{A}$, if its Ricci curvature tensor $\varrho$ is cyclic-parallel that is, if $(\nabla_X \varrho)(X,X) = 0$ for every vector field $X$ tangent to $M$ or, equivalently, if

$$(\nabla_X \varrho)(Y,Z) + (\nabla_Y \varrho)(Z,X) + (\nabla_Z \varrho)(X,Y) = 0$$

for all vector fields $X$, $Y$, $Z$ tangent to $M$.

In dimension three, H. Pedersen and P. Tod ([12]) proved the following result:

**Theorem 1.** *All three-dimensional smooth Riemannian manifolds belonging to class $\mathcal{A}$ are locally homogeneous, and they are either locally symmetric or locally isometric to naturally reductive spaces.*

(Note that earlier, in [9], both Ledger conditions $L_3$, $L_5$ and the real analyticity condition were used for the proof of the conclusion of Theorem 1.)

Now, let us recall the concept of a curvature homogeneous space. A smooth Riemannian manifold $M$ is called *curvature homogeneous* if for any two points $p, q \in M$ there exists a linear isometry $F\colon\ T_pM \to T_qM$ such that $F^*\mathcal{R}_q = \mathcal{R}_p$. This is also equivalent to saying that, locally, there always exists a smooth field of orthonormal frames with respect to which the components of the curvature tensor $\mathcal{R}$ are constant functions (see, for instance, I. M. Singer [14], or the monograph [3]). Hence it is obvious that *all principal Ricci curvatures are constant.* Clearly, any homogeneous manifold is curvature homogeneous. On the other hand, the locally homogeneous Riemannian manifolds in dimensions $\geqslant 3$ form a "negligible" subclass of all curvature homogeneous spaces (see a survey in [3]).

As the first and most extensive step of our classification procedure, we shall look for all 4-dimensional homogeneous spaces of class $\mathcal{A}$. In this direction, F. Podesta and A. Spiro ([13]) published a classification theorem assuming that at most three of the constant Ricci eigenvalues are distinct. In their paper, $(M, g)$ was not necessarily homogeneous but only curvature homogeneous, which is a more general situation. Yet, there was a gap in their main theorem, which we will explain later. (See Appendix.) They also put the question if there are, in dimension four, spaces of class $\mathcal{A}$ with four distinct Ricci eigenvalues. Some years later, P. Bueken and L. Vanhecke ([4]) found a two-parameter family of such spaces. However, their presentation of this family was not explicit and lacked geometrical interpretation (they referred only to computer results, which were not accessible). They also concluded in [4] that all simply connected homogeneous D'Atri spaces in dimension 4 are naturally reductive. But this final result was not completely satisfactory either just because of the gap in [13], and because the new family of spaces in [4] was not described explicitly.

In the present paper, we derive the correct and complete local classification of all 4-dimensional homogeneous spaces of type $\mathcal{A}$ in a simple and explicit form. Our method is based on the classification of Riemannian homogeneous 4-spaces by L. Bérard Bergery ([2]) and on computer support using the program *Mathematica 5.0.*

We shall now formulate our basic result, which will be proved in the next section.

**Classification Theorem.** *Let $(M, g)$ be a four-dimensional homogeneous Riemannian manifold of type $\mathcal{A}$. Then one of the following five cases occurs:*

i) *$M$ is locally symmetric;*

ii) *$(M, g)$ is locally isometric to a Riemannian product $M^3 \times \mathbb{R}$, where $M^3$ is a 3-dimensional Riemannian naturally reductive space with two distinct Ricci curvatures $(\varrho_1, \varrho_2 = \varrho_1, \varrho_3)$, $\varrho_3 \neq \varrho_1$. Thus $M$ is locally isometric to a naturally reductive homogeneous space.*

iii) *$(M, g)$ is locally isometric to a simply connected Lie group $(G, g_\gamma)$ whose Lie algebra $\mathfrak{g}$ is described by*

$$[e_2, e_1] = e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = e_4,$$
$$[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0,$$

*and which is endowed with the left-invariant metric*

$$g_\gamma = \frac{4}{\gamma^2} w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + \gamma^2 w^4 \otimes w^4,$$

*where* $\gamma \in \mathbb{R}^+$ *and* $\{w^i\}$ *is the dual basis of* $\{e_i\}$. *The metrics* $g_\gamma$ *have Ricci eigenvalues* $\varrho_1 = \varrho_2 = \varrho_3 = -\frac{1}{2}\gamma^2$, $\varrho_4 = \frac{1}{2}\gamma^2$ *and are not isometric to one another for different values of* $\gamma$. *Moreover, the Riemannian manifolds* $(G, g_\gamma)$ *are irreducible and not locally symmetric. They are not D'Atri spaces.*

iv) $(M, g)$ *is locally isometric to a simply connected Lie group* $(G, g_{(c,k)})$ *whose Lie algebra* $\mathfrak{g}$ *is described by*

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = \frac{A_+}{4} e_2, \quad [e_2, e_3] = \frac{A_-}{4} e_1,$$
$$[e_1, e_4] = 0, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = 0,$$

*where* $A_\pm = 3 - 3k^2 \pm \sqrt{1 + 2k^2 - 3k^4} \geqslant 0$, $k \in \,]0, 1[\, \setminus \left\{\sqrt{\frac{5}{21}}\right\}$, *and which is endowed with the left-invariant metric*

$$g_{(c,k)} = \frac{1}{c^2}(w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + k w^3 \otimes w^4 + w^4 \otimes w^4),$$

*where* $\{w^i\}$ *is the dual basis of* $\{e_i\}$ *and* $c \in \mathbb{R}^+$ *is another parameter. The metrics* $g_{(c,k)}$ *have four distinct Ricci eigenvalues*

$$\varrho_1 = \frac{c^2}{8}\big(2 - 6k^2 - \sqrt{1 + 2k^2 - 3k^4}\big), \qquad \varrho_2 = \frac{c^2}{8}\big(2 - 6k^2 + \sqrt{1 + 2k^2 - 3k^4}\big),$$
$$\varrho_3 = \frac{c^2}{16}\big(3 - 3k^2 - \sqrt{9 - 2k^2 + 57k^4}\big), \quad \varrho_4 = \frac{c^2}{16}\big(3 - 3k^2 + \sqrt{9 - 2k^2 + 57k^4}\big).$$

*Moreover, the Riemannian manifolds* $(G, g_{(c,k)})$ *are irreducible and not locally symmetric. They are not D'Atri spaces.*

v) $(M, g)$ *is locally isometric to a simply connected Lie group* $(G, g_c)$, *whose Lie algebra* $\mathfrak{g}$ *is described by*

$$[e_1, e_2] = e_3, \quad [e_3, e_1] = \frac{6}{7} e_2, \quad [e_2, e_3] = \frac{2}{7} e_1,$$
$$[e_1, e_4] = 0, \quad [e_2, e_4] = 0, \quad [e_3, e_4] = 0,$$

*and which is endowed with the left-invariant metric*

$$g_c = \frac{1}{c^2}\left(w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + \sqrt{\frac{5}{21}} w^3 \otimes w^4 + w^4 \otimes w^4\right),$$

*where $c \in \mathbb{R}^+$ and $\{w^i\}$ is the dual basis of $\{e_i\}$. The metrics $g_c$ have Ricci eigenvalues $\varrho_1 = \varrho_3 = -\frac{1}{14}c^2$, $\varrho_2 = \frac{3}{14}c^2$, $\varrho_4 = \frac{5}{14}c^2$ and are not isometric to one another for different values of $c$. Moreover, the Riemannian manifolds $(G, g_c)$ are irreducible and not locally symmetric. They are not D'Atri spaces.*

It is well-known that every locally symmetric space is a D'Atri space and that, moreover, it is locally isometric to a naturally reductive homogeneous space. In addition, the Riemannian product spaces $M^3 \times \mathbb{R}$ described in ii) of the Classification Theorem are locally isometric to naturally reductive homogeneous spaces and hence they are D'Atri spaces. On the other hand, we will show that the spaces described in iii), iv) and v) do not satisfy the Ledger condition $L_5$ and thus they cannot be D'Atri spaces. Combining these results with our Classification Theorem, we conclude with

**Main Theorem.** *In dimension* 4, *all simply connected homogeneous D'Atri spaces are naturally reductive spaces (including symmetric spaces as special cases).*

## 2. Proof of the classification theorem

In [2], L. Bérard Bergery published the classification of Riemannian homogeneous 4-spaces. In particular, he obtained

**Proposition 1.** *In dimension* 4, *each simply connected Riemannian homogeneous space $M$ is either symmetric or isometric to a Lie group with a left-invariant metric. In the second case, either $M$ is a solvable group or it is one of the groups* $\mathrm{SU}(2) \times \mathbb{R}$, $\widetilde{\mathrm{Sl}(2, \mathbb{R})} \times \mathbb{R}$.

Now, the main part of our computations is to check which of these spaces are of type $\mathcal{A}$. We shall work at the Lie algebra level and use *Mathematica 5.0* for the computation. Let us start with the non-solvable group case and later we will continue with the solvable case.

### 2.1. Non-solvable case (study of $\mathrm{SU}(2) \times \mathbb{R}$ and $\widetilde{\mathrm{Sl}(2, \mathbb{R})} \times \mathbb{R}$)

Let $\mathfrak{g}_3$ be a unimodular Lie algebra with a scalar product $\langle\, ,\rangle_3$. According to [11, p. 305], there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{g}_3$ such that

$$[f_2, f_3] = af_1, \quad [f_3, f_1] = bf_2, \quad [f_1, f_2] = cf_3, \tag{1}$$

where $a$, $b$, $c$ are real numbers. In the following, we shall study the cases $\mathfrak{g}_3 = \mathfrak{su}(2)$ and $\mathfrak{g}_3 = \mathfrak{sl}(2, \mathbb{R})$ which are characterized by the inequality $abc \neq 0$.

Let now $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}$ be a direct sum, and $\langle\,,\rangle$ a scalar product on $\mathfrak{g}$ defined as follows: we choose a basis $\{f_1, f_2, f_3, f_4\}$ of unit vectors such that $\{f_1, f_2, f_3\}$ is an orthonormal basis of $\mathfrak{g}_3$ satisfying (1) and $f_4$ spans $\mathbb{R}$. Here $\mathbb{R}$ need not be orthogonal to $\mathfrak{g}_3$. In particular, we assume

$$[f_i, f_4] = 0, \quad \langle f_i, f_4\rangle = k_i, \quad i = 1, 2, 3. \tag{2}$$

Here $a$, $b$, $c$, $k_1$, $k_2$, $k_3$ are arbitrary parameters with $\sum\limits_{i=1}^{3} k_i^2 < 1$ due to the positivity of the scalar product. Choosing a convenient orientation of $f_4$, we can always assume that $k_3 \geqslant 0$.

Now we replace the basis $\{f_i\}$ by a new basis $\{e_i\}$ $(i = 1, 2, 3, 4)$, putting

$$e_i = f_i, \ \ i = 1, 2, 3, \quad e_4 = \frac{1}{R}\left(f_4 - \sum_{i=1}^{3} k_i f_i\right) \tag{3}$$

where $R = \sqrt{1 - \sum\limits_{i=1}^{3} k_i^2} > 0$. Then we get an orthonormal basis for which

$$\begin{gathered}
[e_2, e_3] = ae_1, \quad [e_3, e_1] = be_2, \quad [e_1, e_2] = ce_3,\\
[e_1, e_4] = \frac{1}{R}(k_3 b e_2 - k_2 c e_3), \quad [e_2, e_4] = \frac{1}{R}(k_1 c e_3 - k_3 a e_1),\\
[e_3, e_4] = \frac{1}{R}(k_2 a e_1 - k_1 b e_2).
\end{gathered} \tag{4}$$

Next, we consider the simply connected Lie group $G$ with a left invariant Riemannian metric $g$ corresponding to the Lie algebra $\mathfrak{g}$ and the scalar product $\langle\,,\rangle$ on it. Here the vectors $e_i$ determine some left-invariant vector fields on $G$.

According to our construction, the underlying group $G$ is the direct product of the group $\mathrm{SU}(2)$ or $\widetilde{\mathrm{Sl}(2, \mathbb{R})}$ and the multiplicative group $\mathbb{R}^+$.

Now we are going to calculate the expression for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

We know that

$$\begin{aligned}
2g(\nabla_X Z, Y) &= Zg(X, Y) + Xg(Y, Z) - Yg(Z, X)\\
&\qquad - g([Z, X], Y) - g([X, Y], Z) + g([Y, Z], X)
\end{aligned} \tag{5}$$

for every triplet $(X, Y, Z)$ of vectors fields. Then using this formula we obtain by easy calculation

**Lemma 1.**

$$\nabla_{e_i} e_i = 0, \quad i = 1,2,3,4, \tag{6}$$

$$\begin{aligned}
\nabla_{e_1} e_2 &= \frac{(c+b-a)}{2} e_3 + \frac{(a-b)k_3}{2R} e_4, & \nabla_{e_2} e_1 &= \frac{(b-a-c)}{2} e_3 + \frac{(a-b)k_3}{2R} e_4,\\
\nabla_{e_1} e_3 &= \frac{(a-b-c)}{2} e_2 + \frac{(c-a)k_2}{2R} e_4, & \nabla_{e_3} e_1 &= \frac{(a+b-c)}{2} e_2 + \frac{(c-a)k_2}{2R} e_4,\\
\nabla_{e_1} e_4 &= \frac{(b-a)k_3}{2R} e_2 + \frac{(a-c)k_2}{2R} e_3, & \nabla_{e_4} e_1 &= \frac{-(b+a)k_3}{2R} e_2 + \frac{(a+c)k_2}{2R} e_3,\\
\nabla_{e_2} e_3 &= \frac{(a+c-b)}{2} e_1 + \frac{(b-c)k_1}{2R} e_4, & \nabla_{e_3} e_2 &= \frac{(c-a-b)}{2} e_1 + \frac{(b-c)k_1}{2R} e_4,\\
\nabla_{e_2} e_4 &= \frac{(b-a)k_3}{2R} e_1 + \frac{(c-b)k_1}{2R} e_3, & \nabla_{e_4} e_2 &= \frac{(a+b)k_3}{2R} e_1 - \frac{(b+c)k_1}{2R} e_3,\\
\nabla_{e_3} e_4 &= \frac{(a-c)k_2}{2R} e_1 + \frac{(c-b)k_1}{2R} e_2, & \nabla_{e_4} e_3 &= \frac{-(a+c)k_2}{2R} e_1 + \frac{(b+c)k_1}{2R} e_2.
\end{aligned}$$

Now, we denote by $A_{ij}$ the elementary skew-symmetric operators whose corresponding action is given by the formulas $A_{ij}(e_l) = \delta_{il} e_j - \delta_{jl} e_i$. Then, by a lengthy but elementary calculation we get

**Lemma 2.** *The components of the curvature operator are*

$$\begin{aligned}
\mathcal{R}(e_1,e_2) &= \alpha_{1212}A_{12} + \alpha_{1213}A_{13} + \alpha_{1214}A_{14} + \alpha_{1223}A_{23} + \alpha_{1224}A_{24},\\
\mathcal{R}(e_1,e_3) &= \alpha_{1312}A_{12} + \alpha_{1313}A_{13} + \alpha_{1314}A_{14} + \alpha_{1323}A_{23} + \alpha_{1334}A_{34},\\
\mathcal{R}(e_1,e_4) &= \alpha_{1412}A_{12} + \alpha_{1413}A_{13} + \alpha_{1414}A_{14} + \alpha_{1424}A_{24} + \alpha_{1434}A_{34},\\
\mathcal{R}(e_2,e_3) &= \alpha_{2312}A_{12} + \alpha_{2313}A_{13} + \alpha_{2323}A_{23} + \alpha_{2324}A_{24} + \alpha_{2334}A_{34},\\
\mathcal{R}(e_2,e_4) &= \alpha_{2412}A_{12} + \alpha_{2414}A_{14} + \alpha_{2423}A_{23} + \alpha_{2424}A_{24} + \alpha_{2434}A_{34},\\
\mathcal{R}(e_3,e_4) &= \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned} \tag{7}$$

*where the coeficients $\alpha_{ijlm} = g(\mathcal{R}(e_i,e_j)e_l,e_m)$ satisfy the standard symmetries with respect to their indices and*

$$\begin{aligned}
\alpha_{1212} &= \frac{1}{4R^2}((3c^2 - (a-b)^2 - 2c(a+b))R^2 - (a-b)^2k_3^2),\\
\alpha_{1213} &= \frac{1}{4R^2}((a-b)(a-c)k_2k_3),\\
\alpha_{1214} &= \frac{1}{4R}((a-c)(a-b+3c)k_2),\\
\alpha_{1223} &= \frac{1}{4R^2}((a-b)(b-c)k_1k_3),\\
\alpha_{1224} &= \frac{1}{4R}((b-c)(a-b-3c)k_1),
\end{aligned} \tag{8}$$

$$\alpha_{1313} = \frac{1}{4R^2}((3b^2 - (a-c)^2 - 2b(a+c))R^2 - (a-c)^2k_2^2),$$
$$\alpha_{1314} = \frac{1}{4R}((a-b)(a-c+3b)k_3),$$
$$\alpha_{1323} = \frac{1}{4R^2}((a-c)(b-c)k_1k_2),$$
$$\alpha_{1334} = \frac{1}{4R}((c-b)(c-a+3b)k_1),$$
$$\alpha_{1414} = \frac{1}{4R^2}((4c^2 - (a+c)^2)k_2^2 + (4b^2 - (a+b)^2)k_3^2),$$
$$\alpha_{1424} = \frac{1}{4R^2}((c(a+b-3c)+ab)k_1k_2),$$
$$\alpha_{1434} = \frac{1}{4R^2}((b(a+c-3b)+ac)k_1k_3),$$
$$\alpha_{2323} = \frac{1}{4R^2}((3a^2 - (b-c)^2 - 2a(b+c))R^2 - (b-c)^2k_1^2),$$
$$\alpha_{2324} = \frac{1}{4R}((b-a)(3a+b-c)k_3),$$
$$\alpha_{2334} = \frac{1}{4R}((a-c)(3a-b+c)k_2),$$
$$\alpha_{2424} = \frac{1}{4R^2}((4c^2 - (b+c)^2)k_1^2 + (4a^2 - (a+b)^2)k_3^2),$$
$$\alpha_{2434} = \frac{1}{4R^2}((a(-3a+b+c)+bc)k_2k_3),$$
$$\alpha_{3434} = \frac{1}{4R^2}((4b^2 - (b+c)^2)k_1^2 + (4a^2 - (a+c)^2)k_2^2).$$

Further, we obtain easily

**Lemma 3.** *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} \beta_{11} & \frac{(c^2-ab)k_1k_2}{2R^2} & \frac{(b^2-ac)k_1k_3}{2R^2} & \frac{(b-c)^2k_1}{2R} \\ \frac{(c^2-ab)k_1k_2}{2R^2} & \beta_{22} & \frac{(a^2-bc)k_2k_3}{2R^2} & \frac{(a-c)^2k_2}{2R} \\ \frac{(b^2-ac)k_1k_3}{2R^2} & \frac{(a^2-bc)k_2k_3}{2R^2} & \beta_{33} & \frac{(a-b)^2k_3}{2R} \\ \frac{(b-c)^2k_1}{2R} & \frac{(a-c)^2k_2}{2R} & \frac{(a-b)^2k_3}{2R} & \beta_{44} \end{pmatrix} \tag{9}$$

*where*

$$\beta_{11} = \frac{a^2-(b-c)^2}{2} + \frac{(a^2-b^2)k_3^2 + (a^2-c^2)k_2^2}{2R^2},$$
$$\beta_{22} = \frac{b^2-(a-c)^2}{2} + \frac{(b^2-a^2)k_3^2 + (b^2-c^2)k_1^2}{2R^2},$$

$$
\beta_{33} = \frac{c^2-(a-b)^2}{2} + \frac{(c^2-a^2)k_2^2+(c^2-b^2)k_1^2}{2R^2},
$$
$$
\beta_{44} = \frac{-(b-c)^2k_1^2-(a-c)^2k_2^2-(a-b)^2k_3^2}{2R^2}.
$$

Next, the condition for the metric $g$ on $G$ to be cyclic parallel (i.e., of type $\mathcal{A}$) is

$$
(\nabla_X\varrho)(Y,Z)+(\nabla_Y\varrho)(Z,X)+(\nabla_Z\varrho)(X,Y)=0 \tag{10}
$$

for every triplet $(X,Y,Z)$ of vectors fields, where $\varrho$ is the Ricci tensor of type $(0,2)$. This equation has a purely algebraic character because the metric $g$ is left-invariant. Hence, we can substitute for $X,Y,Z$ every triplet chosen from the basis $\{e_1,e_2,e_3,e_4\}$ (with possible repetition).

We obtain, by a lengthy but routine calculation

**Lemma 4.** *The condition* (10) *for the Ricci tensor of type* $(0,2)$ *is equivalent to the system of algebraic equations*

$$
\begin{aligned}
(11)\quad &(1,1,2)\to k_1k_3(b-a)(a-2b+c)=0,\\
&(1,1,3)\to k_1k_2(a-c)(a+b-2c)=0,\\
&(2,2,1)\to k_2k_3(a-b)(2a-b-c)=0,\\
&(2,2,3)\to k_1k_2(c-b)(a+b-2c)=0,\\
&(3,3,1)\to k_2k_3(c-a)(2a-b-c)=0,\\
&(3,3,2)\to k_1k_3(b-c)(2b-a-c)=0,\\
&(4,4,1)\to k_2k_3(2a-b-c)(b-c)=0,\\
&(4,4,2)\to k_1k_3(a-2b+c)(a-c)=0,\\
&(4,4,3)\to k_1k_2(a+b-2c)(b-a)=0,\\
&(1,2,3)\to 2R^2(a-b)(a-c)(b-c)+k_1^2(c-b)(a(b+c)-2bc)\\
&\qquad\qquad +k_2^2(a-c)(b(a+c)-2ac)+k_3^2(b-a)(c(a+b)-2ab)=0,\\
&(1,2,4)\to k_3(R^2(b-a)(2a-c)(2b-c)+k_1^2b(2ab-3ac+2bc-c^2)\\
&\qquad\qquad +k_2^2a(3bc+c^2-2ab-2ac)+k_3^24ab(b-a))=0,\\
&(1,3,4)\to k_2(R^2(c-a)(2a-b)(b-2c)+k_1^2c(3ab-2ac-2bc+b^2)\\
&\qquad\qquad +k_2^24ac(a-c)+k_3^2a(2ab-b^2+2ac-3bc))=0,\\
&(2,3,4)\to k_1(R^2(b-c)(2b-a)(a-2c)+k_1^24bc(c-b)\\
&\qquad\qquad +k_2^2c(2ac+2bc-a^2-3ab)+k_3^2b(a^2-2ab+3ac-2bc))=0,
\end{aligned}
$$

$$
\begin{aligned}
&(1,1,4) \to k_1k_2k_3(a+b+c)(c-b)=0,\\
&(2,2,4) \to k_1k_2k_3(a+b+c)(a-c)=0,\\
&(3,3,4) \to k_1k_2k_3(a+b+c)(b-a)=0.
\end{aligned}
$$

*Here the symbol "$(\alpha,\beta,\gamma)\to$" indicates the substitution of $(e_\alpha,e_\beta,e_\gamma)$ for $(X,Y,Z)$ respectively.*

Now, the goal is to find the values of $a$, $b$, $c$, $k_1$, $k_2$ and $k_3$ which satisfy the system of equations (11) and to study each of these cases.

**Proposition 2.** *The only possible solutions of the system of algebraic equations* (11) *are, up to a re-numeration of the triplet $\{e_1,e_2,e_3\}$, the following ones:*

1. $a=b=c\neq 0$, $k_1$, $k_2$, $k_3$ *arbitrary.*
   *Here three of the four Ricci eigenvalues are equal and $\nabla\mathcal{R}=0$. Hence, the corresponding spaces belong to the case* i) *of the Classification Theorem.*
2. $a=b\neq 0$, $a\neq c\neq 0$, $k_1=k_2=0$, $k_3$ *arbitrary.*
   *In this situation, the corresponding spaces are Riemannian direct products $M^3\times\mathbb{R}$, not locally symmetric, with the Ricci eigenvalues $\varrho_1=\varrho_2=\frac{1}{2}(2ac-c^2)$, $\varrho_3=\frac{1}{2}c^2$, $\varrho_4=0$. Hence, they give the case* ii) *of the Classification Theorem.*
3. $a=\frac{1}{4}cA_-$, $b=\frac{1}{4}cA_+$, $c\neq 0$, $k_1=k_2=0$, $k_3^2\in\,]0,1[\setminus\{\frac{2}{3},\frac{5}{21}\}$, *and* $A_\pm=3-3k_3^2\pm\sqrt{1+2k_3^2-3k_3^4}>0$.
   *For this situation, $(\nabla_{e_4}\mathcal{R})(e_4,e_2)e_4\neq 0$ and all Ricci eigenvalues are distinct. The corresponding spaces belong to the case* iv) *of the Classification Theorem. Moreover, the $L_5$ condition is not satisfied.*
4. $a=\frac{2c}{7}$, $b=\frac{6c}{7}$, $c\neq 0$, $k_1=k_2=0$, $k_3=\sqrt{\frac{5}{21}}$.
   *The corresponding spaces give the case* v) *of the Classification Theorem. Moreover, the $L_5$ condition is not satisfied.*

P r o o f. Because we can re-numerate the basis $\{e_1,e_2,e_3\}$ in an arbitrary way (which implies the corresponding permutation of the symbols $a$, $b$, $c$ and the corresponding re-numeration of the parameters $k_1$, $k_2$, $k_3$), the system (11) is *symmetric* with respect to all such permutations and re-numerations. Then, in order to solve this system of equations, we can just consider the following cases:

A. $k_1k_2k_3\neq 0$,
B. $k_1=k_2=0$, $k_3$ arbitrary,
C. $k_1=0$ and $k_2k_3\neq 0$.

*Case A.* $k_1k_2k_3\neq 0$.

We first divide the formulas $(1,1,2)$ and $(2,2,1)$ by their nonzero coefficients $k_1k_3$, $k_2k_3$ and then subtract them. We obtain the necessary condition $b-a=0$. Because the system (11) is symmetric with respect to all permutations, we get also $b - c = 0$. Hence the only possible solution under the condition $k_1 k_2 k_3 \neq 0$ is $a = b = c \neq 0$.

Now, we can extend this solution also to the case of *arbitrary* $k_1$, $k_2$, $k_3$. The system (11) is still satisfied and we obtain the case 1 of Proposition 2.

In particular, in this case we have the Ricci eigenvalues $\varrho_1 = \varrho_2 = \varrho_3 = \frac{1}{2}a^2$, $\varrho_4 = 0$ and the curvature tensor (7) takes on the form

$$\mathcal{R}(e_1, e_2) = -\frac{1}{4}a^2 A_{12}, \quad \mathcal{R}(e_1, e_3) = -\frac{1}{4}a^2 A_{13}, \quad \mathcal{R}(e_2, e_3) = -\frac{1}{4}a^2 A_{23},$$
$$\mathcal{R}(e_1, e_4) = \mathcal{R}(e_2, e_4) = \mathcal{R}(e_3, e_4) = 0.$$

Moreover, from (6) we get $\nabla_{e_i} e_4 = 0$ for $i = 1, \ldots, 4$ and $e_4$ is a (globally) parallel vector field.

Now, the following lemma is an immediate consequence of the well-known Ambrose-Singer Theorem.

**Lemma 5.** *On a Riemannian manifold $(M, g)$, the Lie algebra $\psi(x)$ of the holonomy group $\Psi(x)$ with the reference point $x \in M$ ("the holonomy algebra") contains the Lie algebra generated by all curvature operators $R(X, Y)$, where $X, Y \in T_x M$.*

Using this lemma we see that the holonomy algebra $\psi(e)$ contains span($A_{12}$, $A_{13}, A_{23}$). On the other hand, the holonomy group $\Psi(e)$ acts trivially on span($e_4$). By the de Rham Decomposition Theorem (see Sections 5, 6 of Chapter IV in [8]), the corresponding Riemannian manifolds are (locally) direct products of a 3-dimensional Lie group and a real line. They are locally symmetric because the 3-dimensional factor is a space of constant curvature.

In conclusion, the corresponding spaces belong to the case i) of our Classification Theorem.

*Case B.* $k_1 = k_2 = 0$, $k_3$ arbitrary.

In this case we have the following system of independent equations:

$$\begin{aligned}
&(1,2,3) \to (a-b)(2(a-c)(b-c) + c(a+b-2c)k^2) = 0,\\
&(1,2,4) \to k(a-b)((2a-c)(2b-c) + c(2a+2b-c)k^2) = 0,
\end{aligned} \tag{12}$$

where we put $k = k_3$. We suppose first that $a - b = 0$. If $a = c$ also holds, we obtain a subcase of the case 1 of Proposition 2. Hence, we can assume $a = b \neq 0$, $a \neq c \neq 0$, $k_1 = k_2 = 0$ and we obtain the case 2 of Proposition 2.

We want to establish the remaining properties. *Mathematica 5.0* shows that this solution has the Ricci eigenvalues $\varrho_1 = \varrho_2 = \frac{1}{2}(2ac - c^2)$, $\varrho_3 = \frac{1}{2}c^2$, $\varrho_4 = 0$. Moreover, the basic curvature operators have the following expression:

$$\mathcal{R}(e_1,e_2)=\frac{1}{4}(3c-4a)cA_{12},\quad \mathcal{R}(e_1,e_3)=-\frac{1}{4}c^2A_{13},\quad \mathcal{R}(e_1,e_4)=0,$$
$$\mathcal{R}(e_2,e_3)=-\frac{1}{4}c^2A_{23},\quad \mathcal{R}(e_2,e_4)=0,\quad \mathcal{R}(e_3,e_4)=0.$$

Then the *Lie algebra* generated by curvature operators is just $\operatorname{span}(A_{12},A_{13},A_{23})$. Analogously to Case A, we conclude that our spaces are direct products of the 3-dimensional Lie group of *nonconstant* curvature and a real line. Hence they are not locally symmetric. The cyclic parallel condition for the whole space implies the cyclic parallel condition for the 3-dimensional factor. According to Theorem 1, the corresponding spaces must be naturally reductive. We obviously obtain the family from the case ii) of our Classification Theorem.

Assume now that $a\neq b$. Then we are left with the equations

$$(13)\qquad \begin{aligned}&(1,2,3)\to(2(a-c)(b-c)+c(a+b-2c)k^2)=0,\\&(1,2,4)\to k((2a-c)(2b-c)+c(2a+2b-c)k^2)=0.\end{aligned}$$

Here we can suppose $k\neq 0$ because otherwise we get $a=c$ or $b=c$, which is, up to a permutation, the case 2 of Proposition 2.

Now, due to $c\neq 0$, *Mathematica 5.0* gives, up to a permutation of the basis, the unique solution depending on two parameters $c$ and $k$

$$(14)\qquad a=\frac{c}{4}\big(3-3k^2-\sqrt{1+2k^2-3k^4}\big),\quad b=\frac{c}{4}\big(3-3k^2+\sqrt{1+2k^2-3k^4}\big).$$

Here we have the standard inequality $k^2<1$ (see the line below the formula (2)) and, due to $k\neq 0$ and $ab\neq 0$, we get the range $k^2\in\,]0,1[\,\setminus\{\frac{2}{3}\}$. The corresponding Ricci eigenvalues are

$$(15)\qquad \begin{aligned}\varrho_1&=\frac{c^2}{8}\big(2-6k^2-\sqrt{1+2k^2-3k^4}\big),\\ \varrho_2&=\frac{c^2}{8}\big(2-6k^2+\sqrt{1+2k^2-3k^4}\big),\\ \varrho_3&=\frac{c^2}{16}\big(3-3k^2-\sqrt{9-2k^2+57k^4}\big),\\ \varrho_4&=\frac{c^2}{16}\big(3-3k^2+\sqrt{9-2k^2+57k^4}\big).\end{aligned}$$

It is clear that these are functions of two independent variables $c$, $k$. Now, *Mathematica 5.0* gives that, due to the assumption $k^2\in\,]0,1[\,\setminus\{\frac{2}{3}\}$, we always have $\varrho_1\neq\varrho_2$, $\varrho_2\neq\varrho_3$, $\varrho_1\neq\varrho_4$, $\varrho_2\neq\varrho_4$, $\varrho_3\neq\varrho_4$. But $\varrho_1=\varrho_3$ can still occur, namely in the case when $k^2 = \frac{5}{21}$. Then we obtain the cases 3 and 4 of Proposition 2. Now, using (7) and (8) we obtain, for all values of $k$, that the space of the curvature operators is $\mathrm{span}(A_{12}, A_{13}, A_{14}, A_{23}, A_{24})$. Hence the Lie algebra generated by these operators is $\mathfrak{so}(4)$. Using Lemma 5 we see that the action of the holonomy algebra on the tangent space $T_eG$ is irreducible and hence the corresponding manifolds are irreducible. Moreover, we can see easily that $(\nabla_{e_4}\mathcal{R})(e_4, e_2)e_4 \neq 0$ (for all values of $k$) and hence *the spaces are not locally symmetric*. Further, if we put $X = e_1 + e_2 + ve_4$, where $v$ is a *nonzero* parameter, *Mathematica 5.0* shows that the Ledger condition $L_5(X) = 0$ can be written in the form $\varphi_1(c,k) + \varphi_2(c,k)v^2 = 0$ and, because $v$ is a free parameter, this implies

$$(16)\quad \varphi_1(c,k) = 59 + 11c^2 - 6c^3 + (250 - 22c^2 + 12c^3)k^2 + (11c^2 - 6c^3)k^4 = 0,$$

$$(17)\quad \varphi_2(c,k) = (-262 + 39c)k^2 - (260 + 39c)k^4 = 0.$$

If $260 + 39c = 0$, the formula (17) leads to a contradiction. Hence $260 + 39c \neq 0$ and $k^2$ can be expressed from (17) in the form $k^2 = -(262 + 39c)/(13(20 + 3c))$. Substituting this into (16), we obtain a cubic equation

$$(18)\qquad 4347200 - 392340c - 1155771c^2 + 544968c^3 = 0.$$

*Mathematica 5.0* says that (18) has only one real solution, namely $c = -1.57074\ldots$ But this gives a negative value for $k^2$, a contradiction. As a conclusion, we always have $L_5(e_1 + e_2 + ve_4) \neq 0$ for *some* $v \neq 0$, and the corresponding spaces do not satisfy the Ledger condition $L_5$.

Note that the case 3 is a family with four distinct Ricci eigenvalues and this is *an explicit presentation* of the family of spaces described by P. Bueken and L. Vanhecke only implicitly in [4]. We conclude that our spaces belong to the case iv) of the Classification Theorem as a generic subfamily. (The exceptional case $k^2 = \frac{2}{3}$ will be added later.)

The case 4 with two coinciding Ricci eigenvalues has been presented in [1] as the only missing family in the Classification Theorem of [13]—see Appendix for more details. In particular, from this solution we obtain the spaces which give the case v) of our Classification Theorem.

*Case C.* $k_1 = 0$ and $k_2k_3 \neq 0$.

First, we suppose that $2a - b - c \neq 0$. Then from the simplified equations $(2,2,1)$ and $(3,3,1)$ of (11) we obtain that $a = b = c$. Hence, the corresponding solution is a particular subcase of the case 1) of Proposition 2. Supposing $2a - b - c = 0$ we obtain the following simpler system of equations:

$$\begin{aligned}(1,2,3) &\to (a-b)(-4(a-b)-3bk_2^2+3(2a-b)k_3^2)=0,\\(1,2,4) &\to (a-b)(-4a+3b-3bk_2^2+3(2a-b)k_3^2)=0,\\(1,3,4) &\to (a-b)(-2a+3b-3bk_2^2+3(2a-b)k_3^2)=0.\end{aligned}$$

If $a-b=0$, we conclude immediately that $a=b=c$. Thus we assume that $a-b\neq 0$. Dividing the equations $(1,2,4)$ and $(1,3,4)$ by the factor $(a-b)$ and subtracting both remaining equations we obtain the necessary condition $a=0$, which is a contradiction to $abc\neq 0$. This concludes the proof of Proposition 2. □

### 2.2. Solvable case

We are going to analyze this case using the following result given by L. Bérard Bergery in [2]:

**Theorem 2.** *In dimension 4, the solvable and simply connected Lie groups are:*

a) *the non-trivial semi-direct products $E(2)\rtimes\mathbb{R}$ and $E(1,1)\rtimes\mathbb{R}$,*
b) *the non-nilpotent semi-direct products $H\rtimes\mathbb{R}$, where $H$ is the Heisenberg group,*
c) *all semi-direct products $\mathbb{R}^3\rtimes\mathbb{R}$.*

As concerns the semidirect products of the form $G=G_3\rtimes\mathbb{R}$ in the above theorem and all possible left-invariant metrics on them, we can construct all of them on the level of Lie algebras as follows: we consider the Lie algebra $\mathfrak{g}_3$ and the vector space $\mathfrak{g}=\mathfrak{g}_3+\mathbb{R}$. Let $\{f_1,\dots,f_4\}$ be any basis of $\mathfrak{g}$ such that $\mathfrak{g}_3=\operatorname{span}\{f_1,f_2,f_3\}$, $\mathbb{R}=\operatorname{span}\{f_4\}$. Let $D$ be an arbitrary derivation of the algebra $\mathfrak{g}_3$ and let us define

$$[f_4,f_i]=Df_i \qquad \text{for } i=1,2,3. \tag{19}$$

(This completes the multiplication table of the algebra $\mathfrak{g}_3$ to the multiplication table of $\mathfrak{g}$). Then we choose any scalar product $\langle,\rangle$ on $\mathfrak{g}$ for which $\{f_1,f_2,f_3\}$ forms an orthonormal triplet but $f_4$ is just a unit vector which need not be orthonormal to $\mathfrak{g}_3$. Thus we have, as in the formula (2), $t\langle f_i,f_4\rangle=k_i$, $i=1,2,3$. Now, all semi-direct products $G_3\rtimes\mathbb{R}$ with left-invariant metrics correspond to various choices of the derivations $D$ of $\mathfrak{g}_3$ and to all scalar products given by the above rule. The algebra of all derivations $D$ of $\mathfrak{g}_3$ will be usually represented in the corresponding matrix form.

Now, we shall study each of the cases from Theorem 2 separately following the construction indicated above and preserving the style of Section 2.1.

**2.2.1. Non-trivial semi-direct products $E(2) \rtimes \mathbb{R}$.** Let $\mathfrak{e}(2)$ be the Lie algebra of $E(2)$ with a scalar product $\langle,\rangle_3$. Then there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{e}(2)$ such that

$$[f_2, f_3] = \gamma f_1, \quad [f_3, f_1] = -\gamma f_2, \quad [f_1, f_2] = 0 \tag{20}$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations $D$ of $\mathfrak{e}(2)$ is

$$\left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ c & d & 0 \end{pmatrix} : \; a, b, c, d \in \mathbb{R} \right\}$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{e}(2) + \mathbb{R}$, where the multiplication table is given by (20) and, according to the general formula (19), also by

$$\begin{gathered} [f_4, f_1] = af_1 + bf_2, \quad [f_4, f_2] = -bf_1 + af_2, \quad [f_4, f_3] = cf_1 + df_2, \\ \langle f_i, f_4 \rangle = k_i, \quad i = 1, 2, 3. \end{gathered} \tag{21}$$

Here $\gamma \neq 0$, $a, b, c, d, k_1, k_2, k_3$ are arbitrary parameters where $\sum\limits_{i=1}^{3} k_i^2 < 1$ due to the positivity of the scalar product. We exclude the case $a = b = c = d = 0$, i.e., the direct product $E(2) \times \mathbb{R}$.

This gives rise to a simply connected group space $(G = E(2) \rtimes \mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by a new basis $\{e_i\}$ as in the formula (3). Thus we get an orthonormal basis for which

$$\begin{gathered} [e_2, e_3] = \gamma e_1, \quad [e_3, e_1] = -\gamma e_2, \quad [e_1, e_2] = 0, \\ [e_4, e_1] = \frac{1}{R}(ae_1 + (b + k_3\gamma)e_2), \quad [e_4, e_2] = \frac{1}{R}(-(b + k_3\gamma)e_1 + ae_2), \\ [e_4, e_3] = \frac{1}{R}((c + k_2\gamma)e_1 + (d - k_1\gamma)e_2). \end{gathered} \tag{22}$$

Next we are going to calculate, in the new basis, the expressions for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

**Lemma 6.**

$$\begin{gathered} \nabla_{e_i} e_i = \frac{a}{R} e_4, \;\; i = 1, 2, \quad \nabla_{e_i} e_i = 0, \;\; i = 3, 4, \quad \nabla_{e_1} e_2 = 0 = \nabla_{e_2} e_1, \\ \nabla_{e_1} e_3 = \frac{c + \gamma k_2}{2R} e_4, \quad \nabla_{e_3} e_1 = -\gamma e_2 + \frac{c + \gamma k_2}{2R} e_4, \end{gathered} \tag{23}$$

$$\nabla_{e_1} e_4 = -\frac{a}{R} e_1 - \frac{c+\gamma k_2}{2R} e_3, \quad \nabla_{e_4} e_1 = -\frac{c+\gamma k_2}{2R} e_3 + \frac{b+\gamma k_3}{R} e_2,$$
$$\nabla_{e_2} e_3 = \frac{d-\gamma k_1}{2R} e_4, \quad \nabla_{e_3} e_2 = \gamma e_1 + \frac{d-\gamma k_1}{2R} e_4,$$
$$\nabla_{e_2} e_4 = -\frac{a}{R} e_2 - \frac{d-\gamma k_1}{2R} e_3, \quad \nabla_{e_4} e_2 = -\frac{d-\gamma k_1}{2R} e_3 - \frac{b+\gamma k_3}{R} e_1,$$
$$\nabla_{e_3} e_4 = -\frac{d-\gamma k_1}{2R} e_2 - \frac{c+\gamma k_2}{2R} e_1, \quad \nabla_{e_4} e_3 = \frac{d-\gamma k_1}{2R} e_2 + \frac{c+\gamma k_2}{2R} e_1.$$

Similarly to Lemma 2 we can now derive

**Lemma 7.** *The components of the curvature operator are*

$$\begin{aligned}
(24)\qquad \mathcal{R}(e_1,e_2) &= \alpha_{1212}A_{12} + \alpha_{1213}A_{13} + \alpha_{1223}A_{23},\\
\mathcal{R}(e_1,e_3) &= \alpha_{1312}A_{12} + \alpha_{1313}A_{13} + \alpha_{1323}A_{23} + \alpha_{1334}A_{34},\\
\mathcal{R}(e_1,e_4) &= \alpha_{1414}A_{14} + \alpha_{1424}A_{24} + \alpha_{1434}A_{34},\\
\mathcal{R}(e_2,e_3) &= \alpha_{2312}A_{23} + \alpha_{2313}A_{13} + \alpha_{2323}A_{23} + \alpha_{2334}A_{34},\\
\mathcal{R}(e_2,e_4) &= \alpha_{2414}A_{14} + \alpha_{2424}A_{24} + \alpha_{2434}A_{34},\\
\mathcal{R}(e_3,e_4) &= \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned}$$

*where the coefficients $\alpha_{ijlm} = g(\mathcal{R}(e_i,e_j)e_l,e_m)$ satisfy the standard symmetries with respect to their indices and*

$$\begin{aligned}
(25)\quad \alpha_{1212} = \frac{a^2}{R^2},\ \alpha_{1213} = \frac{a(d-\gamma k_1)}{2R^2},\ \alpha_{1223} = -\frac{a(c+\gamma k_2)}{2R^2},\ \alpha_{1313} = -\frac{(c+\gamma k_2)^2}{4R^2},\\
\alpha_{1323} = -\frac{(d-\gamma k_1)(c+\gamma k_2)}{4R^2},\ \alpha_{1334} = \frac{\gamma(-d+\gamma k_1)}{2R},\ \alpha_{1414} = \frac{4a^2-(c+\gamma k_2)^2}{4R^2},\\
\alpha_{1424} = -\frac{(d-\gamma k_1)(c+\gamma k_2)}{4R^2},\ \alpha_{1434} = \frac{2a(c+\gamma k_2)+(d-\gamma k_1)(b+\gamma k_3)}{2R^2},\\
\alpha_{2323} = -\frac{(d-\gamma k_1)^2}{4R^2},\ \alpha_{2334} = \frac{\gamma(c+\gamma k_2)}{2R},\ \alpha_{2424} = \frac{4a^2-(d-\gamma k_1)^2}{4R^2},\\
\alpha_{2434} = \frac{2a(d-\gamma k_1)-(c+\gamma k_2)(b+\gamma k_3)}{2R^2},\ \alpha_{3434} = \frac{3((d-\gamma k_1)^2+(c+\gamma k_2)^2)}{4R^2}.
\end{aligned}$$

Further, we obtain easily

**Lemma 8.** *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1,e_2,e_3,e_4\}$ *is of the form*

$$(26)\qquad \begin{pmatrix}
\frac{-4a^2+(c+\gamma k_2)^2}{2R^2} & \frac{(d-\gamma k_1)(c+\gamma k_2)}{2R^2} & \beta_{13} & \frac{\gamma(-d+\gamma k_1)}{2R}\\
\frac{(d-\gamma k_1)(c+\gamma k_2)}{2R^2} & \frac{-4a^2+(d-\gamma k_1)^2}{2R^2} & \beta_{23} & \frac{\gamma(c+\gamma k_2)}{2R}\\
\beta_{13} & \beta_{23} & \frac{-(c+\gamma k_2)^2-(d-\gamma k_1)^2}{2R^2} & 0\\
\frac{\gamma(-d+\gamma k_1)}{2R} & \frac{\gamma(c+\gamma k_2)}{2R} & 0 & \beta_{44}
\end{pmatrix}$$

*where*

$$\beta_{13} = \frac{-3a(c+\gamma k_2) + (-d+\gamma k_1)(b+\gamma k_3)}{2R^2},$$
$$\beta_{23} = \frac{3a(-d+\gamma k_1) + (c+\gamma k_2)(b+\gamma k_3)}{2R^2},$$
$$\beta_{44} = \frac{-4a^2 - (d-\gamma k_1)^2 - (c+\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemma 4:

**Lemma 9.** *The condition* (10) *for the Ricci tensor of type* (0,2) *is equivalent to the system of algebraic equations*

$$
\begin{aligned}
(27)\quad & (1,1,1) \to a(d-\gamma k_1) = 0,\\
& (1,1,2) \to a(c+\gamma k_2) = 0,\\
& (1,1,3) \to (d-\gamma k_1)(c+\gamma k_2) = 0,\\
& (1,1,4) \to -a(c^2 - d^2 + \gamma^2(k_2^2 - k_1^2)) - 2cd(b+\gamma k_3)\\
& \qquad\qquad - 2\gamma k_2(ac + d(b+\gamma k_3)) + 2\gamma k_1((c+\gamma k_2)(b+\gamma k_3) - ad) = 0,\\
& (3,3,1) \to -3a(d-\gamma k_1) + (c+\gamma k_2)(b+\gamma k_3) = 0,\\
& (3,3,2) \to 3a(c+\gamma k_2) + (d-\gamma k_1)(b+\gamma k_3) = 0,\\
& (4,4,1) \to -a(d-\gamma k_1) + (c+\gamma k_2)(b+\gamma k_3) = 0,\\
& (4,4,2) \to a(c+\gamma k_2) + (d-\gamma k_1)(b+\gamma k_3) = 0,\\
& (1,2,3) \to (d-\gamma k_1)^2 - (c+\gamma k_2)^2 = 0,\\
& (1,2,4) \to -2a(d-\gamma k_1)(c+\gamma k_2)\\
& \qquad\qquad + (c+\gamma k_2 + d - \gamma k_1)(c+\gamma k_2 - d + \gamma k_1)(b+\gamma k_3) = 0,\\
& (1,3,4) \to 2a(d-\gamma k_1)(b+\gamma k_3)\\
& \qquad\qquad + (c+\gamma k_2)(-(b+\gamma k_3)^2 + (a^2 + R^2\gamma^2)) = 0,\\
& (2,3,4) \to -2a(c+\gamma k_2)(b+\gamma k_3)\\
& \qquad\qquad + (d-\gamma k_1)(-(b+\gamma k_3)^2 + (a^2 + R^2\gamma^2)) = 0.
\end{aligned}
$$

*Here the symbol "$(\alpha,\beta,\gamma) \to$" indicates the substitution of $(e_\alpha, e_\beta, e_\gamma)$ for $(X,Y,Z)$ respectively.*

Now, our goal is to find the values of $a$, $b$, $c$, $d$, $k_1$, $k_2$, $k_3$ and $\gamma \neq 0$ which satisfy the system of equations (27).

**Proposition 3.** *The unique solution of the system of algebraic equations* (27) *is given by the formula*

$$d = \gamma k_1, \quad c = -\gamma k_2, \quad \gamma \neq 0, \quad a,\ b,\ k_1,\ k_2,\ k_3 \ \textit{arbitrary}. \tag{28}$$

*The corresponding spaces belong to the case* i) *of the Classification Theorem.*

Proof. From the subsystem of (27) formed by the equations $(1,1,3)$ and $(1,2,3)$ we obtain $(d-\gamma k_1) = (c+\gamma k_2) = 0$. Then the remaining equations (27) are automatically satisfied.

Moreover, according to (26), the corresponding spaces have the Ricci eigenvalues $\varrho_1 = \varrho_2 = \varrho_4 = -2a^2/R^2$, $\varrho_3 = 0$ and the curvature tensor (24) takes on the form

$$\mathcal{R}(e_1,e_2) = \frac{a^2}{R^2}A_{12}, \quad \mathcal{R}(e_1,e_4) = \frac{a^2}{R^2}A_{14}, \quad \mathcal{R}(e_2,e_4) = \frac{a^2}{R^2}A_{24},$$
$$\mathcal{R}(e_1,e_3) = \mathcal{R}(e_2,e_3) = \mathcal{R}(e_3,e_4) = 0.$$

Then either each of the spaces is flat (for $a = 0$) or the space of the curvature operators is $\operatorname{span}(A_{12}, A_{14}, A_{24})$. Moreover, from (23) we get $\nabla_{e_i} e_3 = 0$ for all $i = 1, \ldots, 4$ and $e_3$ is a parallel vector field. Using a complete analogue of the proof in Case A of Proposition 2, we conclude that the corresponding spaces belong to the case i) of our Classification Theorem. $\square$

**2.2.2. Non-trivial semi-direct products $E(1,1) \rtimes \mathbb{R}$.** Let $\mathfrak{e}(1,1)$ be the Lie algebra of $E(1,1)$ with a scalar product $\langle , \rangle_3$. Then there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{e}(1,1)$ such that

$$[f_2,f_3] = \gamma f_2, \quad [f_3,f_1] = \gamma f_1, \quad [f_1,f_2] = 0 \tag{29}$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations $D$ of $\mathfrak{e}(1,1)$ is

$$\left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & 0 \end{pmatrix} : a, b, c \in \mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{e}(1,1) + \mathbb{R}$, where the multiplication table is given by (29) and, according to the general formula (19), also by

$$\begin{gathered}[f_4,f_1] = af_1, \quad [f_4,f_2] = af_2, \quad [f_4,f_3] = bf_1 + cf_2, \\ \langle f_i, f_4 \rangle = k_i, \quad i = 1,2,3.\end{gathered} \tag{30}$$

Here $\gamma \neq 0$, $a$, $b$, $c$, $k_1$, $k_2$, $k_3$ are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$, and we exclude the case $a = b = c = 0$.

This gives rise to a simply connected group space $(G = E(1,1) \rtimes \mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by a new basis $\{e_i\}$ as in the formula (3). Then we get an orthonormal basis for which

$$\begin{gathered}
[e_2, e_3] = \gamma e_2, \quad [e_3, e_1] = \gamma e_1, \quad [e_1, e_2] = 0, \\
[e_4, e_1] = \frac{1}{R}((a - k_3\gamma)e_1), \quad [e_4, e_2] = \frac{1}{R}((a + k_3\gamma)e_2), \\
[e_4, e_3] = \frac{1}{R}((b + k_1\gamma)e_1 + (c - k_2\gamma)e_2).
\end{gathered} \tag{31}$$

Now we are going to calculate, in the new basis, the expressions for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

**Lemma 10.**

$$\begin{gathered}
\nabla_{e_1} e_1 = \gamma e_3 + \frac{(a - \gamma k_3)}{R} e_4, \quad \nabla_{e_2} e_2 = -\gamma e_3 + \frac{(a + \gamma k_3)}{R} e_4, \\
\nabla_{e_i} e_i = 0, \ \ i = 3, 4, \quad \nabla_{e_1} e_2 = 0 = \nabla_{e_2} e_1, \\
\nabla_{e_1} e_3 = -\gamma e_1 + \frac{(b + \gamma k_1)}{2R} e_4, \quad \nabla_{e_3} e_1 = \frac{(b + \gamma k_1)}{2R} e_4, \\
\nabla_{e_1} e_4 = -\frac{(b + \gamma k_1)}{2R} e_3 + \frac{(-a + \gamma k_3)}{R} e_1, \quad \nabla_{e_4} e_1 = -\frac{(b + \gamma k_1)}{2R} e_3, \\
\nabla_{e_2} e_3 = \gamma e_2 + \frac{(c - \gamma k_2)}{2R} e_4, \quad \nabla_{e_3} e_2 = \frac{(c - \gamma k_2)}{2R} e_4, \\
\nabla_{e_2} e_4 = -\frac{(c - \gamma k_2)}{2R} e_3 - \frac{(a + \gamma k_3)}{R} e_2, \quad \nabla_{e_4} e_2 = -\frac{(c - \gamma k_2)}{2R} e_3, \\
\nabla_{e_3} e_4 = -\frac{(b + \gamma k_1)}{2R} e_1 - \frac{(c - \gamma k_2)}{2R} e_2, \quad \nabla_{e_4} e_3 = \frac{(b + \gamma k_1)}{2R} e_1 + \frac{(c - \gamma k_2)}{2R} e_2.
\end{gathered} \tag{32}$$

Similarly to Lemma 2 we can now derive

**Lemma 11.** *The components of the curvature operator are*

$$\begin{aligned}
\mathcal{R}(e_1, e_2) &= \alpha_{1212} A_{12} + \alpha_{1213} A_{13} + \alpha_{1214} A_{14} + \alpha_{1223} A_{23} + \alpha_{1224} A_{24}, \\
\mathcal{R}(e_1, e_3) &= \alpha_{1312} A_{12} + \alpha_{1313} A_{13} + \alpha_{1314} A_{14} + \alpha_{1323} A_{23} + \alpha_{1334} A_{34}, \\
\mathcal{R}(e_1, e_4) &= \alpha_{1412} A_{12} + \alpha_{1413} A_{13} + \alpha_{1414} A_{14} + \alpha_{1424} A_{24} + \alpha_{1434} A_{34}, \\
\mathcal{R}(e_2, e_3) &= \alpha_{2312} A_{23} + \alpha_{2313} A_{13} + \alpha_{2323} A_{23} + \alpha_{2324} A_{24} + \alpha_{2334} A_{34}, \\
\mathcal{R}(e_2, e_4) &= \alpha_{2412} A_{12} + \alpha_{2414} A_{14} + \alpha_{2423} A_{23} + \alpha_{2424} A_{24} + \alpha_{2434} A_{34}, \\
\mathcal{R}(e_3, e_4) &= \alpha_{3413} A_{13} + \alpha_{3414} A_{14} + \alpha_{3423} A_{23} + \alpha_{3424} A_{24} + \alpha_{3434} A_{34},
\end{aligned} \tag{33}$$

*where the coeficients $\alpha_{ijlm} = g(\mathcal{R}(e_i, e_j)e_l, e_m)$ satisfy the standard symmetries with respect to their indices and*

$$
\begin{aligned}
\alpha_{1212} &= \frac{a^2 + \gamma^2(-1 + k_1^2 + k_2^2)}{R^2}, \\
\alpha_{1213} &= \frac{(c - \gamma k_2)(a - \gamma k_3)}{2R^2}, \\
\alpha_{1214} &= \frac{-\gamma(c - \gamma k_2)}{2R}, \\
\alpha_{1223} &= \frac{-(b + \gamma k_1)(a + \gamma k_3)}{2R^2}, \\
\alpha_{1224} &= \frac{-\gamma(b + \gamma k_1)}{2R}, \\
\alpha_{1313} &= \frac{4R^2\gamma^2 - (b + \gamma k_1)^2}{4R^2}, \\
\alpha_{1314} &= \frac{\gamma(a - \gamma k_3)}{R}, \\
\alpha_{1323} &= \frac{(b + \gamma k_1)(-c + \gamma k_2)}{4R^2}, \\
\alpha_{1334} &= \frac{\gamma(b + \gamma k_1)}{R}, \\
\alpha_{1414} &= \frac{4(a - \gamma k_3)^2 - (b + \gamma k_1)^2}{4R^2}, \\
\alpha_{1424} &= \frac{(b + \gamma k_1)(-c + \gamma k_2)}{4R^2}, \\
\alpha_{1434} &= \frac{(b + \gamma k_1)(a - \gamma k_3)}{R^2}, \\
\alpha_{2323} &= \frac{4R^2\gamma^2 - (c - \gamma k_2)^2}{4R^2}, \\
\alpha_{2324} &= \frac{-\gamma(a + \gamma k_3)}{R}, \\
\alpha_{2334} &= \frac{\gamma(-c + \gamma k_2)}{R}, \\
\alpha_{2424} &= \frac{4(a + \gamma k_3)^2 - (c - \gamma k_2)^2}{4R^2}, \\
\alpha_{2434} &= \frac{(c - \gamma k_2)(a + \gamma k_3)}{R^2}, \\
\alpha_{3434} &= \frac{3((b + \gamma k_1)^2 + (c - \gamma k_2)^2)}{4R^2}.
\end{aligned} \tag{34}
$$

Further, we obtain easily

**Lemma 12.** *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} \beta_{11} & \frac{(b+\gamma k_1)(c-\gamma k_2)}{2R^2} & \frac{(b+\gamma k_1)(-3a+\gamma k_3)}{2R^2} & \frac{\gamma(b+\gamma k_1)}{2R} \\ \frac{(b+\gamma k_1)(c-\gamma k_2)}{2R^2} & \beta_{22} & \frac{-(c-\gamma k_2)(3a+\gamma k_3)}{2R^2} & \frac{\gamma(-c+\gamma k_2)}{2R} \\ \frac{(b+\gamma k_1)(-3a+\gamma k_3)}{2R^2} & \frac{-(c-\gamma k_2)(3a+\gamma k_3)}{2R^2} & \beta_{33} & \frac{2\gamma^2 k_3}{R} \\ \frac{\gamma(b+\gamma k_1)}{2R} & \frac{\gamma(-c+\gamma k_2)}{2R} & \frac{2\gamma^2 k_3}{R} & \beta_{44} \end{pmatrix} \tag{35}$$

*where*

$$\beta_{11} = \frac{(b+\gamma k_1)^2 - 4a(a-\gamma k_3)}{2R^2},$$
$$\beta_{22} = \frac{(c-\gamma k_2)^2 - 4a(a+\gamma k_3)}{2R^2},$$
$$\beta_{33} = -\frac{aR^2\gamma^2 + (b+\gamma k_1)^2 + (c-\gamma k_2)^2}{2R^2},$$
$$\beta_{44} = -\frac{a(a^2+\gamma^2 k_3^2) + (b+\gamma k_1)^2 + (c-\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemma 4:

**Lemma 13.** *The condition* (10) *for the Ricci tensor of type* $(0,2)$ *is equivalent to the system of algebraic equations*

$$\begin{aligned}
&(1,1,1) \to a(b+\gamma k_1) = 0,\\
&(1,1,2) \to a(c-\gamma k_2) = 0,\\
&(1,1,3) \to -4a^2 + 4\gamma^2(1-k_1^2-k_2^2) + (b+\gamma k_1)^2 + (c-\gamma k_2)^2 = 0,\\
&(1,1,4) \to -4\gamma k_3(a^2-\gamma^2(1-k_1^2-k_2^2))\\
&\qquad\qquad + (c-\gamma k_2)^2(-a+\gamma k_3) + (b+\gamma k_1)^2(a+\gamma k_3) = 0,\\
&(1,2,4) \to 2a(b+\gamma k_1)(c-\gamma k_2) = 0,\\
&(1,3,4) \to (b+\gamma k_1)(a(a+4\gamma k_3) + 3\gamma^2(k_3^2 - R^2)) = 0,\\
&(2,3,4) \to (c-\gamma k_2)(a(a-4\gamma k_3) + 3\gamma^2(k_3^2 - R^2)) = 0,\\
&(3,3,1) \to (b+\gamma k_1)(a+\gamma k_3) = 0,\\
&(3,3,2) \to (c-\gamma k_2)(a-\gamma k_3) = 0,\\
&(4,4,1) \to (b+\gamma k_1)(a+3\gamma k_3) = 0,\\
&(4,4,2) \to (c-\gamma k_2)(-a+3\gamma k_3) = 0.
\end{aligned} \tag{36}$$

*Here the symbol "*$(\alpha,\beta,\gamma)\to$*" indicates the substitution of* $(e_\alpha, e_\beta, e_\gamma)$ *for* $(X,Y,Z)$ *respectively.*

Now, we have

**Proposition 4.** *The unique solution of the system of algebraic equations* (36) *is, up to a re-numeration of the triplet* $\{e_1, e_2, e_3\}$,

$$(37)\qquad a = \gamma\sqrt{1 - k_1^2 - k_2^2}, \quad b = -\gamma k_1, \quad c = \gamma k_2, \quad \gamma \neq 0, \quad k_1, k_2, k_3 \text{ arbitrary.}$$

*The corresponding spaces belong to the case* i) *of the Classification Theorem.*

Proof. Suppose first $a \neq 0$. We obtain the formulas (37) from $(1,1,1)$, $(1,1,2)$ and $(1,1,3)$. Next we suppose $a = 0$. Then we obtain from $(1,1,3)$ that $1 - k_1^2 - k_2^2 \leqslant 0$, which is a contradiction. On the other hand, (36) is automatically satisfied by the solution (37).

Moreover, the corresponding spaces have the Ricci eigenvalues $\varrho_1 = (-2a^2 - k_3 2\gamma a)R^{-2} = \varrho_3$, $\varrho_2 = (-2a^2 + k_3 2\gamma a)R^{-2} = \varrho_4$. In addition, $\nabla\mathcal{R} = 0$, checking by *Mathematica 5.0.* A routine computation shows that, in fact, every space is a direct product $M_2 \times M_2'$ of spaces of constant curvatures $\varrho_1$ and $\varrho_2$ (even for $k_3 = 0$ where $\varrho_1 = \varrho_2$). Hence, the corresponding spaces are locally symmetric and they belong to the case i) of the Classification Theorem. □

**2.2.3. Non-nilpotent semi-direct products $H \rtimes \mathbb{R}$.** Let $\mathfrak{h}$ be the Lie algebra of $H$ (the Heisenberg group) with a scalar product $\langle , \rangle_3$. Then there is an orthonormal basis $\{f_1, f_2, f_3\}$ of $\mathfrak{h}$ such that

$$(38)\qquad [f_3, f_2] = 0, \quad [f_3, f_1] = 0, \quad [f_1, f_2] = \gamma f_3$$

where $\gamma \neq 0$ is a real number. The algebra of all derivations $D$ of $\mathfrak{h}$ is

$$\left\{ \begin{pmatrix} a & b & h \\ c & d & f \\ 0 & 0 & a+d \end{pmatrix} : a, b, c, d, h, f \in \mathbb{R} \right\},$$

when represented in the matrix form.

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{h} + \mathbb{R}$, where the multiplication table is given by (38) and, according to the general formula (19), also by

$$(39)\qquad \begin{gathered} [f_4, f_1] = af_1 + bf_2 + hf_3, \quad [f_4, f_2] = cf_1 + df_2 + ff_3, \\ [f_4, f_3] = (a+d)f_3, \quad \langle f_i, f_4\rangle = k_i, \quad i = 1,2,3. \end{gathered}$$

Here $\gamma \neq 0$, $a$, $b$, $c$, $d$, $f$, $h$, $k_1$, $k_2$, $k_3$ are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$. We exclude the nilpotent case $a = b = c = d = h = 0$. (See [2].)

This gives rise to a simply connected group space $(G = H \rtimes \mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by the new basis $\{e_i\}$ as in the formula (3). Then we get an orthonormal basis for which

$$
\begin{gathered}
[e_1, e_2] = \gamma e_3, \ [e_3, e_2] = [e_3, e_1] = 0, \ [e_4, e_1] = \frac{1}{R}(ae_1 + be_2 + (h + k_2\gamma)e_3), \\
[e_4, e_2] = \frac{1}{R}(ce_1 + de_2 + (f - k_1\gamma)e_3), \ [e_4, e_3] = \frac{1}{R}((a + d)e_3).
\end{gathered} \tag{40}
$$

Now we are going to calculate, in the new basis, the expressions for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

**Lemma 14.**

$$
\begin{gathered}
\nabla_{e_1} e_1 = \frac{a}{R} e_4, \quad \nabla_{e_2} e_2 = \frac{a}{R} e_4, \quad \nabla_{e_3} e_3 = \frac{(a+d)}{R} e_4, \quad \nabla_{e_4} e_4 = 0, \\
\nabla_{e_1} e_2 = \frac{\gamma}{2} e_3 + \frac{(b+c)}{2R} e_4, \quad \nabla_{e_2} e_1 = -\frac{\gamma}{2} e_3 + \frac{(b+c)}{2R} e_4, \\
\nabla_{e_1} e_3 = -\frac{\gamma}{2} e_2 + \frac{(h+\gamma k_2)}{2R} e_4 = \nabla_{e_3} e_1, \quad \nabla_{e_2} e_3 = \frac{\gamma}{2} e_1 + \frac{(f-\gamma k_1)}{2R} e_4 = \nabla_{e_3} e_2, \\
\nabla_{e_1} e_4 = -\frac{a}{R} e_1 - \frac{(b+c)}{2R} e_2 - \frac{(h+\gamma k_2)}{2R} e_3, \quad \nabla_{e_4} e_1 = \frac{(b-c)}{2R} e_2 + \frac{(h+\gamma k_2)}{2R} e_3, \\
\nabla_{e_2} e_4 = -\frac{(b+c)}{2R} e_1 - \frac{d}{R} e_2 - \frac{(f-\gamma k_1)}{2R} e_3, \quad \nabla_{e_4} e_2 = \frac{(-b+c)}{2R} e_1 + \frac{(f-\gamma k_1)}{2R} e_3, \\
\nabla_{e_3} e_4 = -\frac{(h+\gamma k_2)}{2R} e_1 - \frac{(f-\gamma k_1)}{2R} e_2 - \frac{(a+d)}{R} e_3, \\
\nabla_{e_4} e_3 = -\frac{(h+\gamma k_2)}{2R} e_1 - \frac{(f-\gamma k_1)}{2R} e_2.
\end{gathered} \tag{41}
$$

Similarly to Lemma 2 we can now derive

**Lemma 15.** *The components of the curvature operator are*

$$
\begin{aligned}
\mathcal{R}(e_1, e_2) &= \alpha_{1212} A_{12} + \alpha_{1213} A_{13} + \alpha_{1214} A_{14} + \alpha_{1223} A_{23} \\
&\quad + \alpha_{1224} A_{24} + \alpha_{1234} A_{34}, \\
\mathcal{R}(e_1, e_3) &= \alpha_{1312} A_{12} + \alpha_{1313} A_{13} + \alpha_{1314} A_{14} + \alpha_{1323} A_{23} \\
&\quad + \alpha_{1324} A_{24} + \alpha_{1334} A_{34}, \\
\mathcal{R}(e_1, e_4) &= \alpha_{1412} A_{12} + \alpha_{1413} A_{13} + \alpha_{1414} A_{14} + \alpha_{1423} A_{23} \\
&\quad + \alpha_{1424} A_{24} + \alpha_{1434} A_{34},
\end{aligned} \tag{42}
$$

$$
\begin{aligned}
\mathcal{R}(e_2,e_3) &= \alpha_{2312}A_{23} + \alpha_{2313}A_{13} + \alpha_{2314}A_{14} + \alpha_{2323}A_{23} \\
&\quad + \alpha_{2324}A_{24} + \alpha_{2334}A_{34}, \\
\mathcal{R}(e_2,e_4) &= \alpha_{2412}A_{12} + \alpha_{2413}A_{13} + \alpha_{2414}A_{14} + \alpha_{2423}A_{23} \\
&\quad + \alpha_{2424}A_{24} + \alpha_{2434}A_{34}, \\
\mathcal{R}(e_3,e_4) &= \alpha_{3412}A_{12} + \alpha_{3413}A_{13} + \alpha_{3414}A_{14} + \alpha_{3423}A_{23} \\
&\quad + \alpha_{3424}A_{24} + \alpha_{3434}A_{34},
\end{aligned}
$$

*where the coeficients $\alpha_{ijlm} = g(\mathcal{R}(e_i,e_j)e_l,e_m)$ satisfy the standard symmetries with respect to their indices and*

$$
\begin{aligned}
\alpha_{1212} &= \frac{4ad + 3\gamma^2R^2 - (b+c)^2}{4R^2}, \\
\alpha_{1213} &= \frac{2a(f-\gamma k_1) - (b+c)(h+\gamma k_2)}{4R^2}, \\
\alpha_{1214} &= \frac{-3\gamma(h+\gamma k_2)}{4R}, \\
\alpha_{1223} &= \frac{(b+c)(f-\gamma k_1) - 2d(h+\gamma k_2)}{4R^2}, \\
\alpha_{1224} &= \frac{3\gamma(-f+\gamma k_1)}{4R}, \\
\alpha_{1234} &= \frac{-(a+d)\gamma}{2R}, \\
\alpha_{1313} &= \frac{4a(a+d) - R^2\gamma^2 - (h+\gamma k_2)^2}{4R^2}, \\
\alpha_{1323} &= \frac{2(a+d)(b+c) + (-f+\gamma k_1)(h+\gamma k_2)}{4R^2}, \\
\alpha_{1314} &= \frac{-(b+c)\gamma}{4R}, \\
\alpha_{1324} &= \frac{-d\gamma}{2R}, \\
\alpha_{1334} &= \frac{\gamma(f-\gamma k_1)}{4R}, \\
\alpha_{1423} &= \frac{a\gamma}{2R}, \\
\alpha_{1414} &= \frac{4a^2 + (3b-c)(b+c) + 3(h+\gamma k_2)^2}{4R^2}, \\
\alpha_{1424} &= \frac{4(ac+bd) + 3(f-\gamma k_1)(h+\gamma k_2)}{4R^2}, \\
\alpha_{1434} &= \frac{(b-c)(f-\gamma k_1) + 4(a+d)(h+\gamma k_2)}{4R^2}, \\
\alpha_{2323} &= \frac{4d(a+d) - R^2\gamma^2 - (f-\gamma k_1)^2}{4R^2},
\end{aligned}
\tag{43}
$$

$$\alpha_{2324} = \frac{(b+c)\gamma}{4R},$$
$$\alpha_{2334} = \frac{-\gamma(h+\gamma k_2)}{4R},$$
$$\alpha_{2424} = \frac{-(b-3c)(b+c)+4d^2+3(f-\gamma k_1)^2}{4R^2},$$
$$\alpha_{2434} = \frac{4(a+d)(f-\gamma k_1)+(c-b)(h+\gamma k_2)}{4R^2},$$
$$\alpha_{3434} = \frac{4(a+d)^2-(f-\gamma k_1)^2-(h+\gamma k_2)^2}{4R^2}.$$

Further, we obtain easily

**Lemma 16.** *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \frac{\gamma(-f+\gamma k_1)}{2R} \\ \beta_{12} & \beta_{22} & \beta_{23} & \frac{\gamma(h+\gamma k_2)}{2R} \\ \beta_{13} & \beta_{23} & \beta_{33} & 0 \\ \frac{\gamma(-f+\gamma k_1)}{2R} & \frac{\gamma(h+\gamma k_2)}{2R} & 0 & \beta_{44} \end{pmatrix} \tag{44}$$

*where*

$$\beta_{11} = \frac{-4a(a+d)-b^2+c^2-R^2\gamma^2-(h+\gamma k_2)^2}{2R^2},$$
$$\beta_{12} = \frac{(-f+\gamma k_1)(h+\gamma k_2)-a(b+3c)-d(3b+c)}{2R^2},$$
$$\beta_{13} = \tfrac{c(f-\gamma k_1)-(2a+3d)(h+\gamma k_2)}{2R^2},$$
$$\beta_{22} = \frac{b^2-c^2-4d(a+d)-R^2\gamma^2-(f-\gamma k_1)^2}{2R^2},$$
$$\beta_{23} = \frac{(3a+2d)(-f+\gamma k_1)+b(h+\gamma k_2)}{2R^2},$$
$$\beta_{33} = \frac{-4(a+d)^2+R^2\gamma^2+(f-\gamma k_1)^2+(h+\gamma k_2)^2}{2R^2},$$
$$\beta_{44} = \frac{-(b+c)^2-4((a+d)^2-ad)-(f-\gamma k_1)^2-(h+\gamma k_2)^2}{2R^2}.$$

Now we obtain the following analogue of Lemma 4:

**Lemma 17.** *The condition* (10) *for the Ricci tensor of type* $(0,2)$ *is equivalent to the system of algebraic equations*

$$
\begin{aligned}
(45)\ &(1,1,1) \to a(f-\gamma k_1)=0,\\
&(1,1,2) \to (b+c)(f-\gamma k_1)-a(h+\gamma k_2)=0,\\
&(1,1,3) \to a(b+3c)+(3b+c)d=0,\\
&(1,1,4) \to c(b(a-3d)-c(a+d)-(f-\gamma k_1)(h+\gamma k_2))\\
&\qquad\qquad + a(4d^2-R^2\gamma^2+(f-\gamma k_1)^2)=0,\\
&(1,2,3) \to 2a^2+b^2-c^2-2d^2=0,\\
&(1,2,4) \to -(b-c)(a-d)(a+d)+4(b+c)(bc-ad)-(b+c)R^2\gamma^2\\
&\qquad\qquad + b(f-\gamma k_1)^2+c(h+\gamma k_2)^2+(a+d)(-f+\gamma k_1)(h+\gamma k_2)=0,\\
&(1,3,4) \to (a(b+4c)+d(3b+2c))(-f+\gamma k_1)\\
&\qquad\qquad + (c(3b+2c)-4a(a-2d)+d^2)(h+\gamma k_2)=0,\\
&(2,2,1) \to d(f-\gamma k_1)-(b+c)(h+\gamma k_2)=0,\\
&(2,2,2) \to d(h+\gamma k_2)=0,\\
&(2,2,4) \to -b^2(a+d)+bc(d-3a)+d(4a^2-R^2\gamma^2)\\
&\qquad\qquad + (d(h+\gamma k_2)+b(-f+\gamma k_1))(h+\gamma k_2)=0,\\
&(2,3,4) \to (a^2+b(2b+3c)-4d(2a+d))(f-\gamma k_1)\\
&\qquad\qquad - (a(2b+3c)+d(4b+c))(h+\gamma k_2)=0,\\
&(3,3,1) \to (2a+d)(-f+\gamma k_1)+b(h+\gamma k_2)=0,\\
&(3,3,2) \to c(-f+\gamma k_1)+(a+2d)(h+\gamma k_2)=0,\\
&(3,3,4) \to a(b+c)^2+d((b+c)^2-4a(a+d)-h^2)+(a+d)R^2\gamma^2\\
&\qquad\qquad + h(b+c)(f-\gamma k_1)-a(f-\gamma k_1)^2\\
&\qquad\qquad + \gamma k_2((b+c)(f-\gamma k_1)-d(\gamma k_2+2h))=0,\\
&(4,4,1) \to a(-f+\gamma k_1)+c(h+\gamma k_2)=0,\\
&(4,4,2) \to b(-f+\gamma k_1)+d(h+\gamma k_2)=0.
\end{aligned}
$$

*Here the symbol "*$(\alpha,\beta,\gamma)\to$*" indicates the substitution of* $(e_\alpha,e_\beta,e_\gamma)$ *for* $(X,Y,Z)$ *respectively.*

Now, our goal is to find the values of $a$, $b$, $c$, $d$, $f$, $h$, $k_1$, $k_2$, $k_3$ and $\gamma\neq 0$ which satisfy this system of equations and to study each of these cases.

**Proposition 5.** *The only possible solutions of the system of algebraic equations* (45) *are, up to a re-numeration of the triplet* $\{e_1,e_2,e_3\}$, *the following ones:*

1. $a=b=c=d=0$, $\gamma\neq 0$, $h\neq 0$, $f$, $k_1$, $k_2$, $k_3$ *arbitrary.*
2. $a=d=0$, $b=-c\neq 0$, $h=-\gamma k_2$, $f=\gamma k_1$, $\gamma\neq 0$, $k_1$, $k_2$, $k_3$ *arbitrary.*
   *In these two cases, the corresponding spaces are Riemannian direct products $M^3\times\mathbb{R}$, which are not locally symmetric. Hence, they give the case* ii) *of the Classification Theorem.*
3. $a=d=\frac{1}{2}\gamma R$, $b=-c$, $h=-\gamma k_2$, $f=\gamma k_1$, $\gamma\neq 0$, $k_1$, $k_2$, $k_3$ *arbitrary.*
   *In this situation, the corresponding spaces are irreducible Riemannian manifolds with all Ricci eigenvalues equal to $-\frac{3}{2}\gamma^2$. Hence, the corresponding spaces belong to the case* i) *of the Classification Theorem.*
4. $a=-d$, $d^2\leqslant\frac{1}{4}\gamma^2R^2$, $b=c=\frac{1}{2}\sqrt{-4d^2+\gamma^2R^2}$, $h=-\gamma k_2$, $f=\gamma k_1$, $\gamma\neq 0$, $k_1$, $k_2$, $k_3$ *arbitrary.*
   *In this situation, the corresponding spaces are irreducible Riemannian manifolds, not locally symmetric, with the Ricci eigenvalues $\varrho_1=\varrho_2=\varrho_4=-\frac{1}{2}\gamma^2$, $\varrho_3=\frac{1}{2}\gamma^2$. Moreover, they give the case* iii) *of the Classification Theorem and the $L_5$ condition is not satisfied.*

Proof. Let first $b+3c\neq 0$, then from (1,1,3) we get $a=-(3b+c)d/(b+3c)$ and after substitution into (1,2,3) we obtain $(b^2-c^2)(16d^2+(b+3c)^2)=0$. Hence $b^2=c^2$ and from (1,2,3) it follows that $a^2=d^2$. But, if $b=\pm c$, we get from (1,1,3) that $a=\mp d$.

If $b+3c=0$, we get from (1,1,3) that $d(3b+c)=0$, i.e., $8dc=0$. If $c=0$, then $b=0$ and we get again $a^2-d^2=0$. If $d=0$, then from (1,2,3) we obtain $a=0$, $c=b=0$. In conclusion, we only have to study the cases $a=\pm d$, $b=\mp c$.

*Case A.* $a=d$, $b=-c$.

In this case, the system (45) simplifies to

$$\begin{aligned}
&(1,1,1)\to d(f-\gamma k_1)=0,\\
&(1,1,2)\to d(h+\gamma k_2)=0,\\
&(1,1,4)\to d(4d^2-\gamma^2(1-k_1^2-k_2^2-k_3^2)=0,\\
&(3,3,1)\to c(h+\gamma k_2)=0,\\
&(1,1,2)\to c(f-\gamma k_1)=0.
\end{aligned}\tag{46}$$

Now, first we suppose that $d=0$. Then, if $c=0$, we obtain the case 1 of Proposition 5 (note that $h\neq 0$ because otherwise we would have the nilpotent semi-direct product) and, if $c\neq 0$, we obtain the case 2 of Proposition 5.

In both the cases 1 and 2 we obtain $(\nabla_{e_1}\mathcal{R})(e_1,e_2)e_1=\frac{1}{2}\gamma R^{-2}(h^2+2\gamma k_2h+\gamma^2(1-k_1^2-k_3^2))e_3\neq 0$ and the corresponding spaces are not locally symmetric. Further, put $X=((-f+\gamma k_1)/R\gamma)e_1+((h+\gamma k_2)/R\gamma)e_2+e_4$. Then we check easily that $\nabla_{e_i}X=0$ for $i=1,2,3,4$ and $X$ is (globally) parallel. Hence the action of the holonomy group $\Psi(e)$ is trivial on the 1-dimensional subspace $\operatorname{span}(X) \subset T_eG$. Consequently, according to the de Rham theorem, we have $(G, g) = M^3 \times \mathbb{R}$ when $M^3$ is not locally symmetric (and hence irreducible). According to Theorem 1, $M^3$ is naturally reductive and we obtain the case ii) of our Classification Theorem.

On the other hand, if $d \neq 0$, it is clear from (46) that $f = \gamma k_1$, $h = -\gamma k_2$ and

$$(1,1,4) \to d(4d^2 - \gamma^2(1 - k_1^2 - k_2^2 - k_3^2)) = 0.$$

Hence, we obtain the case 3 of Proposition 5. From Lemma 16 we see that we have four coinciding Ricci eigenvalues $-\frac{3}{2}\gamma^2$. Then the corresponding spaces are Einstein and by a well-known theorem of G. R. Jensen (see [7]) they are locally symmetric. Hence, they belong to the case i) of our Classification Theorem.

*Case B.* $a = -d$, $b = c$.

In this case, the system (45) is reduced to

$$\begin{aligned}
(47)\qquad & (1,1,1) \to d(f - \gamma k_1) = 0,\\
& (1,1,4) \to d(4c^2 + 4d^2 - \gamma^2(1 - k_1^2 - k_2^2 - k_3^2)) = 0,\\
& (1,2,4) \to c(4c^2 + 4d^2 - \gamma^2(1 - k_1^2 - k_2^2 - k_3^2)) = 0,\\
& (2,2,2) \to d(h + \gamma k_2) = 0,\\
& (3,3,1) \to c(h + \gamma k_2) = 0,\\
& (3,3,2) \to c(f - \gamma k_1) = 0.
\end{aligned}$$

Note that if we suppose that $(-f + \gamma k_1) \neq 0$ or $(h + \gamma k_2) \neq 0$, we obtain a particular subcase of the case 1. Hence, we can suppose that $f = \gamma k_1$ and $h = -\gamma k_2$. Thus, we obtain only two non-equivalent solutions: either $c = d = 0$, which is a particular subcase of the case 1, or the case 4 of Proposition 5. In the case 4 we have the Ricci eigenvalues $\varrho_1 = \varrho_2 = \varrho_4 = -\frac{1}{2}\gamma^2$, $\varrho_3 = \frac{1}{2}\gamma^2$, and the corresponding spaces are not locally symmetric due to $(\nabla_{e_1}\mathcal{R})(e_1, e_2)e_3 \neq 0$. Now, using (42) and (43), we obtain that the space of the curvature operators is spanned by the five operators $A_{12}$, $A_{13}$, $A_{14}$, $A_{23}$, $A_{24}$. Hence the Lie algebra generated by these operators is $\mathfrak{so}(4)$. We see that the action of the holonomy algebra on the tangent space $T_eG$ is irreducible and hence the corresponding Riemannian manifolds are irreducible. Now, we make the following change of the basis:

$$\begin{aligned}
(48)\qquad & e_1' = \frac{2}{\gamma}e_4, \quad e_2' = e_1\cos(\alpha) + e_2\sin(\alpha),\\
& e_3' = -e_1\sin(\alpha) + e_2\cos(\alpha), \quad e_4' = \gamma e_3,
\end{aligned}$$

where $\alpha$ is an angle satisfying $d\sin(2\alpha) + b\cos(2\alpha) = 0$. (In particular, we should put $\alpha = 0$ if $b = 0$ and $\alpha = \frac{3}{4}\pi$ if $d = 0$). Then the multiplication table for the new basis $\{e'_1, e'_2, e'_3, e'_4\}$ (when using (40)) becomes exactly the same as in the case iii) of our Classification Theorem. We only have to change notation. The corresponding metric is also in accordance with the case iii). What remains is to prove that the condition $L_5$ is not satisfied.

Further, if we put $X = e_2 + ve_4$ where $v$ is a *nonzero* parameter, *Mathematica 5.0* shows that the Ledger condition $L_5(X) = 0$ can be written in the form

$$\varphi_1(b,d) + \varphi_2(b,d)v^2 + \varphi_3(b,d)v^4 = 0$$

and, because $v$ is a free parameter, this implies

$$\begin{aligned}
&(49) \quad \varphi_1(b,d) = -1020 + 364b + 468d - 252bd + (20 - 13b + 30d + 8bd)4d^2 \\
&\qquad\qquad - (61 + 15b - 21d - 14bd)4b^2 = 0, \\
&(50) \quad \varphi_2(b,d) = 3564 - 1208b + 604d - 396bd - (140 + 17b + 16d - 8bd)4d^2 \\
&\qquad\qquad - (51 + 3b + 7d - 13bd)4b^2 = 0, \\
&(51) \quad \varphi_3(b,d) = -16 - 14b - 36d - 18bd - (4 + b - d)2d^2 \\
&\qquad\qquad + (3 + b + d)4b^2 = 0.
\end{aligned}$$

*Mathematica 5.0* affirms that these equations have no common solution. Hence the corresponding spaces do not satisfy the Ledger condition $L_5$ for *some* value $v \neq 0$ and thus, they cannot be D'Atri spaces. This concludes the proof of Proposition 5. $\square$

**2.2.4. Semi-direct products $\mathbb{R}^3 \rtimes \mathbb{R}$.** Let $\mathfrak{r}^3$ be the Lie algebra of $\mathbb{R}^3$ with a scalar product $\langle , \rangle_3$. The algebra of all derivations $D$ of $\mathfrak{r}^3$ is $\mathfrak{gl}(3, \mathbb{R})$. This means that the matrix form of $D$ depends on 9 arbitrary parameters with respect to any fixed orthonormal basis of $\mathfrak{r}^3$. Moreover, if $D$ is fixed, then we can make three convenient rotations in the coordinate planes to obtain a particular orthonormal basis $\{f_1, f_2, f_3\}$ for which the matrix form of $D$ is the sum of a diagonal matrix and a skew-symmetric matrix. In other words, we have the general matrix form

$$D\colon \left\{ \begin{pmatrix} a & b & c \\ -b & f & h \\ -c & -h & p \end{pmatrix} : a, b, c, f, h, p \in \mathbb{R} \right\}$$

depending just on 6 parameters. Moreover, we have

$$(52) \qquad [f_1, f_2] = 0, \quad [f_1, f_3] = 0, \quad [f_2, f_3] = 0.$$

According to the general scheme, we consider the algebra $\mathfrak{g} = \mathfrak{r}^3 + \mathbb{R}$, where the multiplication table is given by (52) and

$$\begin{aligned} (53) \qquad & [f_4, f_1] = af_1 + bf_2 + cf_3, \quad [f_4, f_2] = -bf_1 + ff_2 + hf_3, \\ & [f_4, f_3] = -cf_1 - hf_2 + pf_3, \quad \langle f_i, f_4 \rangle = k_i, \ i = 1, 2, 3. \end{aligned}$$

Here $a$, $b$, $c$, $f$, $h$, $p$, $k_1$, $k_2$, $k_3$ are arbitrary parameters where $\sum_{i=1}^{3} k_i^2 < 1$.

This gives rise to a simply connected group space $(G = \mathbb{R}^3 \rtimes \mathbb{R}, g)$.

Now we replace the basis $\{f_i\}$ by a new basis $\{e_i\}$ as in the formula (3). Then we get an orthonormal basis for which

$$\begin{aligned} (54) \quad & [e_1, e_2] = 0, \quad [e_1, e_3] = 0 \quad [e_2, e_3] = 0, \quad [e_4, e_1] = \frac{1}{R}(ae_1 + be_2 + ce_3), \\ & [e_4, e_2] = \frac{1}{R}(-be_1 + fe_2 + he_3), \quad [e_4, e_3] = \frac{1}{R}(-ce_1 - he_2 + pe_3). \end{aligned}$$

Now we are going to calculate, in the new basis, the expressions for the Levi-Civita connection, the curvature tensor, the Ricci matrix, and the condition for the Ricci tensor to be cyclic parallel.

By an easy calculation we get

**Lemma 18.**

$$\begin{aligned} (55) \qquad & \nabla_{e_1} e_1 = \frac{a}{R} e_4, \quad \nabla_{e_2} e_2 = \frac{f}{R} e_4, \quad \nabla_{e_3} e_3 = \frac{p}{R} e_4, \quad \nabla_{e_4} e_4 = 0, \\ & \nabla_{e_1} e_2 = 0 = \nabla_{e_2} e_1, \quad \nabla_{e_1} e_3 = 0 = \nabla_{e_3} e_1, \quad \nabla_{e_2} e_3 = 0 = \nabla_{e_3} e_2, \\ & \nabla_{e_1} e_4 = -\frac{a}{R} e_1, \quad \nabla_{e_4} e_1 = \frac{b}{R} e_2 + \frac{c}{R} e_3, \quad \nabla_{e_2} e_4 = -\frac{f}{R} e_2, \\ & \nabla_{e_4} e_2 = -\frac{b}{R} e_1 + \frac{h}{R} e_3, \quad \nabla_{e_3} e_4 = -\frac{p}{R} e_3, \quad \nabla_{e_4} e_3 = -\frac{c}{R} e_1 - \tfrac{h}{R} e_2. \end{aligned}$$

Similarly to Lemma 2 we can now derive

**Lemma 19.** *The components of the curvature operator are*

$$\begin{aligned} (56) \quad & \mathcal{R}(e_1, e_2) = \frac{af}{R^2} A_{12}, \quad \mathcal{R}(e_1, e_4) = \frac{a^2}{R^2} A_{14} + \frac{b(f-a)}{R^2} A_{24} + \frac{c(p-a)}{R^2} A_{34}, \\ & \mathcal{R}(e_1, e_3) = \frac{ap}{R^2} A_{13}, \quad \mathcal{R}(e_2, e_4) = \frac{b(f-a)}{R^2} A_{14} + \frac{f^2}{R^2} A_{24} + \frac{h(p-f)}{R^2} A_{34}, \\ & \mathcal{R}(e_2, e_3) = \frac{fp}{R^2} A_{23}, \quad \mathcal{R}(e_3, e_4) = \frac{c(p-a)}{R^2} A_{14} + \frac{h(p-f)}{R^2} A_{24} + \frac{p^2}{R^2} A_{34}. \end{aligned}$$

Further, we obtain easily

**Lemma 20.** *The matrix of the Ricci tensor of type* $(1,1)$ *expressed with respect to the basis* $\{e_1, e_2, e_3, e_4\}$ *is of the form*

$$\begin{pmatrix} -\dfrac{a(a+f+p)}{R^2} & \dfrac{b(a-f)}{R^2} & \dfrac{c(a-p)}{R^2} & 0 \\ \dfrac{b(a-f)}{R^2} & -\dfrac{f(a+f+p)}{R^2} & \dfrac{h(f-p)}{R^2} & 0 \\ \dfrac{c(a-p)}{R^2} & \dfrac{h(f-p)}{R^2} & -\dfrac{p(a+f+p)}{R^2} & 0 \\ 0 & 0 & 0 & -\dfrac{a^2+f^2+p^2}{R^2} \end{pmatrix}. \tag{57}$$

Now we obtain the following analogue of Lemma 4

**Lemma 21.** *The condition* (10) *for the Ricci tensor of type* $(0,2)$ *is equivalent to the system of algebraic equations*

$$\begin{aligned} &(1,1,4) \to -(f+p)a^2 + (f-a)b^2 + (p-a)c^2 + a(f^2+p^2) = 0, \\ &(1,2,4) \to (a+f-2p)ch + (a-f)bp = 0, \\ &(1,3,4) \to (p-a)cf + (a-2f+p)bh = 0, \\ &(2,2,4) \to (a-f)b^2 - (a+p)f^2 + (p-f)h^2 + f(a^2+p^2) = 0, \\ &(2,3,4) \to (f+p-2a)bc + (f-p)ah = 0. \end{aligned} \tag{58}$$

*Here the symbol "*$(\alpha,\beta,\gamma) \to$*" indicates the substitution of* $(e_\alpha, e_\beta, e_\gamma)$ *for* $(X,Y,Z)$ *respectively.*

Now, our goal is to find the values of $a$, $b$, $c$, $f$, $h$, $p$, $k_1$, $k_2$, $k_3$ which satisfy this system of equations and to study each of these cases. Here *Mathematica 5.0* offers just 21 formally different solutions. But, using various numerations *and* various signs of the vectors $e_1$, $e_2$, $e_3$, we see easily that most of the solutions are to one another equivalent, and we can reduce the number of essentially different solutions to five. Then we get

**Proposition 6.** *The only possible solutions of the system of algebraic equations* (58) *are, up to a re-numeration of the triplet* $\{e_1, e_2, e_3\}$*, the following ones:*

1) $p = f = a$, $a$, $b$, $c$, $h$, $k_1$, $k_2$, $k_3$ *arbitrary.*
2) $b = c = f = p = 0$, $a$, $h$, $k_1$, $k_2$, $k_3$ *arbitrary.*
3) $a = b = c = 0$, $p = f$, $f$, $h$, $k_1$, $k_2$, $k_3$ *arbitrary.*
4) $c = h = p = 0$, $b = f = -a$, $a \neq 0$, $k_1$, $k_2$, $k_3$ *arbitrary.*
5) $b = \frac{1}{3}a$, $c = -h = \frac{4}{3}a$, $f = -a$, $p = 0$, $a \neq 0$, $k_1$, $k_2$, $k_3$ *arbitrary.*

For the solution 1) we obtain from (57) that all four Ricci eigenvalues are equal to $-3a^2/R^2$. Then the corresponding spaces are Einstein and by [7] they are locally symmetric. Hence, they belong to the case i) of the Classification Theorem.

For the solution 2) we obtain from (57) that the corresponding spaces have the Ricci eigenvalues $\varrho_1 = \varrho_4 = -a^2/R^2$, $\varrho_2 = \varrho_3 = 0$. From (55) we see that the distribution $\operatorname{span}(e_2, e_3)$ is parallel and hence the holonomy group $\Psi(e)$ acts trivially on it. From the de Rham theorem we see that each space is a direct product $M^2 \times \mathbb{R}^2$, where $M^2$ is of constant curvature $\varrho_1$. Hence, the corresponding spaces are locally symmetric and they belong to the case i) of our Classification Theorem.

For the solution 3) we obtain from (57) that the Ricci eigenvalues are $\varrho_1 = 0$, $\varrho_2 = \varrho_3 = \varrho_4 = -2f^2/R^2$ and from (56) that the curvature tensor takes on the form

$$\mathcal{R}(e_2,e_3) = \frac{f^2}{R^2}A_{23}, \quad \mathcal{R}(e_2,e_4) = \frac{f^2}{R^2}A_{24}, \quad \mathcal{R}(e_3,e_4) = \frac{f^2}{R^2}A_{34},$$
$$\mathcal{R}(e_1,e_2) = \mathcal{R}(e_1,e_3) = \mathcal{R}(e_1,e_4) = 0.$$

We see that each of the spaces is either flat (for $f = 0$) or it is a direct product $M^3 \times \mathbb{R}$ where $M^3$ is a space of constant curvature. In the latter case, the argument is exactly the same as in Case A of Proposition 2. The corresponding spaces belong to the case i) of our Classification Theorem.

Under the hypothesis of the solution 4) we obtain from (57) that the Ricci eigenvalues are $\varrho_1 = \varrho_4 = -2a^2/R^2$, $\varrho_2 = 2a^2/R^2$, $\varrho_3 = 0$. Besides, it is easy to check that $(\nabla_{e_1}\mathcal{R})(e_1,e_2)e_1 \neq 0$ and the curvature tensor (56) takes on the form

$$\mathcal{R}(e_1,e_2) = -\frac{a^2}{R^2}A_{12}, \quad \mathcal{R}(e_1,e_3) = 0, \quad \mathcal{R}(e_1,e_4) = \frac{a^2}{R^2}(A_{14} + 2A_{24}),$$
$$\mathcal{R}(e_2,e_3) = 0, \quad \mathcal{R}(e_2,e_4) = \frac{a^2}{R^2}(2A_{14} + A_{24}), \quad \mathcal{R}(e_3,e_4) = 0.$$

Then the space of the curvature operators is obviously spanned by the three operators $A_{12}$, $A_{14}$, $A_{24}$. In addition, $\nabla_{e_i} e_3 = 0$ for all $i = 1, \ldots, 4$. Consequently, according to Lemma 5 and the de Rham theorem the corresponding manifolds are (not locally symmetric) Riemannian direct products $M^3 \times \mathbb{R}$. Moreover, according to Theorem 1, $M^3$ is naturally reductive and we obtain the case ii) of our Classification Theorem.

Finally, we shall study the solution 5). We obtain from (57) that we have here four distinct Ricci eigenvalues

$$\varrho_1 = \frac{-2a^2}{3R^2}, \quad \varrho_2 = \frac{a^2(1-\sqrt{33})}{3R^2}, \quad \varrho_3 = \frac{a^2(1+\sqrt{33})}{3R^2}, \quad \varrho_4 = \frac{-2a^2}{R^2}.$$

Now, let us introduce a new basis $\{e_1', e_2', e_3', e_4'\}$ by

$$e_1' = -\frac{R\sqrt{3}}{4a}e_4, \quad e_2' = -\frac{R\sqrt{3}}{4a\sqrt{2}}(e_2 - e_1),$$
$$e_3' = -\frac{R}{4a\sqrt{2}}(e_1 + e_2 + 2e_3), \quad e_4' = -\frac{R}{4a\sqrt{3}}(2e_1 + 2e_2 + e_3).$$

Here $\langle e_i', e_i'\rangle = \frac{3}{16}R^2/a^2$ for $i = 1, 2, 3, 4$, the triplet $\{e_1', e_2', e_3'\}$ is orthogonal, $\langle e_3', e_4'\rangle = \frac{3}{16}\sqrt{\frac{2}{3}}R^2/a^2$ and $\langle e_i', e_4'\rangle = 0$ for $i = 1, 2$. Using the multiplication table (54) and the assumptions of the case 5 of Proposition 6, we obtain a new multiplication table

$$[e_1', e_2'] = e_3', \quad [e_1', e_3'] = \frac{1}{2}e_2', \quad [e_2', e_3'] = [e_4', e_1'] = [e_4', e_2'] = [e_4', e_3'] = 0. \tag{59}$$

Now, if we compare this multiplication table and the scalar products $\langle e_i', e_j'\rangle$ with the multiplication table and the family of metrics, $g_{(c,k)}$, in the case iv) of the Classification Theorem, we see that this is exactly the subcase where $k^2 = \frac{2}{3}$ and the parameter $c$ in the metric is equal to $-4a/(R\sqrt{3})$. Notice that it is the particular subcase which was omitted in the case 3 of Proposition 2 for a rather formal reason that it was *not* generated on a non-solvable group $G_3 \times \mathbb{R}$.

## 3. Appendix

In [13], F. Podestà and A. Spiro published the following classification theorem.

**Theorem 3.** *Let $(M, g)$ be a 4-dimensional curvature homogeneous Riemannian manifold of type $\mathcal{A}$, not Einstein, with at most three distinct Ricci principal curvatures. Then just one of the following cases occurs:*

a) *$M$ is locally symmetric;*
b) *$(M, g)$ is locally isometric to a Riemannian product $M^3 \times \mathbb{R}$, where $M^3$ is a 3-dimensional Riemannian space with two distinct Ricci curvatures $(\varrho_1, \varrho_2 = \varrho_1, \varrho_3)$, $\varrho_3 \neq \varrho_1$: $M^3$ is the total space of a Riemannian submersion over a surface $N$ of constant curvature $\varrho_1 + \varrho_3$; the fibres of this submersion are geodesics and the integrability tensor $A$ of the submersion is given by $\sqrt{2\varrho_3}w$, where $w$ is the area form of $N$;*
c) *$(M, g)$ is locally isometric to the simply connected Lie group $(G, g_a)$, whose Lie algebra $\mathfrak{g}$ is described by*

$$[e_1, e_2] = -e_2, \quad [e_1, e_3] = e_3, \quad [e_2, e_3] = e_4,$$
$$[e_1, e_4] = [e_2, e_4] = [e_3, e_4] = 0,$$

*endowed with the left-invariant metric $g_a$ $(a \in \mathbb{R}^+)$,*

$$g_a = \frac{1}{a^2} w^1 \otimes w^1 + w^2 \otimes w^2 + w^3 \otimes w^3 + 4a^2 w^4 \otimes w^4,$$

*$\{w^i\}$ being the dual basis of $\{e_i\}$. The metrics $g_a$ have Ricci eigenvalues $\varrho_1 = \varrho_2 = \varrho_3 = -2a^2$, $\varrho_4 = -\varrho_1 = 2a^2$ and are not isometric to one another for different values of $a$. Moreover, the Riemannian manifolds $(G, g_a)$ are irreducible and not locally symmetric.*

We are going to compare this theorem with our Classification Theorem. The case c) of Theorem 3 is exactly the case iii) of our Classification Theorem. It suffices to put $a = \gamma/2$. Also, applying Theorem 1 to the direct product $M^3 \times \mathbb{R}$ we can see that the case b) of Theorem 3 coincides with the case ii) of our Classification Theorem (and it is simplified herewith). (See also [4].) On the other hand, as we have claimed in [1], Theorem 3 is incomplete because the case v) of our Classification Theorem is missing there.

When the new family of examples was found, we contacted A. Spiro and F. Podestà who confirmed us that there was really a gap in the paper [13] and they asked the present authors kindly to publish the following Erratum: the formula on page 236, line 11, should read correctly

$$d_{34}^2(d_{12}^3 - d_{21}^3) - d_{24}^3(d_{13}^2 - d_{31}^2) = d_{34}^2\Big(d_{12}^3\Big(1 - \frac{\varrho_2 - \varrho_4}{\varrho_3 - \varrho_4}\Big) - 2d_{31}^2 \frac{\varrho_2 - \varrho_4}{\varrho_3 - \varrho_4}\Big) = 0.$$

Several weeks later they sent us a detailed and complete correction of the paper [13] where they recovered the case v) of our Classification Theorem—in a bit different but still equivalent form. Also, they concluded that it was the only missing family. The present authors reproduce here (with some cosmetic changes) the detailed erratum done by F. Podestà and A. Spiro, with their kind consent.

**Erratum** (February 26, 2005)**.** Let $(M, g)$ be a 4-dimensional Riemannian manifold with constant Ricci principal curvatures $\varrho_i$, $i = 1, \ldots, 4$ such that $\varrho_1 = \varrho_2$ and $\varrho_2$, $\varrho_3$, $\varrho_4$ are all distinct. Let $\{e_i\}_{i=1,\ldots,4}$ be a fixed set of vector fields which gives an orthonormal frame at any point of $M$ such that the Ricci tensor $S$ is diagonal in such a frame, i.e. $S(e_i, e_j) = \varrho_i \delta_{ij}$. Finally, we denote by $d_{ij}^k$ the Christoffel symbols of the Levi-Civita connection with respect to the frame field $\{e_i\}_{i=1,\ldots,4}$, i.e. the smooth functions $d_{ij}^k = g(\nabla_{e_i} e_j, e_k)$. Notice that $d_{ij}^k = -d_{ik}^j$ by orthonormality of the frame field $\{e_i\}_{i=1,\ldots,4}$. The gap in the proof of Theorem 3 concerns the analysis of Subcase 1.1 of class $\mathcal{A}$ (see p. 234 of [13]).

Under the hypothesis of Subcase 1.1 of [13], there exists an orthonormal frame field $\{e_i\}_{i=1,\ldots,4}$ in a neighborhood of any point $p \in M$ such that $d_{ij}^k$ are all vanishing except for the following functions:

$$(60)\ d^1_{32} = -d^2_{31} = -A, \quad d^3_{21} = -d^1_{23} = \frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3}A, \quad d^3_{12} = -d^2_{13} = -\frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3}A,$$
$$d^4_{32} = -d^2_{34} = f, \quad d^4_{23} = -d^3_{24} = -\frac{\varrho_2 - \varrho_4}{\varrho_3 - \varrho_4}f,$$

where $A$, $f$ are smooth functions, $A > 0$ and $f$ nonzero.

Now, we consider the Jacobi identity

$$[e_1, [e_3, e_4]] + [e_3, [e_4, e_1]] + [e_4, [e_1, e_3]] = 0$$

and the inner product of both sides with the vector field $e_3$. After that, we write each Lie bracket by means of the identities

$$[e_i, e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i = (d^k_{ij} - d^k_{ji})e_k,$$

obtaining from the previous claim that

$$(d^3_{12} - d^3_{21})d^2_{34} + (-d^3_{24})(d^2_{13} - d^2_{31}) = fA\frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3}\left(3 - \frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3}\right) = 0.$$

Since $\varrho_i - \varrho_j \neq 0$ for any $i, j = 2, 3, 4$, we immediately get the following necessary relation between the Ricci eigenvalues $\varrho_i$:

$$(61) \qquad \frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3} = 3 \quad \text{or, equivalently,} \quad 2\varrho_4 + \varrho_2 - 3\varrho_3 = 0.$$

On the other hand, $A$, $\varrho_i$ and $f$ must satisfy the relations (3.2) of [13], i.e. the expressions which give the components of the Ricci curvature tensor in terms of the Christoffel symbols $d^k_{ij}$. Substituting the expressions (60) and (61) into those relations, we get that $A$, $\varrho_i$ and $f$ satisfy the equations

$$(62) \qquad \varrho_1 = \varrho_2 = -2\frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3}A^2 = -6A^2, \quad \varrho_3 = 2\left(\frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3}\right)^2 A^2 = 18A^2,$$
$$\varrho_4 = 2\frac{\varrho_4 - \varrho_2}{\varrho_4 - \varrho_3}f^2 = 6f^2.$$

From (62) it follows immediately that $A$ and $f = d^4_{32}$ are constants and, changing $e_4$ into $-e_4$, there is no loss of generality if we assume that $f > 0$. Moreover, substituting (62) into (61), we obtain that

$$(63) \qquad f = \sqrt{5}A \quad \text{and hence that} \quad \varrho_4 = 30A^2.$$

According to (60), all Christoffel symbols are constant and the vector fields $e_i$, $i = 1,2,3,4$, generate a 4-dimensional Lie algebra $\mathfrak{g}$ whose Lie brackets can be easily computed as follows:

$$
\begin{gathered}
[e_1,e_2] = -6Ae_3, \quad [e_1,e_3] = 2Ae_2, \quad [e_1,e_4] = 0,\\
[e_2,e_3] = -2Ae_1 - 4A\sqrt{5}e_4, \quad [e_2,e_4] = 3A\sqrt{5}e_3, \quad [e_3,e_4] = -A\sqrt{5}e_2.
\end{gathered} \tag{64}
$$

We constructed a new family of spaces of class $\mathcal{A}$ and, obviously, this is the only missing family in our Theorem 2 in [13].

Now, let us introduce a new basis $\{e_1', e_2', e_3', e_4'\}$ by

$$
\begin{gathered}
e_1' = -\frac{1}{2\sqrt{21}A}e_2, \quad e_2' = -\frac{1}{2\sqrt{21}A}e_3,\\
e_3' = -\frac{1}{42A}\big(e_1 + 2\sqrt{5}e_4\big), \quad e_4' = -\frac{\sqrt{21}}{126A}\big(\sqrt{5}e_1 - 2e_4\big).
\end{gathered}
$$

Here $\langle e_i', e_i'\rangle = \frac{1}{84}A^{-2}$ for $i = 1,2,3,4$, the triplet $\{e_1', e_2', e_3'\}$ is orthogonal, $\langle e_3', e_4'\rangle = \frac{1}{84}A^{-2}\sqrt{\frac{5}{21}}$ and $\langle e_i', e_4'\rangle = 0$ for $i = 1,2$. Using the multiplication table (64) we obtain a new multiplication table

$$
\begin{gathered}
[e_1', e_2'] = e_3', \quad [e_3', e_1'] = \frac{6}{7}e_2', \quad [e_2', e_3'] = \frac{2}{7}e_1',\\
[e_4', e_1'] = [e_4', e_2'] = [e_4', e_3'] = 0.
\end{gathered} \tag{65}
$$

Now, if we compare this multiplication table and the scalar products $\langle e_i', e_j'\rangle$ with the multiplication table and the family of metrics, $g_c$, in the case v) of the Classification Theorem, we see that we obtain exactly the same family of spaces via the substitution $c = -2\sqrt{21}A$.

Therefore, the classification by F. Podestà and A. Spiro should be now corrected as follows:

**Theorem 4.** *Let $(M,g)$ be a 4-dimensional curvature homogeneous Riemannian manifold of type $\mathcal{A}$, not Einstein, with at most three distinct Ricci principal curvatures. Then just one of the following cases holds:* a) *$(M,g)$ is locally symmetric, or one of the cases* b), c) *from Theorem* 3 *occurs, or the case* d), *namely the family described in the case* v) *of the Classification Theorem from Section* 1 *occurs.*

Note that, in the case of at most three distinct Ricci eigenvalues, the corrected result by Podestà and Spiro is stronger than our classification result because the homogeneity is replaced by the weaker assumption of curvature homogeneity.

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