Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 58 (2008), No. 1, 93–111

Persistent URL: http://dml.cz/dmlcz/128248

Czechoslovak Mathematical Journal, 58 (133) (2008), 93–111

POSITIVE TOEPLITZ OPERATORS BETWEEN THE PLURIHARMONIC BERGMAN SPACES

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(Received November 28, 2005)

Abstract. We study Toeplitz operators between the pluriharmonic Bergman spaces for positive symbols on the ball. We give characterizations of bounded and compact Toeplitz operators taking a pluriharmonic Bergman space b^p into another b^q for $1 < p, q < \infty$ in terms of certain Carleson and vanishing Carleson measures.

Keywords: Toeplitz operators, pluriharmonic Bergman spaces, Carleson measure

MSC 2000: 47B35, 31B05

1. Introduction

Let B be the open unit ball of the complex n-space $\mathbb{C}^n$. For $1 \leqslant p < \infty$, the pluriharmonic Bergman space $b^p = b^p(B)$ is the space of all complex-valued pluriharmonic functions u on B such that

$$\|u\|_p = \left\{ \int_B |u|^p \, \mathrm{d}V \right\}^{1/p} < \infty,$$

where V denotes the normalized Lebesgue volume measure on B. It is well known that b^p is a closed subspace of $L^p = L^p(B, V)$, and hence is a Banach space. In particular, b^2 is a Hilbert space. We will write Q for the Hilbert space orthogonal projection from L^2 onto b^2. Each point evaluation is a bounded linear functional on b^2. Hence, for each $z \in B$, there exists a unique function $R_z \in b^2$ which has the reproducing property

$$u(z) = \int_B u(w) \overline{R_z(w)} \, \mathrm{d}V(w) \quad (z \in B) \tag{1.1}$$

This research was supported by KOSEF (R01-2003-000-10243-0) and Korea University Grant.

for all $u \in b^2$. More explicitly, the kernel R_z is given by

$$R_z(w) = \frac{1}{(1 - w \cdot \overline{z})^{n+1}} + \frac{1}{(1 - z \cdot \overline{w})^{n+1}} - 1 \quad (w \in B). \tag{1.2}$$

Here and subsequently $z \cdot \overline{w} = z_1\overline{w}_1 + \ldots + z_n\overline{w}_n$ denotes the Hermitian inner product for points $z, w \in \mathbb{C}^n$. Moreover, using reproducing properties, we have

$$\|R_z(w)\|_2^2 = R(z, z) \approx \frac{1}{(1 - |z|)^{n+1}}. \tag{1.3}$$

For $\varphi \in L^2$ and for each $z \in B$ we have

$$Q\varphi(z) = \int_B \varphi(w) R_z(w) \, \mathrm{d}V(w). \tag{1.4}$$

From (1.2), one can see that

$$|R(z, w)| \leqslant \frac{C}{|1 - z \cdot \overline{w}|^{n+1}} \quad (z, w \in B) \tag{1.5}$$

so that $R_z \in L^\infty$. Thus, the orthogonal projection Q can be extended to an integral operator, by means of (1.4), from L^1 into the space of all pluriharmonic functions on B. If $1 < p < \infty$, then Q is a bounded projection from L^p onto b^p. The integral transform can be extended to $\mathcal{M}$, the space of all complex Borel measures on B. Namely, for each $\mu \in \mathcal{M}$, the integral

$$Q\mu(z) = \int_B R(z, w) \, \mathrm{d}\mu(w) \quad (z \in B)$$

defines a pluriharmonic function on B. For $\mu \in \mathcal{M}$, the Toeplitz operator T_μ with symbol μ is defined by

$$T_\mu u = Q(u \, \mathrm{d}\mu)$$

for $u \in b^\infty$. In case $\mu = f \, \mathrm{d}V$, we will write $T_\mu = T_f$. Note that T_μ is densely defined on b^p for each $1 < p < \infty$.

Toeplitz operators acting on holomorphic Bergman spaces have been well studied. Especially, positive symbols of bounded and compact Toeplitz operators from a Bergman space into itself are completely characterized in terms of Carleson type measures as in [7]. The analogous characterizations for harmonic Bergman spaces on the ball have been obtained in [5] and then for the pluriharmonic Bergman space on the ball in [3]. In the setting of the upper half-space, bounded and compact positive Toeplitz operators from a Bergman space into another are characterized in [1] and the analogous results have been obtained on smooth bounded domains in [2]. In this paper, we prove analogous results for the pluriharmonic Bergman space on the ball.

In Section 2, we collect some basic facts which we need in later sections. In Section 3, we consider certain averaging functions and Berezin transforms. Section 4 is devoted to characterizations of bounded and compact Toeplitz operators in terms of Carleson and vanishing Carleson measures.

2. Preliminaries

In this section we collect some basic facts which we need in later sections.

Notation. We use the notation $A \approx B$ if $A \lesssim B$ and $B \lesssim A$ by writing $A \lesssim B$ for positive quantities A and B if the ratio A/B has a positive upper bound. Constants will be explicitly denoted by the same letter C, often with subscripts indicating dependency, which may change at each occurrence. For $1 < p < \infty$, we use p' to denote the conjugate exponent of p, i.e., $1/p + 1/p' = 1$. We also use the usual inner product notation

$$\langle u, v \rangle = \int_B u\bar{v} \, \mathrm{d}V$$

where $u\bar{v} \in L^1$.

For $z, w \in B$, $z \neq 0$, define

$$\Phi_z(w) = \frac{z - |z|^{-2}(w \cdot \bar{z})z - \sqrt{1 - |z|^2}\{w - |z|^{-2}(w \cdot \bar{z})z\}}{1 - w \cdot \bar{z}}$$

and $\Phi_0(w) = -w$. Then each Φ_z is a biholomorphic self-map of B and $\Phi_z \circ \Phi_z = \mathrm{id}$.

For $z \in B$ and r, $0 < r < 1$, the pseudohyperbolic ball $E_r(z)$ with center z and radius r is defined by

$$E_r(z) = \Phi_z(rB).$$

Since Φ_z is an involution, $w \in E_r(z)$ if and only if $|\Phi_z(w)| < r$. Note that $V(E_r(z)) \approx (1 - |z|)^{n+1}$. Recall that the well-known identity

$$1 - \Phi_z(w) \cdot \overline{\Phi_z(a)} = \frac{(1 - |z|^2)(1 - w \cdot \bar{a})}{(1 - w \cdot \bar{z})(1 - z \cdot \bar{a})}$$

holds for all $w, a \in \overline{B}$ (see Theorem 2.2.2 of [8]). In particular, it is a consequence of this that

$$|1 - w \cdot \bar{z}| \approx 1 - |z| \approx 1 - |w| \tag{2.1}$$

whenever $w \in E_r(z)$; see Lemma 2.1 of [3].

Lemma 2.1. *There exist some $r_0 \in (0,1)$ and a constant $C > 0$ such that*

$$C^{-1} \leqslant R(z,w)(1-|z|)^{n+1} \leqslant C$$

whenever $w \in E_{r_0}(z)$ and $z \in B$.

Proof. See Lemma 2.2 of [3]. □

In what follows, r_0 will always the number provided in Lemma 2.1.

Lemma 2.2. *Let $1 < p < \infty$. Then there is a constant C such that*

$$C^{-1} \leqslant \|R(z,\cdot)\|_p (1-|z|)^{(1-1/p)(n+1)} \leqslant C$$

for every $z \in B$.

Proof. See Lemma 2.3 of [3]. □

3. Averaging function and Berezin transform

For a positive Borel measure μ on B (simply $\mu \geqslant 0$) and $r \in (0,1)$, the averaging function $\hat{\mu}_r$ of μ over the pseudohyperbolic balls $E_r(z)$ is defined by

$$\hat{\mu}_r(z) = \frac{\mu(E_r(z))}{V(E_r(z))} \quad (z \in B).$$

Also, for $1 < t < \infty$, we define the Berezin t-transform $\tilde{\mu}_t$ on B by

$$\tilde{\mu}_t(z) = \int_B |r_{z,t}|^t \, \mathrm{d}\mu \quad (z \in B)$$

where

$$r_{z,t}(w) = \frac{R(z,w)}{\|R(z,\cdot)\|_t} \quad (w \in B)$$

is the L^t-normalized reproducing kernel. In case $\mathrm{d}\mu = f \, \mathrm{d}V$ for $f \in L^1$, we will write $\hat{\mu}_r = \hat{f}_r$ and $\tilde{\mu}_t = \tilde{f}_t$ for simplicity.

We start with the L^p-boundedness of the averaging operator.

Proposition 3.1. *The averaging operator $f \mapsto \hat{f}_r$ is bounded on L^p for each $1 \leqslant p \leqslant \infty$ and $0 < r < 1$.*

Proof. Clearly, the averaging operator is bounded on L^∞. So, we now assume $1 \leqslant p < \infty$ and let $0 < r < 1$. For $f \in L^p$, we have by Jensen's inequality and (2.1) that

$$\begin{aligned}\int_B |\hat{f}_r(z)|^p \,\mathrm{d}V(z) &\leqslant \int_B \left\{\frac{1}{V(E_r(z))} \int_{E_r(z)} |f(w)| \,\mathrm{d}V(w)\right\}^p \mathrm{d}V(z) \\ &\lesssim \int_B \frac{1}{(1-|z|)^{n+1}} \int_{E_r(z)} |f(w)|^p \,\mathrm{d}V(w) \,\mathrm{d}V(z) \\ &\approx \int_B \int_{E_r(z)} \frac{1}{(1-|w|)^{n+1}} |f(w)|^p \,\mathrm{d}V(w) \,\mathrm{d}V(z) \\ &\leqslant \int_B |f(w)|^p \frac{1}{(1-|w|)^{n+1}} \int_{E_r(w)} 1 \,\mathrm{d}V(z) \,\mathrm{d}V(w) \\ &\approx \int_B |f(w)|^p \,\mathrm{d}V(w).\end{aligned}$$

The proof is complete. □

We also need the following submean-value type inequality for averaging functions of positive finite Borel measures μ on B (we simply write $\mu \geqslant 0$).

Lemma 3.2. *Let $r, \varepsilon \in (0,1)$. Then there exists a constant $C_{r,\varepsilon}$ such that $\hat{\mu}_r(z) \leqslant C_{r,\varepsilon} \widehat{(\hat{\mu}_\varepsilon)}_r(z)$ for $\mu \geqslant 0$ and $z \in B$.*

Proof. See Lemma 3.1 of [3]. □

Combining the above with Proposition 3.1, we see that L^p-behavior of $\hat{\mu}_r$ of a given measure $\mu \geqslant 0$ is independent of r.

Proposition 3.3. *Let $1 \leqslant p \leqslant \infty$ and $\mu \geqslant 0$. If $\hat{\mu}_\varepsilon \in L^p$ for some $\varepsilon \in (0,1)$, then $\hat{\mu}_r \in L^p$ for all $r \in (0,1)$.*

Proof. By Lemma 3.2, we have $\hat{\mu}_r \lesssim \widehat{[\hat{\mu}_\varepsilon]}_r$ for each fixed $r, \varepsilon \in (0,1)$. Thus, the result follows from Proposition 3.1. □

Given $r \in (0,1)$ and a sequence $\{w_i\}$ in B, we say that $\{w_i\}$ is r-*separated* if the sets $E_r(w_i)$ are pairwise disjoint. Next, we need a decomposition of B whose proof is essentially the same as the ball version of that for the covering Lemma of [6]. So, we omit the details.

Lemma 3.4. *Let $r \in (0,1)$. Then there exists a sequence $\{z_i\}$ in B satisfying the following conditions:*

(a) *$\{z_i\}$ is an $r/6$-separated sequence.*

(b) $\bigcup\limits_i E_{r/3}(z_i) = B$.

(c) *There is a positive integer $N = N(n,r)$ such that each point in B belongs to at most N of the balls $E_r(z_i)$.*

Note that $|z_i| \to 1$ as $i \to \infty$. Whenever we use expressions like $\hat{\mu}_r(z_i)$ in what follows, the sequence $\{z_i\} = \{z_i(r)\}$ will always refer to the sequence chosen in Lemma 3.4.

Proposition 3.5. *Let $1 \leqslant p < \infty$ and $r, \varepsilon \in (0,1)$. Then, for any $\mu \geqslant 0$, we have $\hat{\mu}_\varepsilon \in L^p$ if and only if $\sum\limits_i |\hat{\mu}_r(z_i)|^p (1-|z_i|)^{n+1} < \infty$.*

P r o o f. First, assume $\hat{\mu}_\varepsilon \in L^p$. By Lemma 3.2 and Jensen's inequality, we have

$$\sum_i |\hat{\mu}_r(z_i)|^p (1-|z_i|)^{n+1} \lesssim \sum_i \int_{E_r(z_i)} |\hat{\mu}_\varepsilon|^p \,\mathrm{d}V \leqslant N \int_B |\hat{\mu}_\varepsilon|^p \,\mathrm{d}V < \infty,$$

where N is the positive integer provided by Lemma 3.4.

Conversely, suppose $\sum\limits_i |\hat{\mu}_r(z_i)|^p (1-|z_i|)^{n+1} < \infty$. For $z \in B$ and $w \in E_{r/3}(z)$, we note that $E_{r/3}(w) \subset E_r(z)$. It follows from (2.1) that

$$\hat{\mu}_{r/3}(w) \leqslant \hat{\mu}_r(z_i) \frac{V(E_r(z_i))}{V(E_{r/3}(w))} \lesssim \hat{\mu}_r(z_i), \quad w \in E_{r/3}(z_i)$$

for $i = 1, 2, \ldots$. Thus, we have

$$\begin{aligned}
\int_B |\hat{\mu}_{r/3}|^p \,\mathrm{d}V &\leqslant \sum_i \int_{E_{r/3}(z_i)} |\hat{\mu}_{r/3}(w)|^p \,\mathrm{d}V(w) \\
&\lesssim \sum_i |\hat{\mu}_r(z_i)|^p V(E_{r/3}(z_i)) \\
&\approx \sum_i |\hat{\mu}_r(z_i)|^p (1-|z_i|)^{n+1}.
\end{aligned}$$

So, we have $\hat{\mu}_{r/3} \in L^p$. Now, by Proposition 3.3, we have $\hat{\mu}_\varepsilon \in L^p$. The proof is complete. □

Next, we prove the L^p-boundedness of Berezin transforms. To that purpose, we need the following fact (see Proposition 1.4.10 of [8]).

Lemma 3.6. *For $-1 < \alpha < \infty$, c real and $z \in B$, let*

$$I_{\alpha,c}(z) = \int_B \frac{(1-|w|)^\alpha}{|1 - z \cdot \overline{w}|^{n+1+\alpha+c}} \,\mathrm{d}V(w)$$

for $z \in B$. Then the following estimates hold:

$$I_{\alpha,c}(z) \approx \begin{cases} 1 & \text{if } c < 0, \\ \log \dfrac{1}{1-|z|^2} & \text{if } c = 0, \\ \dfrac{1}{(1-|z|^2)^c} & \text{if } c > 0 \end{cases}$$

as $|z| \to 1$.

Proposition 3.7. *Let $1 < p \leqslant \infty$ and $1 < t < \infty$. Then the Berezin t-transform $f \mapsto \tilde{f}_t$ is bounded on L^p.*

Proof. The case $p = \infty$ is clear. Now, let $f \in L^1$. By Hölder's inequality, Lemma 2.2 and (1.5), we have

$$\begin{aligned} |\tilde{f}_t(z)|^p &\lesssim \left(\int_B \frac{(1-|z|)^{(t-1)(n+1)} |f(w)|}{|1 - z \cdot \overline{w}|^{(n+1)t}} \,\mathrm{d}V(w) \right)^p \\ &\leqslant (1-|z|)^{p(t-1)(n+1)} \left(\int_B \frac{(1-|w|)^{t/p'} |f(w)|^p}{|1 - z \cdot \overline{w}|^{(n+1)t}} \,\mathrm{d}w \right) \\ &\quad \times \left(\int_B \frac{(1-|w|)^{-t/p}}{|1 - z \cdot \overline{w}|^{(n+1)t}} \,\mathrm{d}w \right)^{p/p'} \\ &\lesssim (1-|z|)^{(t-1)(n+1)-t/p'} \int_B \frac{(1-|w|)^{t/p'} |f(w)|^p}{|1 - z \cdot \overline{w}|^{(n+1)t}} \,\mathrm{d}V(w) \end{aligned}$$

for $z \in B$ where the last inequality holds by Lemma 3.6. Thus, it follows from Fubini's theorem and Lemma 3.6 that

$$\begin{aligned} \int_B |\tilde{f}_t(z)|^p \,\mathrm{d}V(z) &\lesssim \int_B (1-|z|)^{(t-1)(n+1)-t/p'} \int_B \frac{(1-|w|)^{t/p'} |f(w)|^p}{|1 - z \cdot \overline{w}|^{(n+1)t}} \,\mathrm{d}V(w) \,\mathrm{d}V(z) \\ &= \int_B |f(w)|^p (1-|w|)^{t/p'} \int_B \frac{(1-|z|)^{(t-1)(n+1)-t/p'}}{|1 - z \cdot \overline{w}|^{(n+1)t}} \,\mathrm{d}V(z) \,\mathrm{d}V(w) \\ &\lesssim \int_B |f(w)|^p \,\mathrm{d}V(w). \end{aligned}$$

The proof is complete. □

We now turn to relations between L^p-behavior of averaging functions and Berezin transforms. We first prove the following.

Lemma 3.8. *Given $r \in (0,1)$, there is a constant C_r such that*

$$\int_B f \,\mathrm{d}\mu \leqslant C_r \int_B f \hat{\mu}_r \,\mathrm{d}V$$

for all $f \geqslant 0$ subharmonic on B and $\mu \geqslant 0$.

Proof. Fix $r \in (0,1)$. Let $\mu \geqslant 0$ and f be a positive subharmonic function. By subharmonicity and Fubini's theorem, we have

$$\int_B f(z)\,\mathrm{d}\mu(z) \leqslant \int_B \frac{1}{V(E_r(z))} \int_{E_r(z)} f(w)\,\mathrm{d}V(w)\,\mathrm{d}\mu(z) \lesssim \int_B f(w)\widehat{\mu_r}(w)\,\mathrm{d}V(w).$$

The proof is complete. □

Lemma 3.9. *Let $r \in (0,1)$ and $1 < t < \infty$, there exists a constant $C = C_{r,t}$ such that $\hat{\mu}_r \leqslant C\tilde{\mu}_t$ for any $\mu \geqslant 0$.*

Proof. See Lemma 3.3 of [3]. □

Proposition 3.10. *Let $r \in (0,1)$ and $1 < t < \infty$. Suppose $\mu \geqslant 0$ and $1 < p \leqslant \infty$. Then, $\hat{\mu}_r \in L^p$ if and only if $\tilde{\mu}_t \in L^p$.*

Proof. First, suppose $\hat{\mu}_r \in L^p$. Applying Lemma 3.8 to functions $f = |r_{z,t}|^t$, we obtain $\tilde{\mu}_t \lesssim \widetilde{[\hat{\mu}_r]}_t$ and thus $\tilde{\mu}_t \in L^p$ by Proposition 3.7. Conversely, if $\tilde{\mu}_t \in L^p$, then by Lemma 3.9, we have $\hat{\mu}_r \in L^p$ for $0 < r < 1$. This completes the proof. □

4. Carleson measure

To characterize Toeplitz operators, we need the notion of certain Carleson measures. Let $1 < p, q < \infty$. Given $\mu \geqslant 0$, we say that μ is a (p,q)-*Carleson measure* if there exists a constant C such that

$$\left\{\int_B |f|^q \,\mathrm{d}\mu\right\}^{1/q} \leqslant C\|f\|_p$$

for all $f \in b^p$. In other words, μ is a (p,q)-Carleson measure if and only if the inclusion $i_{p,q}\colon b^p \to L^q(\mu)$ is bounded. Carleson measures in various settings have been well studied as in [5], [7] and the references therein. In this section, we also characterize (p,q)-Carleson measures in terms of L^p-behavior of the averaging functions and Berezin transforms. We first consider the case where $p \leqslant q$.

Theorem 4.1. *Assume $1 < p \leqslant q < \infty$, $s = p/q$, $1/s < t < \infty$ and $\varepsilon, r \in (0,1)$. Suppose $\mu \geqslant 0$. Then the following conditions are all equivalent:*

(1) *μ is a (p,q)-Carleson measure.*

(2) $\sup\limits_{z\in B} \tilde{\mu}_t(z)(1-|z|)^{(n+1)(1-1/s)} < \infty$.

(3) $\sup\limits_{z\in B} \hat{\mu}_\varepsilon(z)(1-|z|)^{(n+1)(1-1/s)} < \infty$.

(4) $\sup\limits_{i} \hat{\mu}_r(z_i)(1-|z_i|)^{(n+1)(1-1/s)} < \infty$.

Proof. First, suppose (1) and show (2) with $t = q$. Let $z \in B$. By Lemma 2.2 we have

$$|r_{z,p}|^q = \Big(\frac{\|R(z,\cdot)\|_q}{\|R(z,\cdot)\|_p}\Big)^q |r_{z,q}|^q \approx (1-|z|)^{n(1-q/p)}|r_{z,q}|^q.$$

Integrating with respect to $\mathrm{d}\mu$, we obtain

$$\tilde{\mu}_q(z)(1-|z|)^{(n+1)(1-q/p)} \approx \int_B |r_{z,p}|^q \,\mathrm{d}\mu. \tag{4.1}$$

Since $\|r_{z,p}\|_p = 1$ and μ is a (p,q)-Carleson measure, the above shows that (2) holds for $t = q$.

Next, by Lemma 3.9, we have (2) $\Rightarrow$ (3).

The implication (3) $\Rightarrow$ (4) can also be easily seen from Lemma 3.2.

Now, suppose (4) and show (1). Let $u \in b^p$. Since $|u|^p$ is plurisubharmonic, we have

$$|u(w)|^p \lesssim \frac{1}{(1-|w|)^{n+1}} \int_{E_{r/3}(w)} |u|^p \,\mathrm{d}V$$

for $w \in B$. This, together with Lemma 2.1, yields

$$\begin{aligned}\sup_{w\in E_{r/3}(z)} |u(w)|^p &\leqslant \sup_{w\in E_{r/3}(z)} \frac{1}{(1-|w|)^{n+1}} \int_{E_{r/3}(w)} |u|^p \,\mathrm{d}V \\ &\lesssim \frac{1}{(1-|z|)^{n+1}} \int_{E_r(z)} |u|^p \,\mathrm{d}V\end{aligned}$$

for all $z \in B$. Hence, we have

$$\begin{aligned}\int_{E_{r/3}(z)} |u|^q \,\mathrm{d}\mu &\lesssim \frac{\mu(E_r(z))}{(1-|z|)^{q(n+1)/p}} \Big\{\int_{E_r(z)} |u|^p \,\mathrm{d}V\Big\}^{q/p} \\ &\approx \hat{\mu}_r(z)(1-|z|)^{(n+1)(1-q/p)} \Big\{\int_{E_r(z)} |u|^p \,\mathrm{d}V\Big\}^{q/p}\end{aligned}$$

for $z \in B$. Note that $q/p \geqslant 1$. Let $M = \sup_i \hat{\mu}_r(z_i)(1-|z_i|)^{(n+1)(1-1/s)}$. It follows from Lemma 3.4 that

$$\begin{aligned}
\int_B |u|^q \,\mathrm{d}\mu &\leqslant \sum_i \int_{E_{r/3}(z_i)} |u|^q \,\mathrm{d}\mu \\
&\lesssim \sum_i \hat{\mu}_r(z_i)(1-|z_i|)^{(n+1)(1-1/s)} \biggl\{ \int_{E_r(z_i)} |u|^p \,\mathrm{d}V \biggr\}^{q/p} \\
&\leqslant M \sum_i \biggl\{ \int_{E_r(z_i)} |u|^p \,\mathrm{d}V \biggr\}^{q/p} \\
&\leqslant M \biggl\{ \sum_i \int_{E_r(z_i)} |u|^p \,\mathrm{d}V \biggr\}^{q/p} \\
&\leqslant N^{q/p} M \|u\|_p^q,
\end{aligned}$$

where N is the number provided by Lemma 3.4. Hence, μ is a (p,q)-Carleson measure, as desired.

Finally, suppose (1) and show (2) for general t. We have seen that Condition (1) is equivalent to Condition (3), which does not depend on particular values of p and q, but depends on the ratio $s = p/q$. Therefore, we may take $p = st > 1$ and $q = t$ in order to see that (1) implies (2) for general t. The proof is complete. $\square$

Let $1 < p < \infty$ and $\{z_i\}$ be a sequence in B. For a sequence $\lambda = \{\lambda_i\} \in \ell^p$, we let $S(\lambda)$ be the function defined by

$$S(\lambda)(z) = \sum \lambda_i (1-|z_i|)^{(n+1)(1-1/p)} R(z, z_i), \quad z \in B. \tag{4.2}$$

Proposition 4.2. *Let $1 < p < \infty$. Then $S\colon \ell^p \to b^p$ is bounded whenever $\{z_i\}$ is r-separated for some r.*

Proof. Note that for $z, w, z_i \in \overline{B}$, we have

$$|1 - z \cdot \overline{w}|^{1/2} \leqslant |1 - z \cdot \overline{z}_i|^{1/2} + |1 - z_i \cdot \overline{w}|^{1/2}$$

by Proposition 5.1.2 of [8]. This, together with (2.1), gives the following estimate

$$|1 - z \cdot \overline{w}| \lesssim |1 - z \cdot \overline{z}_i|, \quad w \in E_r(z_i).$$

Therefore, by (1.5) we have

$$|R(z, z_i)| \lesssim \frac{1}{|1 - z \cdot \overline{z}_i|^{n+1}} \lesssim \frac{1}{|1 - z \cdot \overline{w}|^{n+1}}$$

for all $w \in E_r(z_i)$ and $z \in B$. It follows that

$$|S(\lambda)(z)| \lesssim \sum |\lambda_i|(1-|z_i|)^{(n+1)(1-1/p)} V(E_r(z_i))^{-1} \int_{E_r(z_i)} \frac{1}{|1-z\cdot\overline{w}|^{n+1}}\,\mathrm{d}V(w)$$

for $z \in B$. Thus, setting

$$u = \sum |\lambda_i|(1-|z_i|)^{(n+1)(1-1/p)} V(E_r(z_i))^{-1}\chi_{E_r(z_i)},$$

we have

$$|S(\lambda)| \lesssim \int_B \frac{u(w)}{|1-z\cdot\overline{w}|^{n+1}}\,\mathrm{d}V(w).$$

For $u \in L^1$, we define Ψ by

$$\Psi u(z) = \int_B \frac{u(w)}{|1-z\cdot\overline{w}|^{n+1}}\,\mathrm{d}V(w), \quad z \in B.$$

Note that Ψ is bounded on L^p. Thus, since $\{z_i\}$ is r-separated, we have

$$\|S(\lambda)\|_p^p \lesssim \|u\|_p^p = \sum |\lambda_i|^p(1-|z_i|)^{(n+1)(p-1)} V(E_r(z_i))^{1-p} \approx \sum |\lambda_i|^p,$$

which shows that $S\colon \ell^p \to L^p$ is bounded and the series in (4.2) converges in norm. Since each term is pluriharmonic, the series converges uniformly on every compact subsets of B. It follows that S maps ℓ^p into b^p. The proof is complete. $\square$

In order to characterize (p,q)-Carleson measures for $q < p$, we will utilize Luecking's idea in [4]. To do so, we first need Khinchine's inequality. Recall that the Rademacher functions ψ_i are defined by

$$\psi_0(t) = \begin{cases} 1 & \text{if } 0 \leqslant t-[t] < 1/2, \\ -1 & \text{if } 1/2 \leqslant t-[t] < 1, \end{cases}$$

and $\psi_i(t) = \psi_0(2^i t)$ for positive integers i. Then Khinchine's inequality is the following.

Lemma 4.3 (Khinchine's inequality)**.** *For $0 < p < \infty$, there exists a constant C_p such that*

$$C_p^{-1}\Big(\sum_{k=1}^m |\lambda_k|^2\Big)^{p/2} \leqslant \int_0^1 \Big|\sum_{k=1}^m \lambda_k\psi_k(t)\Big|^p\,\mathrm{d}t \leqslant C_p\Big(\sum_{k=1}^m |\lambda_k|^2\Big)^{p/2}$$

for all $m \geqslant 1$ and complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_m$.

We now characterize (p,q)-Carleson measures for the case $q < p$.

Theorem 4.4. *Let $1 < q < p < \infty$, $s = p/q$, $1 < t < \infty$ and $r, \varepsilon \in (0,1)$. Suppose $\mu \geqslant 0$. Then the following conditions are equivalent:*

(1) *μ is a (p,q)-Carleson measure.*

(2) $\sum\limits_i |\hat{\mu}_r(z_i)|^{s'}(1-|z_i|)^{(n+1)} < \infty$.

(3) $\hat{\mu}_\varepsilon \in L^{s'}$.

(4) $\tilde{\mu}_t \in L^{s'}$.

Proof. Assume (1) and show (2). First, consider $r = r_0$. Corresponding to each $\{\lambda_i\} \in \ell^p$, we put

$$u = \sum \lambda_i (1-|z_i|)^{(n+1)(1-1/p)} R(\cdot, z_i).$$

Since $\{z_i\}$ is $r/6$-separated, by Proposition 4.2, we have $\|u\|_p \lesssim \|(\lambda_i)\|_{\ell^p}$ and hence, by Assumption (1),

$$\int_B \Big|\sum \lambda_i (1-|z_i|)^{(n+1)(1-1/p)} R(w, z_i)\Big|^q \, \mathrm{d}\mu(w) \lesssim \Big(\sum |\lambda_i|^p\Big)^{q/p}.$$

In the above inequality, replace λ_i with $\psi_i(t)\lambda_i$ and then integrate with respect to t from 0 to 1. Then, after making use of Fubini's theorem and Khinchine's inequality, the result becomes

$$\int_B \Big(\sum |\lambda_i|^2 (1-|z_i|)^{2(n+1)(1-1/p)} |R(w,z_i)|^2\Big)^{q/2} \, \mathrm{d}\mu(w) \lesssim \Big(\sum |\lambda_i|^p\Big)^{q/p}. \tag{4.3}$$

Since $\chi_{E_r(z_i)} \lesssim (1-|z_i|)^{(n+1)} |R(\cdot, z_i)|$ for all i by Lemma 2.1, it follows from (4.3) that

$$\begin{aligned}
&\sum |\lambda_i|^q \hat{\mu}_r(z_i)(1-|z_i|)^{(n+1)(1-q/p)} \\
&\qquad \lesssim \int_B \sum |\lambda_i|^q (1-|z_i|)^{-(n+1)q/p} \chi_{E_r(z_i)} \, \mathrm{d}\mu \\
&\qquad \lesssim \max\{N^{1-q/2}, 1\} \int_B \Big(\sum |\lambda_i|^2 \chi_{E_r(z_i)} (1-|z_i|)^{-2(n+1)/p}\Big)^{q/2} \, \mathrm{d}\mu \\
&\qquad \lesssim \int_B \Big(\sum |\lambda_i|^2 (1-|z_i|)^{2(n+1)(1-1/p)} |R(w,z_i)|^2\Big)^{q/2} \, \mathrm{d}\mu(w) \\
&\qquad \lesssim \Big(\sum |\lambda_i|^p\Big)^{q/p},
\end{aligned}$$

where N is the number provided in Lemma 3.4. This shows that

$$\sum |b_i| \hat{\mu}_r(z_i)(1-|z_i|)^{(n+1)/s'} \lesssim \Big(\sum |b_i|^s\Big)^{1/s}$$

for all $\{b_i\} \in \ell^s$. Thus, a duality argument yields (2) for $r = r_0$. Now, an application of Proposition 3.5 shows that (2) holds for a given r.

The implication (2) $\Rightarrow$ (3) is also a consequence of Proposition 3.5.

Now, assume (3) and show (1). Using Lemma 3.8 and Hölder's inequality, we have

$$\int_B |u|^q \,\mathrm{d}\mu \lesssim \int_B |u|^q \hat{\mu}_\varepsilon \,\mathrm{d}V \leqslant \|u\|_p^q \|\hat{\mu}_\varepsilon\|_{s'}$$

for $u \in b^p$ so that (1) holds.

Finally, the equivalence (3) $\Leftrightarrow$ (4) is a consequence of Proposition 3.10. The proof is complete. $\square$

For $\mu \geqslant 0$ and $1 < p, q < \infty$, we say that μ is a *vanishing* (p,q)*-Carleson measure* if the inclusion $i_{p,q}\colon\ b^p \to L^q(\mu)$ is compact, or equivalently, if

$$\int_B |u_j|^q \,\mathrm{d}\mu \to 0$$

whenever $u_j \to 0$ weakly in b^p. Note that the kernels $r_{z,p}$ converges to 0 weakly in b^p as $|z| \to 1$ for each $1 < p < \infty$; see Lemma 3.10 of [3].

Now, we characterize vanishing (p,q)-Carleson measures. We first consider the case $p \leqslant q$.

Theorem 4.5. *Let $\mu \geqslant 0$. Assume $1 < p \leqslant q < \infty$, $s = p/q$, $1/s < t < \infty$, and $r, \varepsilon \in (0,1)$. Then the following conditions are equivalent:*

(1) *μ is a vanishing (p,q)-Carleson measure.*

(2) *$\tilde{\mu}_t(z)(1-|z|)^{(n+1)(1-1/s)} \to 0$ as $|z| \to 1$.*

(3) *$\hat{\mu}_\varepsilon(z)(1-|z|)^{(n+1)(1-1/s)} \to 0$ as $|z| \to 1$.*

(4) *$\hat{\mu}_r(z_i)(1-|z_i|)^{(n+1)(1-1/s)} \to 0$ as $i \to \infty$.*

Proof. First, suppose (1) and show (2) with $t = q$. Since $r_{z,p} \to 0$ weakly in b^p as $|z| \to 1$, it follows from (4.1) that (2) holds for $t = q$.

Next, by Lemma 3.9, we have (2) $\Rightarrow$ (3) for a given ε.

The implication (3) $\Rightarrow$ (4) follows from Lemma 3.2 as before, because $|z_i| \to 1$ as $i \to \infty$.

Now, assume (4) and show (1). Let $\{u_k\}$ be a sequence converging to 0 weakly in b^p. Let $M_j = \sup\limits_{i \geqslant j} \hat{\mu}_r(z_i)(1-|z_i|)^{(n+1)(1-1/s)}$ for positive integers j. By the proof of (4) $\Rightarrow$ (1) of Theorem 4.1, we have

$$\int_B |u_k|^q \,\mathrm{d}\mu \lesssim \sum_{i<j} \int_{E_r(z_i)} |u_k|^q \,\mathrm{d}\mu + M_j \|u_k\|_p^q \tag{4.4}$$

for each j and k. Since $u_k \to 0$ weakly in b^p, one can easily see that $u_k \to 0$ uniformly on compact subsets of B and $\{u_k\}$ is bounded in L^p-norm. Thus, fixing j and taking the limit $k \to \infty$ in (4.4), we obtain

$$\limsup_k \int_B |u_k|^q \, \mathrm{d}\mu \lesssim M_j$$

for each j. Note that we have $M_j \to 0$ as $j \to \infty$ by assumption. Thus, taking the limit $j \to \infty$, we conclude that

$$\limsup_k \int_B |u_k|^q \, \mathrm{d}\mu = 0.$$

Namely, μ is a vanishing (p,q)-Carleson measure, as desired.

Finally, as in the proof of Theorem 4.1, one can see that (1) implies (2) for general t. The proof is complete. □

The case $p > q$ is a little bit more subtle and we have the following.

Theorem 4.6. *Let $\mu \geqslant 0$ and assume $1 < q < p < \infty$. Then the following conditions are equivalent:*

(1) *μ is a (p,q)-Carleson measure.*

(2) *μ is a vanishing (p,q)-Carleson measure.*

Proof. We only need to prove (1) $\Rightarrow$ (2). So assume μ is a (p,q)-Carleson measure. Take any sequence $\{u_j\}$ converging weakly to 0 in b^p. Then $\{u_j\}$ is a bounded sequence in b^p and $u_j \to 0$ on each compact subset of B. Let K be any compact subset of B and μ_K be the restriction of μ to $B \setminus K$. Let $s = p/q$ and fix $\varepsilon \in (0,1)$. Then, as in the proof of (3) $\Rightarrow$ (1) of Theorem 4.4, we have

$$\int_{B\setminus K} |u_j|^q \, \mathrm{d}\mu = \int_B |u_j|^q \, \mathrm{d}\mu_K \lesssim \|u_j\|_p^q \|\hat{\mu}_{K,\varepsilon}\|_{s'}$$

for all j. Therefore, letting $M = \sup_j \|u_j\|_p^q < \infty$, we have by assumption

$$\int_B |u_j|^q \, \mathrm{d}\mu \lesssim \left\{ \int_K |u_j|^p \, \mathrm{d}V \right\}^{q/p} + M \|\hat{\mu}_{K,\varepsilon}\|_{s'}.$$

Take the limit $j \to \infty$. Since u_j converges to 0 uniformly on K, we have

$$\limsup_j \int_B |u_j|^q \, \mathrm{d}\mu \lesssim M \|\hat{\mu}_{K,\varepsilon}\|_{s'}.$$

Note that $\hat{\mu}_{K,\varepsilon} \to 0$ as K increases to B. Also, we have $|\hat{\mu}_{K,\varepsilon}|^{s'} \leqslant |\hat{\mu}_\varepsilon|^{s'} \in L^1$ by Theorem 4.4. Thus, an application of the dominated convergence theorem yields

$$\limsup_j \int_B |u_j|^q \,\mathrm{d}\mu = 0,$$

and therefore μ is a vanishing (p,q)-Carleson measure. This completes the proof. $\square$

By Theorem 4.1 and Theorem 4.4, the notion of (vanishing) (p,q)-Carleson measures depends only on the ratio p/q. Thus we will simply say (vanishing) s-Carleson measures in what follows. Having characterized s-Carleson measures, we turn to characterizations of positive Toeplitz operators. In characterizing bounded positive Toeplitz operators, the key step is the justification of the equality

$$\langle T_\mu u, v\rangle = \int_B u\bar{v}\,\mathrm{d}\mu, \quad u,v \in b^\infty, \tag{4.5}$$

which enables us to make a connection between Carleson measures and positive Toeplitz operators.

Lemma 4.7. *Let $\mu \geqslant 0$, $u,v \in b^\infty$. If $T_\mu u \in b^1$, then we have* (4.5).

P r o o f. Note that the kernel function has the symmetry

$$R(rz,w) = R(z,rw), \quad z,w \in B, \ 0 \leqslant r \leqslant 1$$

and $R(z,w)$ is bounded on $|z| \leqslant r < 1$. Let $\mu \geqslant 0$ and assume that $T_\mu u \in b^1$. Thus, for $u,v \in b^\infty$, we have by Fubini's theorem and the dominated convergence theorem

$$\begin{aligned}
\langle T_\mu u, v\rangle &= \lim_{r\to1} \int_{rB} \overline{v(z)} \int_B u(w)R(z,w)\,\mathrm{d}\mu(w)\,\mathrm{d}V(z)\\
&= \lim_{r\to1} \int_B u(w) \int_{rB} \overline{v(z)}R(z,w)\,\mathrm{d}V(z)\,\mathrm{d}\mu(w)\\
&= \lim_{r\to1} r^{2n} \int_B u(w) \int_B \overline{v(rz)}R(rz,w)\,\mathrm{d}V(z)\,\mathrm{d}\mu(w)\\
&= \lim_{r\to1} r^{2n} \int_B u(w) \int_B \overline{v(rz)}R(z,rw)\,\mathrm{d}V(z)\,\mathrm{d}\mu(w)\\
&= \lim_{r\to1} r^{2n} \int_B u(w)\overline{v(r^2w)}\,\mathrm{d}\mu(w) = \int_B u\bar{v}\,\mathrm{d}\mu.
\end{aligned}$$

This completes the proof. $\square$

We now characterize bounded (resp. compact) Toeplitz operators in terms of (resp. vanishing) s-Carleson measures. We first consider the case $p \leqslant q$.

Theorem 4.8. *Let $1 < p \leqslant q < \infty$, $1/s = 1 - 1/q + 1/p$ and $\mu \geqslant 0$. Then the following conditions are equivalent:*
(1) *$T_\mu\colon b^p \to b^q$ is bounded (resp. compact).*
(2) *μ is an s-Carleson measure (resp. vanishing).*

Proof. Assume (1) and show (2). First, assume that $T_\mu\colon b^p \to b^q$ is bounded. Let $z \in B$ and take $r = r_0$, where r_0 is the number provided by Lemma 2.1. Then we have

$$\mu(E_r(z)) \lesssim (1-|z|)^{2(n+1)} \int_B |R(z,w)|^2 \, \mathrm{d}\mu(w) = (1-|z|)^{2(n+1)} T_\mu[R(z,\cdot)](z),$$

and therefore

$$\hat{\mu}_r(z) \lesssim (1-|z|)^{n+1} T_\mu[R(z,\cdot)](z) \approx (1-|z|)^{(n+1)(1/p)} T_\mu r_{z,p}(z).$$

On the other hand, since point evaluation is continuous on b^q, we have

$$|T_\mu r_{z,p}(z)| \lesssim (1-|z|)^{-(n+1)/q} \|T_\mu r_{z,p}\|_q.$$

Combining these estimates, we have

$$\hat{\mu}_r(z)(1-|z|)^{(n+1)(1-1/s)} \lesssim \|T_\mu r_{z,p}\|_q. \tag{4.6}$$

Now, since $\|r_{z,p}\|_p = 1$, we see that

$$\hat{\mu}_r(z)(1-|z|)^{(n+1)(1-1/s)} \lesssim \|T_\mu\|,$$

where $\|T_\mu\|$ denote the operator norm of $T_\mu\colon b^p \to b^q$. This is true for all $z \in B$ and the constants abbreviated above are independent of z. Hence, μ is an s-Carleson measure by Theorem 4.1.

Recall that $r_{z,p} \to 0$ weakly in b^p as $|z| \to 1$. Hence, if $T_\mu\colon b^p \to b^q$ is compact, then we have by (4.6)

$$\hat{\mu}_r(z)(1-|z|)^{(n+1)(1-1/s)} \lesssim \|T_\mu r_{z,p}\|_q \to 0$$

as $|z| \to 1$. Hence, μ is a vanishing s-Carleson measure by Theorem 4.5.

Now, assume (2) and show (1). First, assume μ is an s-Carleson measure. Note that the function $w \mapsto \int_B |R(z,w)| \, \mathrm{d}V(z)$ is subharmonic on B. Fix $r \in (0,1)$. Since $s < 1$, $\hat{\mu}_r(w) \lesssim (1-|w|^2)^{(n+1)(1/s-1)} \lesssim 1$ by Theorem 4.1. Thus, by (1.5) and Lemma 3.8, we have

$$\begin{aligned} \int_B\int_B |R(z,w)| \, \mathrm{d}V(z) \, \mathrm{d}\mu(w) &\lesssim \int_B \hat{\mu}_r(w) \int_B \frac{1}{|1-z\cdot\overline{w}|^{n+1}} \, \mathrm{d}V(z) \, \mathrm{d}V(w) \\ &\lesssim \int_B\int_B \frac{1}{|1-z\cdot\overline{w}|^{n+1}} \, \mathrm{d}V(z) \mathrm{d}V(w). \end{aligned} \tag{4.7}$$

We therefore have by Lemma 3.6, for $u \in b^\infty$

$$\|T_\mu u\|_1 \lesssim \|u\|_\infty \int_B \int_B \frac{1}{|1 - z \cdot \overline{w}|^{n+1}} \,\mathrm{d}V(z)\,\mathrm{d}V(w) < \infty.$$

Since $1/s = 1/q' + 1/p$, we note that p/s is the conjugate exponent of q'/s. Hence, for $u, v \in b^\infty$, we have by Lemma 4.7 and Hölder's inequality

$$(4.8)\quad |\langle T_\mu u, v\rangle| = \left|\int_B u\overline{v}\,\mathrm{d}\mu\right| \leqslant \left\{\int_B |u|^{p/s}\,\mathrm{d}\mu\right\}^{s/p} \left\{\int_B |v|^{q'/s}\,\mathrm{d}\mu\right\}^{s/q'} \lesssim \|u\|_p \|v\|_{q'}.$$

The last inequality holds by our assumption μ is an s-Carleson measure. Now, a duality argument shows the boundedness of $T_\mu\colon b^p \to b^q$, because b^∞ is dense in b^p.

Next, assume μ is a vanishing s-Carleson measure. Let $\{u_j\}$ be a sequence of functions such that $u_j \to 0$ weakly in b^p. Then we have

$$\int_B |u_j|^{p/s}\,\mathrm{d}\mu \to 0.$$

Hence, by (4.8) and a duality argument, we obtain

$$\|T_\mu u_j\|_q \lesssim \left\{\int_B |u_j|^{p/s}\,\mathrm{d}\mu\right\}^{s/p} \to 0.$$

Therefore, $T_\mu\colon b^p \to b^q$ is compact. The proof is complete. □

Now, we turn to the case $q < p$. In this case we will prove that bounded Toeplitz operators are all compact.

Lemma 4.9. *Let $s > 1$ and $\mu \geqslant 0$. Then the following conditions are equivalent:*
(1) *$T_\mu\colon b^p \to b^q$ is bounded whenever $1 < q < p < \infty$ and $1/s = 1 - 1/q + 1/p$.*
(2) *$T_\mu\colon b^{2s} \to b^{2s/(2s-1)}$ is bounded.*
(3) *μ is an s-Carleson measure.*

Proof. The implication (1) ⇒ (2) is trivial.

Assume (2) and show (3). Let $p = 2s$. Then, by the proof of (1) ⇒ (2) of Theorem 4.8, we have

$$\hat{\mu}_r(z)(1 - |z|)^{(n+1)(1-1/s)} \lesssim \|T_\mu\|$$

for any $z \in B$, where $\|T_\mu\|$ denotes the operator norm of $T_\mu\colon b^{2s} \to b^{2s/(2s-1)}$. Hence, μ is an s-Carleson measure.

Finally, assume μ is an s-Carleson measure. Let $u, v \in b^\infty$. We only need to show that $T_\mu u \in b^1$. The rest of the proof is exactly the same as the proof of (2) ⇒

(1) of Theorem 4.8. Note that (4.7) is still available. It follows from Lemma 3.6, Theorem 4.4 and Hölder's inequality that

$$\begin{aligned}
&\int_B \hat{\mu}_r(w) \int_B \frac{1}{|1 - z \cdot \overline{w}|^{n+1}} \, \mathrm{d}V(z) \, \mathrm{d}V(w) \\
&\qquad \lesssim \int_B \hat{\mu}_r(w) \log \frac{1}{1 - |w|^2} \, \mathrm{d}V(w) \\
&\qquad \leqslant \|\hat{\mu}_r\|_{s'} \Big\{ \int_B \Big(\log \frac{1}{1 - |w|^2} \Big)^s \, \mathrm{d}V(w) \Big\}^{1/s} < \infty.
\end{aligned}$$

So, we have the implication (3) $\Rightarrow$ (1). The proof is complete. □

Theorem 4.10. *Let $1 < q < p < \infty$ and $s > 1$. Then the following conditions are equivalent:*

(1) $T_\mu\colon b^p \to b^q$ *is compact whenever* $1/s = 1 - 1/q + 1/p$.
(2) $T_\mu\colon b^p \to b^q$ *is bounded whenever* $1/s = 1 - 1/q + 1/p$.
(3) $T_\mu\colon b^{2s} \to b^{2s/(2s-1)}$ *is compact.*
(4) $T_\mu\colon b^{2s} \to b^{2s/(2s-1)}$ *is bounded.*
(5) μ *is a vanishing s-Carleson measure.*
(6) μ *is an s-Carleson measure.*

Proof. By Lemma 4.9 we have the equivalences (2) $\Leftrightarrow$ (4) $\Leftrightarrow$ (6). By Theorem 4.6, we have the equivalence (5) $\Leftrightarrow$ (6). The implications (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are trivial. Also, we have (5) $\Rightarrow$ (1), as in the proof of (2) $\Rightarrow$ (1) of Theorem 4.8. The proof is complete. □

Ackowledgement. The author gratefully thanks professor Boo Rim Choe for several helpful comments about the material in this paper.

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