Czechoslovak Mathematical Journal

Basudeb Dhara; Rajendra K. Sharma

Derivations with power central values on Lie ideals in prime rings

Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 1, 147–153

Persistent URL: `http://dml.cz/dmlcz/128251`

Terms of use:

© Institute of Mathematics AS CR, 2008

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* `http://dml.cz`

Czechoslovak Mathematical Journal, 58 (133) (2008), 147–153

DERIVATIONS WITH POWER CENTRAL VALUES ON LIE IDEALS IN PRIME RINGS

Basudeb Dhara, Kharagpur and R. K. Sharma, Delhi

(Received January 15, 2006)

Abstract. Let R be a prime ring of char $R \neq 2$ with a nonzero derivation d and let U be its noncentral Lie ideal. If for some fixed integers $n_1 \geqslant 0, n_2 \geqslant 0, n_3 \geqslant 0$, $(u^{n_1}[d(u),u]u^{n_2})^{n_3} \in Z(R)$ for all $u \in U$, then R satisfies S_4, the standard identity in four variables.

Keywords: prime ring, derivation, extended centroid, martindale quotient ring

MSC 2000: 16W25, 16R50, 16N60

1. Introduction

Throughout this paper R always denotes a prime ring with center $Z(R)$, extended centroid C and two-sided Martindale quotient ring Q. For $x, y \in R$, set $[x,y]_1 = [x,y] = xy - yx$ and $[x,y]_2 = [[x,y],y]$.

A well-known result proved by Posner [11] states that for a derivation d of R, if $[d(x),x] \in Z(R)$ for all $x \in R$ then either $d = 0$ or R is commutative. In [9] Lanski generalized the result of Posner to a Lie ideal. Lanski proved that if U is a noncommutative Lie ideal of R and $d \neq 0$ is a derivation of R such that $[d(x),x] \in Z(R)$ for all $x \in U$, then either R is commutative, or char $R = 2$ and R satisfies S_4, the standard identity in four variables. Carini and Filippis [2] studied the case $[d(u),u]^n \in Z(R)$ for all u in a noncentral Lie ideal of R. They showed that if U is a noncentral Lie ideal of R with char $R \neq 2$ and d a nonzero derivation of R such that $[d(u),u]^n \in Z(R)$ for all $u \in U$ then R satisfies S_4. Here we shall prove that the same conclusion of Carini and Filippis holds if $(u^{n_1}[d(u),u]u^{n_2})^{n_3} \in Z(R)$ for all u in a noncentral Lie ideal of R with char $R \neq 2$.

2. Main results

First we prove some lemmas.

Lemma 2.1. *Let $R = M_k(F)$ be the set of all $k \times k$ matrices over a field F of characteristic $\neq 2$. If for some $b \in R$, $([b,[x,y]]_2[x,y]^n)^t = 0$ for all $x, y \in R$, where t ($\geqslant 0$), n ($\geqslant 1$) are fixed integers and t is even, then $b \in F.I_k$.*

Proof. Let $b = (b_{ij})_{k\times k}$. We choose $x = e_{12}$, $y = e_{21}$ and then compute $[x,y] = e_{11} - e_{22}$,

$$[b,[x,y]]_2 = \begin{pmatrix} 0 & 4b_{12} & b_{13} & \dots & b_{1k} \\ 4b_{21} & 0 & b_{23} & \dots & b_{2k} \\ b_{31} & b_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k1} & b_{k2} & 0 & \dots & 0 \end{pmatrix},$$

$$[b,[x,y]]_2[x,y]^n = \begin{pmatrix} 0 & (-1)^n 4b_{12} & 0 & \dots & 0 \\ 4b_{21} & 0 & 0 & \dots & 0 \\ b_{31} & (-1)^n b_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k1} & (-1)^n b_{k2} & 0 & \dots & 0 \end{pmatrix},$$

$$e_{11}([b,[x,y]]_2[x,y]^n)^t = (-1)^{tn/2} 4^t b_{12}^{t/2} b_{21}^{t/2} e_{11}.$$

Now the assumption $e_{11}([b,[x,y]]_2[x,y]^n)^t = 0$ implies that one of b_{12} and b_{21} must be zero. So without loss of generality we assume that $b_{12} = 0$. Now choose $x = e_{11}, y = e_{12} - e_{21}$ and then compute

$$[x,y]^n = \begin{cases} I_2, & \text{if } n \text{ is even,} \\ e_{12} + e_{21}, & \text{if } n \text{ is odd,} \end{cases}$$

$$[b,[x,y]]_2 = \begin{pmatrix} 2(b_{11}-b_{22}) & -2b_{21} & b_{13} & \dots & b_{1k} \\ 2b_{21} & -2(b_{11}-b_{22}) & b_{23} & \dots & b_{2k} \\ b_{31} & b_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k1} & b_{k2} & 0 & \dots & 0 \end{pmatrix}.$$

If n is even then

$$[b,[x,y]]_2[x,y]^n = \begin{pmatrix} 2(b_{11}-b_{22}) & -2b_{21} & 0 & \dots & 0 \\ 2b_{21} & -2(b_{11}-b_{22}) & 0 & \dots & 0 \\ b_{31} & b_{32} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k1} & b_{k2} & 0 & \dots & 0 \end{pmatrix}$$

and $e_{11}([b,[x,y]]_2[x,y]^n) = 2(b_{11}-b_{22})e_{11} - 2b_{21}e_{12}$. If n is odd then

$$[b,[x,y]]_2[x,y]^n = \begin{pmatrix} -2b_{21} & 2(b_{11}-b_{22}) & 0 & \dots & 0 \\ -2(b_{11}-b_{22}) & 2b_{21} & 0 & \dots & 0 \\ b_{32} & b_{31} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ b_{k2} & b_{k1} & 0 & \dots & 0 \end{pmatrix}$$

and $e_{11}([b,[x,y]]_2[x,y]^n) = -2b_{21}e_{11} + 2(b_{11}-b_{22})e_{12}$. Thus whether n is even or odd both cases give $e_{11}([b,[x,y]]_2[x,y]^n)^t = (-)^{tn/2}2^t\{(b_{11}-b_{22})^2 - b_{21}^2\}^{t/2}e_{11}$. The assumption $e_{11}([b,[x,y]]_2[x,y]^n)^t = 0$ implies $(b_{11}-b_{22})^2 - b_{21}^2 = 0$.

On the other hand, by choosing $x = e_{11}$, $y = e_{12} + e_{21}$ we obtain in a similar manner as earlier that $(b_{11}-b_{22})^2 + b_{21}^2 = 0$. The addition and subtraction of $(b_{11}-b_{22})^2 - b_{21}^2 = 0$ and $(b_{11}-b_{22})^2 + b_{21}^2 = 0$ implies $b_{21} = 0$ and $b_{11} = b_{22}$.

In this way we can prove for any $i \neq j$ that $b_{ij} = b_{ji} = 0$ and $b_{ii} = b_{jj}$. Thus $b \in F.I_k$. □

Lemma 2.2. *Let $R = M_k(F)$ be the set of all $k \times k$ matrices over a field F of characteristic $\neq 2$ and $k \geqslant 3$. If for some $b \in R$, $([x,y]^m[b,[x,y]]_2[x,y]^n)^t \in Z(R)$ for all $x, y \in R$, where $t(\geqslant 0), n(\geqslant 0), m(\geqslant 0)$ are fixed integers, then $b \in F.I_k$.*

Proof. Let $b = (b_{ij})_{k\times k}$. We choose $x = e_{12}$, $y = e_{21}$ and then compute $[x,y] = e_{11} - e_{22}$,

$$([x,y]^m[b,[x,y]]_2[x,y]^n)^t = \begin{cases} \alpha^{t/2}\beta^{t/2}I_2, & \text{if } t \text{ is even,} \\ \alpha^{(t-1)/2}\beta^{(t-1)/2}(\alpha e_{12} + \beta e_{21}), & \text{if } t \text{ is odd} \end{cases}$$

where $\alpha = (-1)^n 4b_{12}$ and $\beta = (-1)^m 4b_{21}$. Since $k \geqslant 3$, $([x,y]^m[b,[x,y]]_2[x,y]^n)^t \in Z(R)$ implies that at least one of α and β must be zero, i.e., b_{12} or b_{21} is equal to zero.

Let $b_{12} = 0$. Now choose $x = e_{11}, y = e_{12} - e_{21}$ and then compute

$$[x,y]^n = \begin{cases} I_2, & \text{if } n \text{ is even,} \\ e_{12} + e_{21}, & \text{if } n \text{ is odd.} \end{cases}$$

If both n and m are odd integers then

$$([x,y]^m[b,[x,y]]_2[x,y]^n)^t = \begin{cases} \lambda^{t/2}I_2, & \text{if } t \text{ is even,} \\ \lambda^{(t-1)/2}\{-2(b_{11}-b_{22})e_{11} + 2b_{21}e_{12} \\ -2b_{21}e_{21} + 2(b_{11}-b_{22})e_{22}\}, & \text{if } t \text{ is odd} \end{cases}$$

where $\lambda = 4\{(b_{11} - b_{22})^2 - b_{21}^2\}$. Since $k \geqslant 3$, the assumption $([x,y]^m[b,[x,y]]_2 [x,y]^n)^t \in Z(R)$ implies that $\lambda = 0$ for t even and $-2(b_{11} - b_{22})\lambda^{(t-1)/2} = 0$, $2b_{21}\lambda^{(t-1)/2} = 0$ for t odd, which gives $\lambda^{(t+1)/2} = 0$, i.e., $\lambda = 0$. If n is odd and m is even then

$$([x,y]^m[b,[x,y]]_2[x,y]^n)^t = \begin{cases} (-\lambda)^{t/2} I_2, & \text{if } t \text{ is even,} \\ (-\lambda)^{(t-1)/2}\{-2b_{21}e_{11} + 2(b_{11} - b_{22})e_{12} \\ -2(b_{11} - b_{22})e_{21} + 2b_{21}e_{22}\}, & \text{if } t \text{ is odd} \end{cases}$$

is in the center of R, again implying $\lambda = 0$. Similarly, for any choice of n and m, even or odd, we get $\lambda = 0$.

By the same process as above, we obtain by choosing $x = e_{11}, y = e_{12} + e_{21}$ that $\mu = 4\{(b_{11} - b_{22})^2 + b_{21}^2\} = 0$. Hence $0 = \lambda \pm \mu$ leads to $b_{21} = 0$. Thus for any $i \neq j$, $b_{ij} = b_{ji} = 0$, i.e., b is diagonal. So let $b = \sum_{i=1}^{k} b_{ii}e_{ii}$. For any F-automorphism θ of R, b^θ enjoy the same property as b does, namely, $([x,y]^m[b^\theta,[x,y]]_2[x,y]^n)^t \in Z(R)$ for all $x, y \in R$. Hence, b^θ must also be diagonal. For each $j \neq 1$ we have $(1 + e_{1j})b(1 - e_{1j}) = \sum_{i=1}^{k} b_{ii}e_{ii} + (b_{jj} - b_{11})e_{1j}$ diagonal. Therefore, $b_{jj} = b_{11}$ and so $b \in F.I_k$. □

We are now in a position to prove our first theorem.

Theorem 2.3. *Let R be a prime ring of char $R \neq 2$, d a derivation of R and U a noncentral Lie ideal of R. If for some fixed integers $n_1 \geqslant 0, n_2 \geqslant 0, n_3 \geqslant 0$, $(u^{n_1}[d(u),u]u^{n_2})^{n_3} = 0$ for all $u \in U$, then $d = 0$.*

P r o o f. By virtue of our assumption, we can write $([d(u),u]u^{n_1+n_2})^{n_3+1} = 0$. Let $m = n_1 + n_2$ and choose an even integer $t \geqslant n_3 + 1$. Then we have $([d(u),u]u^m)^t = 0$ for all $u \in U$. For $m = 0$, the result holds true by [2, Lemma 1.1]. So we are to deal with the case $m \geqslant 1$.

Now since char $R \neq 2$ and U is noncentral, by [1, Lemma 1] $[U,U] \neq 0$ and $0 \neq [I,R] \subseteq U$, where I is the ideal of R generated by $[U,U]$. So $[I,I] \subseteq U$. Hence without loss of generality we can assume $U = [I,I]$. By our assumption we have

$$(1) \qquad ([d[x,y],[x,y]][x,y]^m)^t = 0$$

for all $x, y \in I$, which implies

$$([[d(x),y] + [x,d(y)],[x,y]][x,y]^m)^t = 0$$

for all $x, y \in I$.

If d is not Q-inner then by Kharchenko's theorem [7],

$$([[u,y]+[x,v],[x,y]][x,y]^m)^t = 0$$

for all $x,y,u,v \in I$. By Chuang [3, Theorem 2], this generalized polynomial identity (GPI) is also satisfied by Q and hence by R as well. In this case it is a polynomial identity and hence there exists a field F such that $R \subseteq M_k(F)$ and R and $M_k(F)$ satisfy the same polynomial identities [5, Theorem 2, p. 57 and Lemma 1, p. 89]. Suppose $k \geqslant 2$. If we choose $x = e_{12}, y = e_{21}, u = e_{22}, v = e_{11}$, then we get the contradiction

$$0 = ([[u,y]+[x,v],[x,y]][x,y]^m)^t = 2^t(e_{21} + (-1)^m e_{12})^t \neq 0.$$

Therefore, $k = 1$ and so R is commutative, contradicting the fact that U is noncentral.

Now if d is Q-inner, i.e., $d(x) = [b,x]$ for all $x \in R$ and for some $b \in Q$, then (1) becomes

$$([[b,[x,y]]_2[x,y]^m)^t = 0$$

for all $x,y \in I$. By Chuang [3, Theorem 2], this GPI is also satisfied by Q, i.e.,

$$f(x,y) = ([[b,[x,y]]_2[x,y]^m)^t = 0 \tag{2}$$

for all $x,y \in Q$.

In the case the center C of Q is infinite, we have $f(x,y) = 0$ for all $x,y \in Q \otimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \otimes_C \overline{C}$ are prime and centrally closed [4, Theorem 2.5 and 3.5], we may replace R by Q or $Q \otimes_C \overline{C}$ according to whether C is finite or infinite. Thus we may assume that R is centrally closed over C (i.e., $RC = R$) which is either finite or algebraically closed, and $f(x,y) = 0$ for all $x,y \in R$.

Now suppose that $d \neq 0$. Then $b \notin C$ and so the GPI $([[b,[x,y]]_2[x,y]^m)^t$ is nontrivial in R. By Martindale's theorem [10], R is then a primitive ring having nonzero socle H with C as the associated division ring. Hence by Jacobson's theorem [6, p. 75] R is isomorphic to a dense ring of linear transformations of some vector space V over C, and H consists of the linear transformations in R of finite rank. Assume first that V is finite dimensional over C. Then the density of R on V implies that $R \cong M_k(C)$ where $k = \dim_C V$. By Lemma 2.1 we have $b \in Z(R)$ implying $d = 0$, a contradiction. Assume next that V is infinite dimensional over C. Then for any $e = e^2 \in H$ we have $e\,\mathrm{Re} \cong M_k(C)$ with $k = \dim_C Ve$. Since $b \notin C$, b does not centralize the nonzero ideal H of R, so $bh_0 \neq h_0 b$ for some

$h_0 \in H$. By Litoff's theorem [9, p. 280] there exists an idempotent $e \in H$ such that h_0, $h_0 b$, bh_0 are all in $e\,\mathrm{Re}$. We have $e\,\mathrm{Re} \cong M_k(C)$ where $k = \dim_C Ve$. Since R satisfies the GPI $e([[b,[exe,eye]]_2[exe,eye]^m)^t e = 0$, the subring $e\,\mathrm{Re}$ satisfies the GPI $([[ebe,[x,y]]_2[x,y]^m)^t = 0$. Then by Lemma 2.1, $ebe \in Z(e\,\mathrm{Re})$. Thus

$$bh_0 = ebh_0 = ebeh_0 = h_0ebe = h_0be = h_0b,$$

a contradiction. Thus the proof of the theorem is complete. □

Theorem 2.4. *Let R be a prime ring of char $R \neq 2$, d a nonzero derivation of R and U a noncentral Lie ideal of R. If for some fixed integers $n_1 \geqslant 0$, $n_2 \geqslant 0$, $n_3 \geqslant 0$, $(u^{n_1}[d(u),u]u^{n_2})^{n_3} \in Z(R)$ for all $u \in U$, then R satisfies S_4, the standard identity in four variables.*

Proof. Since char $R \neq 2$ and U is noncentral, by [1, Lemma 1] there exists an ideal I of R such that $0 \neq [I,R] \subseteq U$ and $[U,U] \neq 0$. Let J be any nonzero two-sided ideal of R. Then it is easy to check that $V = [I,J^2] \subseteq U$ is a noncentral Lie ideal of R. If for each $v \in V$, $(v^{n_1}[d(v),v]v^{n_2})^{n_3} = 0$, then by Theorem 2.3 $d = 0$, which contradicts our assumption. Hence for some $v \in V$, $0 \neq (v^{n_1}[d(v),v]v^{n_2})^{n_3} \in J \cap Z(R)$, since $d(V) \subseteq J$. Thus $J \cap Z(R) \neq 0$. Now let K be a nonzero two-sided ideal of R_Z, the ring of central quotients of R. Since $K \cap R$ is a nonzero two-sided ideal of R, $(K \cap R) \cap Z(R) \neq 0$. Therefore, K contains an invertible element in R_Z and so R_Z is a simple ring with identity 1.

Moreover, without loss of generality, we may assume that $U = [I,I]$. Thus I satisfies the generalized differential identity

(3) $$[([x_1,x_2]^{n_1}[d[x_1,x_2],[x_1,x_2]][x_1,x_2]^{n_2})^{n_3},x_3].$$

If d is not Q-inner then by Kharchenko's theorem [7],

(4) $$[([x_1,x_2]^{n_1}[[y_1,x_2]+[x_1,y_2],[x_1,x_2]][x_1,x_2]^{n_2})^{n_3},x_3] = 0$$

for all $x_1,x_2,x_3,y_1,y_2 \in I$. By Chuang [3], this GPI of (4) is also satisfied by Q and hence by R as well. By localizing R at $Z(R)$, we obtain that $[([x_1,x_2]^{n_1}[[y_1,x_2]+[x_1,y_2],[x_1,x_2]][x_1,x_2]^{n_2})^{n_3},x_3]$ is also an identity of R_Z. Since R and R_Z satisfy the same polynomial identities, in order to prove that R satisfies S_4, we may assume that R is a simple ring with 1 and $[R,R] \subseteq U$. Thus R satisfies the identity (4). Now putting $y_1 = [b,x_1] = \delta(x_1)$ and $y_2 = [b,x_2] = \delta(x_2)$ for some $b \notin Z(R)$, where δ is an inner derivation induced by some $b \in R$, we obtain that R satisfies

$$[([x_1,x_2]^{n_1}[\delta[x_1,x_2],[x_1,x_2]][x_1,x_2]^{n_2})^{n_3},x_3] = 0.$$

Thus by Martindale's theorem [10], R is a primitive ring with a minimal right ideal, whose commuting ring D is a division ring which is finite dimensional over $Z(R)$. However, since R is simple with 1, R must be Artinian. Hence $R = D_{k'}$, the ring of $k' \times k'$ matrices over D, for some $k' \geqslant 1$. Again, by [8, Lemma 2], it follows that there exists a field F such that $R \subseteq M_k(F)$, the ring of $k \times k$ matrices over the field F, and $M_k(F)$ satisfies

$$[([x_1, x_2]^{n_1}[\delta[x_1, x_2], [x_1, x_2]][x_1, x_2]^{n_2})^{n_3}, x_3] = 0.$$

If $k \geqslant 3$, then by Lemma 2.2 we have $b \in Z(R)$, a contradiction. Thus $k = 2$, that is, R satisfies S_4. $\square$

Similarly, we can draw the same conclusion in the case d is a Q-inner derivation induced by some $b \in Q$.

References

[1] *J. Bergen, I. N. Herstein and J. W. Keer*: Lie ideals and derivations of prime rings. J. Algebra *71* (1981), 259–267.

[2] *L. Carini and V. D. Filippis*: Commutators with power central values on a Lie ideal. Pacific J. Math. *193* (2000), 269–278.

[3] *C. L. Chuang*: GPI's having coefficients in Utumi quotient rings. Proc. Amer. Math. Soc. *103* (1988), 723–728.

[4] *T. S. Erickson, W. S. Martindale III and J. M. Osborn*: Prime nonassociative algebras. Pacific J. Math. *60* (1975), 49–63.

[5] *N. Jacobson*: PI-algebras, an Introduction. Lecture notes in Math., 441, Springer Verlag, New York, 1975.

[6] *N. Jacobson*: Structure of Rings. Amer. Math. Soc. Colloq. Pub., 37, Amer. Math. Soc., Providence, RI, 1964.

[7] *V. K. Kharchenko*: Differential identity of prime rings. Algebra and Logic. *17* (1978), 155–168.

[8] *C. Lanski*: An engel condition with derivation. Proc. Amer. Math. Soc. *118* (1993), 731–734.

[9] *C. Lanski*: Differential identities, Lie ideals, and Posner's theorems. Pacific J. Math. *134* (1988), 275–297.

[10] *W. S. Martindale III*: Prime rings satisfying a generalized polynomial identity. J. Algebra *12* (1969), 576–584.

[11] *E. C. Posner*: Derivation in prime rings. Proc. Amer. Math. Soc. *8* (1957), 1093–1100.

Authors' addresses: Basudeb Dhara, Department of Mathematics, Indian Institute of Technology, Kharagpur, Kharagpur-721302, India, e-mail: basu_dhara@yahoo.com; R. K. Sharma, Department of Mathematics, Indian Institute of Technology, Delhi, Hauz Khas, New Delhi-110016, India, e-mail: rksharma@maths.iitd.ernet.in.