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Dagmar Medková

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# THE NEUMANN PROBLEM FOR THE LAPLACE EQUATION ON GENERAL DOMAINS

Dagmar Medková, Praha

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*Abstract.* The solution of the weak Neumann problem for the Laplace equation with a distribution as a boundary condition is studied on a general open set $G$ in the Euclidean space. It is shown that the solution of the problem is the sum of a constant and the Newtonian potential corresponding to a distribution with finite energy supported on $\partial G$. If we look for a solution of the problem in this form we get a bounded linear operator. Under mild assumptions on $G$ a necessary and sufficient condition for the solvability of the problem is given and the solution is constructed.

*Keywords*: Laplace equation, Neumann problem, potential, boundary integral equation method

*MSC 2000*: 31B10, 35J05

## 1. Introduction

The boundary integral equation method is very useful in studying boundary value problems. It is used to look for a solution of the Neumann problem for the Laplace equation in the form of a single layer potential. The original problem is transferred to the problem $Tf = g$, where $g$ is the boundary condition, $f$ is an unknown density of the single layer potential and the integral operator $T$ is a Fredholm operator with index 0 on the space of boundary conditions.

First, we must know that the corresponding integral operator is bounded on the space of boundary conditions. Therefore, the choice of the space of boundary conditions restricts our choice of a class of open sets. If we look for a classical solution we choose open sets with Ljapunov boundary and $\alpha$-Hölder functions on the boundary

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as the space of boundary conditions. It is usual to deal with open sets $G \subset \mathbb{R}^m$ with Lipschitz boundary if we look for a solution in the sense of the nontangential limits for boundary conditions from $L_p(\partial G)$ (a class of strong solutions). Necessary and sufficient conditions for the corresponding integral operator to be bounded on $L_p(\partial G)$ are unknown. Nevertheless, the surface measure must make sense. So, it is natural to restrict ourselves to the case when the $(m-1)$-dimensional Hausdorff measure of the boundary is finite. If we study weak solutions with real measures on the boundary as boundary conditions then the corresponding integral operator is a bounded linear operator on the space of real measures on the boundary if and only if the cyclic variation of the domain is bounded (see [11]). The class of such open sets is very rich but most of the sets with $C^1$ boundary are not included (see [18]). For open sets with Lipschitz boundary the Neumann problem for the Laplace equation with a boundary condition from the Sobolev space $H^s$ is studied, too. But to study the Neumann problem with a boundary condition from $H^s(\partial G)$ for a general open set $G$ might be a bit problematic. O. Steinbach and W. Wendland studied in [23] the Neumann problem for a class of elliptic systems including the Laplace equation for open sets $G \subset \mathbb{R}^3$ with compact connected Lipschitz boundary and boundary conditions from the Sobolev space $H^{-1/2}(\partial G)$. They equipped the space $H^{-1/2}(\partial G)$ with a special norm equivalent to the original norm and proved that the corresponding integral operator has very nice properties in this space. For such open sets $G$ the space $H^{-1/2}(\partial G)$ is precisely the space of all distributions with finite energy supported on the boundary and Steinbach-Wendland's norm is the energy norm. The space of all distributions with finite energy supported on the boundary of $G$ equipped with the energy norm is a well-defined Hilbert space for each open set $G$. In this paper we shall study for which open sets $G$ the integral operator corresponding to the Neumann problem for the Laplace equation is bounded on the space of all distributions with finite energy supported on the boundary of $G$. Surprisingly, for all.

Now we have another question. Is this space of boundary conditions sufficiently rich? Is this space nontrivial for each open set with compact boundary? If $G$ is an open set with compact boundary then the space of all distributions with finite energy supported on $\partial G$ is trivial if and only if the Newtonian capacity of $\partial G$ is zero. In this case the Lebesgue measure of the complement of $G$ is equal to zero. Since the cyclic variation of such open set $G$ is bounded we can use the integral equation method for real measures on the boundary of $G$ as boundary values (see [11]). The corresponding integral operator is the identity operator. Therefore the single layer potential corresponding to the boundary condition is a solution of the Neumann problem for the Laplace equation. Thus, the open sets for which the space of all distributions with finite energy supported on the boundary is trivial are not interesting because we can solve the Neumann problem for such open sets.

The space of all distributions with finite energy supported on the boundary is relatively rich for reasonable open sets. If $G$ has compact locally Lipschitz boundary then the space of all distributions with finite energy supported on $\partial G$ contains all functions from $L_2(\partial G)$. If $G \subset \mathbb{R}^m$ and the boundary of $G$ is a compact subset of finitely many Lipschitz surfaces then the space of all distributions with finite energy supported on $\partial G$ contains all functions from $L_p(\partial G)$ with $p > m-1$.

This paper studies the solvability of the weak Neumann problem for the Laplace equation on a general open set $G \subset \mathbb{R}^m$. (If $G$ is a bounded Lipschitz domain then the formulation of the problem coincides with the definition of the weak solution of the problem in the Sobolev space $W_2^1(G)$ (see Remark 2.3 4).) Unlike the formulation of the problem in [11], this new formulation ensures the uniqueness of a solution (up to adding a locally constant function). It is shown that each solution is the sum of a constant and the Newtonian potential corresponding to a distribution with finite energy supported on $\partial G$. This enables us to look for a solution in the form of the Newtonian potential $\mathscr{U}\mathscr{B}$ corresponding to a distribution $\mathscr{B}$ with finite energy supported on $\partial G$. We get a bounded operator $N^G\mathscr{U}$ on the space $\mathscr{E}(\partial G)$ of all distributions with finite energy supported on $\partial G$. If $\mathscr{B} \in \mathscr{E}(\partial G)$ then the distribution $N^G\mathscr{U}\mathscr{B}$ represents the normal derivative of the Newtonian potential $\mathscr{U}\mathscr{B}$ and solving the Neumann problem for the Laplace equation with boundary condition $\mathscr{F}$ transposes to solving the equation $N^G\mathscr{U}\mathscr{B} = \mathscr{F}$. It is shown that $N^G\mathscr{U}$ is a positive nonexpansive operator. Under the mild condition that the range of the operator is closed (which is fulfilled for bounded $W_2^1$-extendible open sets with slits of zero Lebesgue measure) a necessary and sufficient condition for the solvability of the Neumann problem is given. Moreover, it is proved that for $\mathscr{G} = \sum (I - N^G\mathscr{U})^j \mathscr{F}$ the Newtonian potential $\mathscr{U}\mathscr{G}$ is a solution of the Neumann problem with boundary condition $\mathscr{F}$. This generalizes the result of O. Steinbach and W. Wendland ([23]) for bounded open sets $G \subset \mathbb{R}^3$ with connected Lipschitz boundary and my result ([15]) for piecewise-smooth bounded domains in $\mathbb{R}^3$. We remark that nontangentially accessible domains are $W_2^1$-extendible. The explicit solution of the Neumann problem for the Laplace equation on nontangentially accessible domains is a new result.

## 2. Formulation of the problem

Let $G$ be an open set in $\mathbb{R}^m$, $m > 2$. If $h$ is a complex harmonic function on $G$ such that

$$\int_H |\nabla h| \, \mathrm{d}\mathscr{H}_m < \infty$$

for all bounded open subsets $H$ of $G$ we define the weak normal derivative $N^G h$ of $h$ as the complex distribution

$$N^G h(\varphi) = \int_G \nabla\varphi \cdot \nabla h \, \mathrm{d}\mathscr{H}_m$$

for $\varphi \in \mathscr{D}$ (= the space of all compactly supported infinitely differentiable real-valued functions in $\mathbb{R}^m$). Here $\mathscr{H}_k$ is the $k$-dimensional Hausdorff measure normalized so that $\mathscr{H}_k$ is the Lebesgue measure in $\mathbb{R}^k$. The distribution $N^G h$ is called the *weak normal derivative of* $h$. It is shown in [11] that $N^G h$ is a distribution supported on $\partial G$, the boundary of $G$.

The following problem was studied in [11]: For a real measure $\mu$ on $\partial G$ find a harmonic function $u$ on $G$ such that $N^G u = \mu$. Since there is no restriction on the growth of $u$ at infinity this Neumann problem for the Laplace equation is evidently not uniquely solvable up to an additive constant on unbounded domains. But the uniqueness up to an additive constant does not hold for bounded domains, too, as the following example shows:

**Example 2.1.** Let $H \subset \mathbb{R}^m$ be a nonempty bounded domain with Ljapunov boundary (i.e., of class $C^{1+\alpha}$). Fix $x = [x_1, x_2, \ldots, x_m] \in H$. Fix $r > 0$ such that $\Omega_r(x) = \{y \in \mathbb{R}^m;\ |y - x| < r\} \subset H$. Denote $S = \{[y_1, \ldots, y_m];\ y_m = x_m\}$, $\Gamma = S \cap \operatorname{cl} \Omega_{r/2}(x)$, where $\operatorname{cl} M$ means the closure of a set $M$. Fix $\Psi \in \mathscr{D}$ such that $\Psi(x) \neq 0$ and $\operatorname{spt} \Psi \cap S \subset \Omega_{r/4}(x)$, where $\operatorname{spt} \Psi$ means the support of $\Psi$. Put

$$v(z) = \int_S \Psi(y) \frac{y_m - z_m}{|z - y|^m} \, \mathrm{d}\mathscr{H}_{m-1}(y)$$

for $z \in \mathbb{R}^m \setminus \Gamma$. The double layer potential $v$ is a harmonic function on $\mathbb{R}^m \setminus \Gamma$, extendible to a function from the class $C^1$ on the sets $E_+ \equiv \{[y_1, \ldots, y_m];\ y_m \geqslant x_m\}$, $E_- \equiv \{[y_1, \ldots, y_m];\ y_m \leqslant x_m\}$ (see [17], § 15). Put $G = H \setminus \Gamma$. Then $|\nabla v| \in L_\infty(G) \subset L_1(G)$. Denote by $n^M$ the unit exterior normal of $M$ for $M = G \cap E_\pm$. Since the normal derivative of the double layer potential $v$ has no jump on $S$ (see [17], § 15) we have for $\varphi \in \mathscr{D}$ by Green's formula

$$\begin{aligned} N^G v(\varphi) &= \int_{G \cap E_+} \nabla v \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}_m + \int_{G \cap E_-} \nabla v \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}_m \\ &= \int_{\partial(G \cap E_+)} \varphi n^{G \cap E_+} \cdot \nabla v \, \mathrm{d}\mathscr{H}_{m-1} + \int_{\partial(G \cap E_-)} \varphi n^{G \cap E_-} \cdot \nabla v \, \mathrm{d}\mathscr{H}_{m-1} \\ &= \int_{\partial H} \varphi n^H \cdot \nabla v \, \mathrm{d}\mathscr{H}_{m-1}. \end{aligned}$$

Denote by $\mathscr{H}$ the restriction of $\mathscr{H}_{m-1}$ to $\partial H$, $\mu = (n^H \cdot \nabla v)\mathscr{H}$. Fix $\varphi \in \mathscr{D}$ such that $\varphi \equiv 1$ on a neighbourhood of $\operatorname{cl} H$. Then

$$\mu(\partial H) = \int_{\partial H} \varphi n^H \cdot \nabla v \, \mathrm{d}\mathscr{H}_{m-1} = N^G v(\varphi) = 0.$$

According to [11], Theorem 5.12 there is a real measure $\nu$ supported by $\partial H$ such that $N^H(\mathscr{U}\nu) = \mu$, where $\mathscr{U}\nu$ is the single layer potential of $\nu$. Put $u = \mathscr{U}\nu - v$. If $\varphi \in \mathscr{D}$ then

$$\begin{aligned} N^G u(\varphi) &= N^H(\mathscr{U}\nu)(\varphi) - N^G v(\varphi) \\ &= \int_{\partial H} \varphi n^H \cdot \nabla v \, \mathrm{d}\mathscr{H}_{m-1} - \int_{\partial H} \varphi n^H \cdot \nabla v \, \mathrm{d}\mathscr{H}_{m-1} = 0. \end{aligned}$$

Therefore $u$ is a solution of the Neumann problem for the Laplace equation on $G$ with zero boundary condition. Since the single layer potential $\mathscr{U}\nu$ is continuous on $H$ and the double layer potential $v$ has a nonzero jump at $x$ (see [17], § 15), the function $u$ has a nonzero jump at $x$. Therefore the function $u$ is not constant. Consequently, there is a nonconstant solution of the Neumann problem for the Laplace equation on $G$ with zero boundary condition.

So, to ensure the uniqueness of a solution of the problem up to the addition of a locally constant function we must suppose that the solution has no jump at slits and we must restrict the growth of the solution at the infinity.

**Notation 2.2.** We denote by $L_{2,\mathrm{loc}}(G)$ the class of all complex measurable functions in $G$ that are in $L_2(K)$ for every compact subset $K$ of $G$. Denote by $L_2^1(G)$ the space of all functions in $L_{2,\mathrm{loc}}(G)$ for which all generalized derivatives of order 1 are in $L_2(G)$. Denote $W_2^1(G) = L_2^1(G) \cap L_2(G)$. The space $W_2^1(G)$ is equipped with the norm

$$\|u\|_{W_2^1(G)} = \sqrt{\int_G (|u|^2 + |\nabla u|^2) \, \mathrm{d}\mathscr{H}_m}.$$

**Weak Neumann problem for the Laplace equation.** *Let $\mathscr{F}$ be a complex distribution supported on $\partial G$. We say that $u$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if $u$ is a complex harmonic function in $G$ extendible to a function from $L_2^1(\mathbb{R}^m)$ such that $N^G u = \mathscr{F}$.*

If $v \in L_2^1(\mathbb{R}^m)$ and

$$\int_G \nabla\varphi \cdot \nabla v \, \mathrm{d}\mathscr{H}_m = \mathscr{F}(\varphi)$$

for each $\varphi \in \mathscr{D}$, where $\mathscr{F}$ is a distribution supported on $\partial G$, then there is $u \in L^1_2(\mathbb{R}^m)$ harmonic in $G$ such that $u(x) = v(x)$ for $\mathscr{H}_m$ almost all $x \in \mathbb{R}^m$ (see [21], Chapter II, Lemma 6.1 and [20], p. 162). Evidently $N^G u = \mathscr{F}$. The requirement that $u$ be harmonic in $G$ means only that we choose a suitable representation of a function in $L^1_2(\mathbb{R}^m)$.

The point of this definition is following: Denote by $\Gamma = \partial G \setminus \partial(\operatorname{cl} G)$ the slits in $G$. Then $u$ is a solution of the Laplace equation in $G$ wich has no jump on $\Gamma$. The distribution $\mathscr{F}$ represents on $\partial G \setminus \Gamma$ the normal derivative of $u$ and on $\Gamma$ the jump of the normal derivative of $u$. (See the following example.)

**Example 2.3.** Let $H, \tilde{H} \subset \mathbb{R}^m$ be nonempty bounded domains with Ljapunov boundary such that $\operatorname{cl} \tilde{H} \subset H$. Let $\Gamma$ be a closed subset of $\partial \tilde{H}$. Put $G = H \setminus \Gamma$. Let $u$ be a harmonic function on $G$, continuous on $\operatorname{cl} G$ such that $\nabla u$ is continuously extendible to the sets $\operatorname{cl} \tilde{H}$ and $\operatorname{cl}(H \setminus \tilde{H})$. Denote for $x \in \Gamma$

$$(\nabla u)_+(x) = \lim_{y \in \tilde{H},\, y \to x} \nabla u(y), \quad (\nabla u)_-(x) = \lim_{y \in H \setminus \operatorname{cl} \tilde{H},\, y \to x} \nabla u(y).$$

If $\varphi \in \mathscr{D}$ then Green's formula yields

$$\begin{aligned} N^G u(\varphi) &= \int_{\tilde{H}} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}_m + \int_{H \setminus \tilde{H}} \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}_m \\ &= \int_{\partial(H \setminus \tilde{H})} (\nabla u \cdot n^{H \setminus \operatorname{cl} \tilde{H}}) \varphi \, \mathrm{d}\mathscr{H}_{m-1} + \int_{\partial \tilde{H}} (\nabla u \cdot n^{\tilde{H}}) \varphi \, \mathrm{d}\mathscr{H}_{m-1} \\ &= \int_{\partial H} (\nabla u \cdot n^H) \varphi \, \mathrm{d}\mathscr{H}_{m-1} + \int_{\Gamma} [(\nabla u)_+ - (\nabla u)_-] \cdot n^{\tilde{H}} \varphi \, \mathrm{d}\mathscr{H}_{m-1}. \end{aligned}$$

**Remark 2.4.** Suppose that $G$ is a bounded domain with locally Lipschitz boundary. Denote by $H^{1/2}(\partial G)$ the space of traces of fuctions from $W^1_2(G)$ and by $H^{-1/2}(\partial G)$ the dual space of $H^{1/2}(\partial G)$ (see [14], Chapter 3). If $\mathscr{F} \in H^{-1/2}(G)$ we say that $u$ is a weak solution in $W^1_2(G)$ of the Neumann problem for the Laplace equation with the boundary condition $\mathscr{F}$ if $u \in W^1_2(G)$ and

$$\int_G \nabla u \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}_m = \mathscr{F}(\varphi)$$

for each $\varphi \in W^1_2(G)$ (see [14], p. 128). Now we show that $u$ is a weak solution of the Neumann problem for the Laplace equation in $W^1_2(G)$ if and only if $u$ is a solution of the corresponding weak Neumann problem for the Laplace equation.

Suppose first that $u$ is a weak solution in $W^1_2(G)$ of the Neumann problem for the Laplace equation with the boundary condition $\mathscr{F} \in H^{-1/2}(\partial G)$. The functions from $W_2^1(G)$ are considered as the equivalence classes of functions which differ on a set of zero Lebesgue measure. We can choose arbitrary representative from this class. Since $u$ is a solution of the Laplace equation in the sense of distributions in $G$ (see [4], Appendix A, Remark 6) we can choose a representative $u$ which is harmonic in $G$ (see [20], Chapter III, § 3). Since $G$ has locally Lipschitz boundary and $u \in W_2^1(G)$ the function $u$ is extendible to a function from $L_2^1(\mathbb{R}^m)$ (see [8], Theorem A). Since $\mathscr{F} \in H^{-1/2}(\partial G)$ it is well known that $\mathscr{F}$ is a distribution. (Since $\mathscr{D} \subset W_2^1(G)$ we deduce that $\mathscr{D} \subset H^{1/2}(\partial G)$ and $\mathscr{F}(\varphi)$ is well defined for each $\varphi \in \mathscr{D}$. If $\varphi, \varphi_j \in \mathscr{D}$ are supported in a compact set $K$ and $\varphi_j \to \varphi$ and $\nabla\varphi_j \to \nabla\varphi$ uniformly then $\varphi_j \to \varphi$ in $W_2^1(G)$ and thus $\varphi_j \to \varphi$ in $H^{1/2}(\partial G)$ (see [14], Theorem 3.38). Since $\mathscr{F}$ is a continuous fuctional in $H^{1/2}(\partial G)$ we have $\mathscr{F}(\varphi_j) \to \mathscr{F}(\varphi)$.) Since $\mathscr{F}(\varphi)$ depends only on the restriction of $\varphi$ to $\partial G$ the distribution $\mathscr{F}$ is supported on $\partial G$. Since $\mathscr{D} \subset W_2^1(G)$ the function $u$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$.

Suppose now that $u$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ where $\mathscr{F}$ is a distribution supported on $\partial G$. Then $u \in W_2^1(G)$. If we denote

$$H(\varphi) = \int_G \nabla\varphi \cdot \nabla u \,\mathrm{d}\mathscr{H}_m$$

for $\varphi \in W_2^1(G)$ then Hölder's inequality yields that

$$|H(\varphi)| \leqslant \|u\|_{W_2^1(G)} \|\varphi\|_{W_2^1(G)}.$$

Since $\mathscr{D}$ is dense in $W_2^1(G)$ (see [14], Theorem 3.29) we have

$$|\mathscr{F}(\varphi)| = |H(\varphi)| \leqslant \|u\|_{W_2^1(G)} \inf_{\psi \in W_2^1(G);\ \psi|\partial G = \varphi} \|\psi\|_{W_2^1(G)} = \|u\|_{W_2^1(G)} \|\varphi\|_{H^{1/2}(\partial G)}$$

for each $\varphi \in \mathscr{D}$. Since $\mathscr{D}$ is dense in $H^{1/2}(\partial G)$ there is a unique continuous extension of $\mathscr{F}$ onto $H^{1/2}(\partial G)$. Thus $\mathscr{F} \in H^{-1/2}(\partial G)$ and

$$\int_G \nabla\varphi \cdot \nabla u = \mathscr{F}(\varphi)$$

for each $\varphi \in W_2^1(G)$.

## 3. Uniqueness

**Theorem 3.1.** *Let $u$, $v$ be two solutions of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. Then $w = u - v$ is locally constant in $G$, i.e., $w$ is constant on each component of $G$.*

Proof. We can suppose that $u$ and $v$ are real. According to [21], Chapter I, Lemma 1.1 there is a sequence of functions $\varphi_n \in \mathscr{D}$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^m} |\nabla w - \nabla \varphi_n|^2 \,\mathrm{d}\mathscr{H}_m = 0.$$

Since the function $w$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with zero boundary condition we have

$$\int_G |\nabla w|^2 \,\mathrm{d}\mathscr{H}_m = \lim_{n\to\infty} \int_G \nabla w \cdot \nabla \varphi_n \,\mathrm{d}\mathscr{H}_m = 0.$$

Since $\nabla w = 0$ in $G$ the function $w$ is locally constant in $G$. $\square$

## 4. Representability of solutions by potentials

For $x, y \in \mathbb{R}^m$ denote

$$h_x(y) = \begin{cases} \dfrac{1}{(m-2)A} |x-y|^{2-m} & \text{for } x \neq y, \\ \infty & \text{for } x = y, \end{cases}$$

where $A$ is the area of the unit sphere in $\mathbb{R}^m$. For a closed set $F$ denote by $C'(F)$ the space of all finite complex Borel measures with support in $F$. For $\mu \in C'(\mathbb{R}^m)$, $\mu \geqslant 0$, denote

$$\mathscr{U}\mu(x) = \int_{\mathbb{R}^m} h_x(y) \,\mathrm{d}\mu(y), \quad x \in \mathbb{R}^m \tag{4.1}$$

the Newtonian potential corresponding to $\mu$. According to [12], Theorem 1.11

$$\mathscr{U}\mu(x) = \lim_{r\to 0_+} (\mathscr{H}_m(\Omega_r(x)))^{-1} \int_{\Omega_r(x)} \mathscr{U}\mu \,\mathrm{d}\mathscr{H}_m. \tag{4.2}$$

For a compact set $K \subset \mathbb{R}^m$ we define the Newtonian capacity of $K$ as

$$\operatorname{cap}(K) = \sup\{\mu(\mathbb{R}^m);\ \mu \in C'(K);\ \mu \geqslant 0,\ \mathscr{U}\mu \leqslant 1\}.$$

There is a positive constant $c_m$ depending only on $m$ such that

$$\operatorname{cap}(K) = c_m \inf\left\{ \int_{\mathbb{R}^m} |\nabla\varphi|^2 \, \mathrm{d}\mathscr{H}_m \colon \; \varphi \in \mathscr{D}, \, \varphi \geqslant 1 \text{ on } K \right\}$$

(see [12], Chapter II and [1], p. 18). For an open $G \subset \mathbb{R}^m$ define

$$\operatorname{cap}(G) = \sup\{\operatorname{cap}(K); \; K \subset G, \; K \text{ compact}\}.$$

Since

$$\operatorname{cap}(K) = \inf\{\operatorname{cap}(G); \; K \subset G, \; G \text{ open}\}$$

for each compact $K$ (see [12], Theorem 2.5), we can define the exterior Newtonian capacity

$$\operatorname{cap}(E) = \inf\{\operatorname{cap}(G); \; E \subset G, \; G \text{ open}\}$$

for arbitrary $E \subset \mathbb{R}^m$. We say that a condition A is fulfilled quasieverywhere (and write q.e.) if it is fulfilled outside some set $M$ with $\operatorname{cap}(M) = 0$. (Note that $\mathscr{H}_{m-1}(E) = 0$ for each $E \subset \mathbb{R}^m$ with $\operatorname{cap}(E) = 0$ by [12], Theorem 3.13.)

Let $f$ be a function defined quasieverywhere on $\mathbb{R}^m$. We say that $f$ is quasicontinuous if for every $\varepsilon > 0$ there is an open set $G$ such that $\operatorname{cap}(G) < \varepsilon$ and the restriction of $f$ to $\mathbb{R}^m \setminus G$ is continuous. If $\mu \in C'(\mathbb{R}^m)$, $\mu \geqslant 0$ and $\mathscr{U}\mu \not\equiv \infty$ then $\mathscr{U}\mu$ is quasicontinuous by [12], Theorem 1.4 and [2], Theorem 5.5.8.

Now we define the Bessel capacity. If $K \subset \mathbb{R}^m$ is a compact set denote

$$C_{1,2}(K) = \inf\left\{ \int_{\mathbb{R}^m} [|\varphi|^2 + |\nabla\varphi|^2] \, \mathrm{d}\mathscr{H}_m ; \; \varphi \in \mathscr{D}, \; \varphi \geqslant 1 \text{ on } K \right\}.$$

For an open set $G \subset \mathbb{R}^m$ define

$$C_{1,2}(G) = \sup\{C_{1,2}(K); \; K \subset G, \; K \text{ compact}\}.$$

Since

$$C_{1,2}(K) = \inf\{C_{1,2}(G); \; K \subset G, \; G \text{ open}\}$$

for each compact $K$ we can define the exterior Bessel capacity

$$C_{1,2}(E) = \inf\{C_{1,2}(G); \; E \subset G, \; G \text{ open}\}$$

for arbitrary $E \subset \mathbb{R}^m$. Clearly, $\operatorname{cap}(E) \leqslant c_m C_{1,2}(E)$.

We say that a condition A is fulfilled $(1,2)$-quasieverywhere (and write (1,2)-q.e.) if it is fulfilled outside some set $M$ with $C_{1,2}(M) = 0$. If A is fulfilled (1,2)-q.e. then it is fulfilled q.e. because $C_{1,2}(M) = 0$ implies $\operatorname{cap}(M) = 0$.

Let $f$ be a function defined $(1,2)$-quasieverywhere on $\mathbb{R}^m$. We say that $f$ is $(1,2)$-quasicontinuous if for every $\varepsilon > 0$ there is an open set $G$ such that $C_{1,2}(G) < \varepsilon$ and the restriction of $f$ to $\mathbb{R}^m \setminus G$ is continuous. Remark that a $(1,2)$-quasicontinuous function is quasicontinuous.

Now we define the Newtonian potential for a suitable class of distributions. Denote by $S$ the linear space of all $f \in C^\infty(\mathbb{R}^m)$ such that

$$\lim_{|x|\to\infty} |x|^n D^\alpha f(x) = 0$$

for each integer $n$ and each multiindex $\alpha$. The sequence $f_k$ is said to converge to $f$ in $S$ if $(1+|x|^n)D^\alpha f_k(x)$ converges uniformly to $(1+|x|^n)D^\alpha f(x)$ for each integer $n$ and each multiindex $\alpha$ as $k \to \infty$. Denote by $S^*$ the dual space of $S$. If $\mathscr{F} \in S^*$ is a real measure absolutely continuous with respect to the Lebesgue measure then we identify its density with $\mathscr{F}$.

For $f \in S$ define the Fourier transform $\hat{f}$ of $f$ by

$$\hat{f}(x) = \int_{\mathbb{R}^m} f(y) \mathrm{e}^{-2\pi \mathrm{i} x\cdot y} \,\mathrm{d}\mathscr{H}_m(y),$$

where $x \cdot y$ denotes the scalar product of $x$ and $y$. Then the mapping $f \mapsto \hat{f}$ is an isomorphism of $S$. For $\mathscr{F} \in S^*$ denote $\widehat{\mathscr{F}}(\varphi) = \mathscr{F}(\hat{\varphi})$ for each $\varphi \in S$. Then $\widehat{\mathscr{F}} \in S^*$ is the so-called Fourier transform of $\mathscr{F}$.

Denote by $\mathscr{E}$ the space of all complex distributions $\mathscr{F} = \mathscr{F}_1 + \mathrm{i}\mathscr{F}_2$, where $\mathscr{F}_1, \mathscr{F}_2 \in S^*$, such that the Fourier transform $\widehat{\mathscr{F}} = \widehat{\mathscr{F}_1} + \mathrm{i}\widehat{\mathscr{F}_2}$ of $\mathscr{F}$ is absolutely continuous with respect to the Lebesgue measure and

$$\|\mathscr{F}\|_{\mathscr{E}} = \sqrt{\int \frac{|\widehat{\mathscr{F}}(x)|^2}{|x|^2} \,\mathrm{d}\mathscr{H}_m(x)} < \infty.$$

Recall that $\|\mathscr{F}\|_{\mathscr{E}}$ is called the energy of $\mathscr{F}$. Then $\mathscr{E}$ equipped with the energy $\|\mathscr{F}\|_{\mathscr{E}}$ as a norm is a complex Hilbert space with the scalar product

$$(\mathscr{F}, \mathscr{G})_{\mathscr{E}} = \int \frac{\widehat{\mathscr{F}}(x)\overline{\widehat{\mathscr{G}}(x)}}{|x|^2} \,\mathrm{d}\mathscr{H}_m(x).$$

(Here $\overline{\widehat{\mathscr{G}}(x)}$ denotes complex conjugate of $\widehat{\mathscr{G}}(x)$.) If $\mu \in \mathscr{C}'(\mathbb{R}^m)$ then $\mu \in \mathscr{E}$ if and only if $\int \mathscr{U}|\mu|(x) \,\mathrm{d}|\mu|(y) < \infty$. (Here $|\mu|$ denotes the variarion of $\mu$.) The space $\mathscr{E} \cap \mathscr{C}'(\mathbb{R}^m)$ is dense in $\mathscr{E}$.

For each $\mathscr{F} \in \mathscr{E}$ there is a unique complex distribution $\mathscr{U}_{\mathscr{F}} = \mathscr{G}_1 + \mathrm{i}\mathscr{G}_2$ with $\mathscr{G}_1, \mathscr{G}_2 \in S^*$ such that $\widehat{\mathscr{U}_{\mathscr{F}}}(x) = \widehat{\mathscr{F}}(x)|x|^{-2}$. The complex distribution $\mathscr{U}_{\mathscr{F}}$ is a complex measure which is absolutely continuous with respect to the Lebesgue measure and $\mathscr{U}_{\mathscr{F}} \in L^1_2(\mathbb{R}^m)$ by [12], Theorem 6.4. Denote by $\mathscr{U}\mathscr{F}$ the quasicontinuous representant of $\mathscr{U}_{\mathscr{F}}/(4\pi^2)$ (the so called Newtonian potential of $\mathscr{F}$) (see [12], p. 435 or [3], p. 155 and [3], Chap. II, § 2). According to [3], chap. II, § 2, [1], Theorem 6.2.1 and [1], Theorem 6.1.4 we can even suppose that $\mathscr{U}\mathscr{F}$ is $(1,2)$-quasicontinuous and

$$\mathscr{U}\mathscr{F}(x) = \lim_{r \to 0_+} (\mathscr{H}_m(\Omega_r(x)))^{-1} \int_{\Omega_r(x)} \mathscr{U}\mathscr{F} \, \mathrm{d}\mathscr{H}_m \tag{4.3}$$

at each $x \in \mathbb{R}^m$ for which the limit on the right-hand side exists. Then $\mathscr{U}\mathscr{F}$ is determined quasieverywhere on $\mathbb{R}^m$ and the equality

$$(\mathscr{F}, \nu)_{\mathscr{E}} = 4\pi^2 \int \mathscr{U}\mathscr{F} \, \mathrm{d}\overline{\nu}$$

holds for each $\nu \in \mathscr{E} \cap \mathscr{C}'(\mathbb{R}^m)$, where $\overline{\nu}$ denotes the complex conjugate of $\nu$ (see [12], Theorem 6.2). According to [12], Theorem 6.4,

$$\|\mathscr{F}\|_{\mathscr{E}} = \sqrt{\int |\nabla \mathscr{U}\mathscr{F}|^2 \, \mathrm{d}\mathscr{H}_m}.$$

If spt $\mathscr{F}$, the support of $\mathscr{F}$, is compact then

$$\mathscr{U}_{\mathscr{F}} = h_0 * \mathscr{F},$$

where $h_0 * \mathscr{F}$ denotes the convolution of the distributions $h_0$ and $\mathscr{F}$ (see [12], p. 434) and

$$\mathscr{U}\mathscr{F}(x) = \mathscr{F}(h_x) \quad \text{for } x \in \mathbb{R}^m \setminus \operatorname{spt} \mathscr{F}.$$

If $\mathscr{F} \in C'(\mathbb{R}^m)$, $\mathscr{F} \geqslant 0$ then $\mathscr{U}\mathscr{F}$ is given by (4.1) (see (4.2), [3], p. 155 and [3], Chap. II, § 2).

For a closed set $K$ denote by $\mathscr{E}(K)$ the space of all distributions from $\mathscr{E}$ supported on $K$ with the energy $\|\cdot\|_{\mathscr{E}}$ as the norm. Then $\mathscr{E}(K)$ is a complex Hilbert space (see [3], p. 121).

**Theorem 4.1.** *Let $u$ be a solution of the weak Neumann problem for the Laplace equation in $G$. Then there are $\mathscr{B} \in \mathscr{E}(\partial G)$ and a complex number $a$ such that $u = \mathscr{U}\mathscr{B} + a$ in $G$.*

Proof. We may suppose that $u \in L^1_2(\mathbb{R}^m)$. According to [3], p. 155 there are $\mathscr{G} \in \mathscr{E}$ and a complex number $a$ such that $u = \mathscr{U}\mathscr{G} + a$ almost everywhere. Since $u$ is continuous in $G$ and $\mathscr{U}\mathscr{G}$ fulfils (4.3) we have

$$\mathscr{U}\mathscr{G}(x) = \lim_{r \to 0_+} (\mathcal{H}_m(\Omega_r(x)))^{-1} \int_{\Omega_r(x)} (u(y) - a) \, \mathrm{d}\mathscr{H}_m = u(x) - a$$

for each $x \in G$. According to [3], p. 158 we have $\Delta \mathscr{U}\mathscr{G} = -\mathscr{G}$. Since $\mathscr{U}\mathscr{G}$ is harmonic in $G$ we obtain $\operatorname{spt} \mathscr{G} \subset \mathbb{R}^m \setminus G$.

Denote by $\mathscr{B}$ the orthogonal projection of $\mathscr{G}$ to $\mathscr{E}(\operatorname{cl} G)$. Then $\mathscr{U}\mathscr{B} = \mathscr{U}\mathscr{G}$ on $G$ by [3], Chapitre I, Théorème 4. Since $\mathscr{U}\mathscr{B}$ is harmonic in $G$ we deduce that $\operatorname{spt} \mathscr{B} \subset \mathbb{R}^m \setminus G$. Therefore $\mathscr{B} \in \mathscr{E}(\partial G)$ and $u = a + \mathscr{U}\mathscr{B}$. □

## 5. The necessary condition for the solvability

Since every solution of the weak Neumann problem for the Laplace equation in $G$ has the form $\mathscr{U}\mathscr{B} + a$ where $\mathscr{B} \in \mathscr{E}(\partial G)$ and $\mathscr{U}\mathscr{B} + a$, $\mathscr{U}\mathscr{B}$ are solutions of the same problem, we shall look for a solution in the form $\mathscr{U}\mathscr{B}$ with $\mathscr{B} \in \mathscr{E}(\partial G)$.

**Lemma 5.1.** *Let $\mathscr{F} \in \mathscr{E}$, $\varphi \in \mathscr{D}$. Then $\Delta\varphi \in \mathscr{E}$ and $\mathscr{F}(\varphi) = -(\mathscr{F}, \Delta\varphi)_{\mathscr{E}}$.*

Proof. According to [12], p. 100 and [12], Theorem 6.2 we have $\varphi = -\mathscr{U}\Delta\varphi$ and $\Delta\varphi \in \mathscr{E}$. Put

$$a_n = \int_{\Omega_{1/n}(0)} \exp\Big(|x|^2 - \frac{1}{n^2}\Big)^{-1} \, \mathrm{d}\mathscr{H}_m(x),$$

$$\begin{cases} \varrho_n(x) = a_n^{-1} \exp\Big(|x|^2 - \dfrac{1}{n^2}\Big)^{-1} & \text{for } |x| < \dfrac{1}{n}, \\ 0 & \text{for } |x| \geqslant \dfrac{1}{n}. \end{cases}$$

Then $\mathscr{F} * (\varrho_n \mathscr{H}_m) \in \mathscr{E} \cap \mathscr{C}'(\mathbb{R}^m)$ and $\mathscr{F} * (\varrho_n \mathscr{H}_m) \to \mathscr{F}$ in $\mathscr{E}$ as $n \to \infty$ (see [12], Lemma 6.4 and p. 34). Since $\varrho_n \mathscr{H}_m \to \delta_0$ in the sense of distributions as $n \to \infty$, where $\delta_0$ is the Dirac measure, [12], Lemma 0.7 yields that $\mathscr{F} * (\varrho_n \mathscr{H}_m) \to \mathscr{F}$ in the sense of distributions as $n \to \infty$.

Since $\mathscr{F} * (\varrho_n \mathscr{H}_m)$ is a real measure, [12], Theorem 6.2 yields

$$\begin{aligned} \mathscr{F}(\varphi) &= \lim_{n\to\infty} [\mathscr{F} * (\varrho_n \mathscr{H}_m)](\varphi) \\ &= \lim_{n\to\infty} \int \varphi \, \mathrm{d}[\mathscr{F} * (\varrho_n \mathscr{H}_m)] \\ &= -\lim_{n\to\infty} \int \mathscr{U}\Delta\varphi \, \mathrm{d}[\mathscr{F} * (\varrho_n \mathscr{H}_m)] \\ &= -\lim_{n\to\infty} (\mathscr{F} * (\varrho_n \mathscr{H}_m), \Delta\varphi)_{\mathscr{E}} = -(\mathscr{F}, \Delta\varphi)_{\mathscr{E}}. \end{aligned}$$

□

**Lemma 5.2.** *Let $M \subset \mathbb{R}^m$ be a Borel set. If $\mathscr{F} \in \mathscr{E}$ then there is a unique distribution $J_M\mathscr{F} \in \mathscr{E}$ such that*

$$\int_M \nabla \mathscr{U}\mathscr{G} \cdot \nabla \overline{\mathscr{U}\mathscr{F}} \, \mathrm{d}\mathscr{H}_m = (\mathscr{G}, J_M\mathscr{F})_{\mathscr{E}}$$

*for each $\mathscr{G} \in \mathscr{E}$. The operator $J_M\colon \mathscr{F} \mapsto J_M\mathscr{F}$ is a bounded linear positive operator on $\mathscr{E}$ with $\|J_M\| \leqslant 1$. Moreover, $J_M(\mathscr{E}) \subset \mathscr{E}(\mathrm{cl}\, M)$.*

Proof. Fix $\mathscr{F} \in \mathscr{E}$. If $\mathscr{G} \in \mathscr{E}$ then [12], Theorem 6.4 and Hölder's inequality yield that

$$\left| \int_M \nabla \mathscr{U}\mathscr{G} \cdot \nabla \overline{\mathscr{U}\mathscr{F}} \, \mathrm{d}\mathscr{H}_m \right| \leqslant \|\mathscr{F}\|_{\mathscr{E}} \|\mathscr{G}\|_{\mathscr{E}}.$$

Since

$$\mathscr{G} \mapsto \int_M \nabla \mathscr{U}\mathscr{G} \cdot \nabla \overline{\mathscr{U}\mathscr{F}} \, \mathrm{d}\mathscr{H}_m$$

is a bounded linear functional on the Hilbert space $\mathscr{E}$ there is unique $J_M\mathscr{F} \in \mathscr{E}$ such that

$$\int_M \nabla \mathscr{U}\mathscr{G} \cdot \nabla \overline{\mathscr{U}\mathscr{F}} \, \mathrm{d}\mathscr{H}_m = (\mathscr{G}, J_M\mathscr{F})_{\mathscr{E}}$$

for each $\mathscr{G} \in \mathscr{E}$. Since $|(\mathscr{G}, J_M\mathscr{F})_{\mathscr{E}}| \leqslant \|\mathscr{F}\|_{\mathscr{E}} \|\mathscr{G}\|_{\mathscr{E}}$ for each $\mathscr{G} \in \mathscr{E}$ the operator $J_M$ is a bounded linear operator on $\mathscr{E}$ with $\|J_M\| \leqslant 1$. If $\mathscr{F} \in \mathscr{E}$ then

$$(\mathscr{F}, J_M\mathscr{F})_{\mathscr{E}} = \int_M |\nabla \mathscr{U}\mathscr{F}|^2 \, \mathrm{d}\mathscr{H}_m \geqslant 0.$$

Therefore $J_M \geqslant 0$.

Let now $\varphi \in \mathscr{D}$, $\mathrm{spt}\, \varphi \cap \mathrm{cl}\, M = \emptyset$, $\mathscr{F} \in \mathscr{E}$. According to Lemma 5.1

$$J_M\mathscr{F}(\varphi) = -(J_M\mathscr{F}, \Delta\varphi)_{\mathscr{E}} = -\overline{(\Delta\varphi, J_M\mathscr{F})_{\mathscr{E}}} = -\int_M \nabla \mathscr{U}(\Delta\varphi) \cdot \nabla \mathscr{U}\mathscr{F} \, \mathrm{d}\mathscr{H}_m.$$

Since $-\mathscr{U}(\Delta\varphi) = \varphi = 0$ on a neighbourhood of $M$ by [12], p. 100, we obtain $J_M\mathscr{F}(\varphi) = 0$. Thus $\mathrm{spt}\, J_M\mathscr{F} \subset \mathrm{cl}\, M$. □

**Lemma 5.3.** *If $\mathscr{F} \in \mathscr{E}(\partial G)$ then $\mathscr{U}\mathscr{F} \in L_2^1(\mathbb{R}^m)$, $\mathscr{U}\mathscr{F}$ is harmonic on $G$ and $N^G\mathscr{U}\mathscr{F} = J_G\mathscr{F} \in \mathscr{E}(\partial G)$.*

Proof. $\mathscr{U}\mathscr{F} \in L_2^1(\mathbb{R}^m)$ by [12], Theorem 6.4. According to [3], p. 158 we have $\Delta\mathscr{U}\mathscr{F} = -\mathscr{F}$. Since $\mathrm{spt}\, \mathscr{F} \subset \partial G$ we obtain that $\Delta\mathscr{U}\mathscr{F} = 0$ in $G$. According to [20], § 16 there is a harmonic function $u$ in $G$ such that $\mathscr{U}\mathscr{F}(x) = u(x)$ at $\mathscr{H}_m$ almost all $x \in G$. Using (4.3) we deduce that $\mathscr{U}\mathscr{F} = u$ in $G$.

Fix $\varphi \in \mathscr{D}$. Then $\varphi = -\mathscr{U}(\Delta\varphi)$ by [12], p. 100. According to Lemma 5.1

$$J_G\mathscr{F}(\varphi) = -(J_G\mathscr{F}, \Delta\varphi)_{\mathscr{E}} = \int_G \nabla[-\mathscr{U}(\Delta\varphi)] \cdot \nabla\mathscr{U}\mathscr{F} \,\mathrm{d}\mathscr{H}_m = \int_G \nabla\varphi \cdot \nabla\mathscr{U}\mathscr{F} \,\mathrm{d}\mathscr{H}_m.$$

Thus $J_G\mathscr{F} = N^G\mathscr{U}\mathscr{F}$. Using Lemma 5.2 we get $\operatorname{spt} N^G\mathscr{U}\mathscr{F} \subset \operatorname{cl} G$. Since $\mathscr{U}\mathscr{F}$ is a harmonic function in $G$, [21], Chapter I, Lemma 6.1 yields that $N^G\mathscr{U}\mathscr{F}(\varphi) = 0$ for each $\varphi \in \mathscr{D}$ with $\operatorname{spt}\varphi \subset G$. This gives $\operatorname{spt} N^G\mathscr{U}\mathscr{F} \subset \partial G$. □

**Proposition 5.4.** *Denote $N^G\mathscr{U}\colon \mathscr{F} \mapsto N^G\mathscr{U}\mathscr{F}$ for $\mathscr{F} \in \mathscr{E}(\partial G)$. Then $N^G\mathscr{U}$ is a bounded selfadjoint operator on $\mathscr{E}(\partial G)$, $0 \leqslant N^G\mathscr{U} \leqslant I$, where $I$ denotes the identity operator. Moreover* $\operatorname{Ker} N^G\mathscr{U} = (N^G\mathscr{U}(\mathscr{E}(\partial G)))^\perp$. *(Here* $\operatorname{Ker} N^G\mathscr{U}$ *denotes the kernel of the operator $N^G\mathscr{U}$ and $(N^G\mathscr{U}(\mathscr{E}(\partial G)))^\perp$ is the orthogonal complement of $N^G\mathscr{U}(\mathscr{E}(\partial G))$ in $\mathscr{E}(\partial G)$.)*

Proof. $N^G\mathscr{U}$ is a bounded positive operator on $\mathscr{E}(\partial G)$ by Lemma 5.3 and Lemma 5.2. If $\mathscr{F} \in \mathscr{E}(\partial G)$ then $(\mathscr{F}, (I - N^G\mathscr{U})\mathscr{F})_{\mathscr{E}(\partial G)} = (\mathscr{F}, J_{\mathbb{R}^m\setminus G}\mathscr{F})_{\mathscr{E}(\partial G)} \geqslant 0$ by Lemma 5.3 and Lemma 5.2. Therefore $N^G\mathscr{U} \leqslant I$. Since $N^G\mathscr{U}$ is positive, it is selfadjoint (see [19], p. 295). Since $N^G\mathscr{U}$ is selfadjoint we have $\operatorname{Ker} N^G\mathscr{U} = (N^G\mathscr{U}(\mathscr{E}(\partial G)))^\perp$ (see [6], Satz 70.3). □

**Remark 5.5.** Suppose that $G$ is a Lipschitz domain with compact boundary. Denote by $\mathscr{H}$ the restriction of $\mathscr{H}_{m-1}$ to $\partial G$. Then there exists the exterior unit normal $n^G(x)$ of $G$ at $\mathscr{H}$ almost all $x \in \partial G$. Let $\mathscr{F} = f\mathscr{H} \in \mathscr{E}(\partial G)$, where $f \in L_p(\mathscr{H})$, $1 < p < \infty$. Then the limit

$$g(x) = \lim_{\varepsilon\to 0_+} \int_{\partial G\setminus\Omega_\varepsilon(x)} n^G(x) \cdot \nabla h_y(x) f(y) \,\mathrm{d}\mathscr{H}(y)$$

exists for $\mathscr{H}$ almost all $x \in \partial G$ and $N^G\mathscr{U}\mathscr{F} = (\frac{1}{2}f + g)\mathscr{H}$ (see [24]).

**Remark 5.6.** Suppose that $G$ is an open set with compact boundary and finite perimeter. (If $\mathscr{H}_{m-1}(\partial G) < \infty$ then $G$ has finite perimeter.) If $z \in \mathbb{R}^m$ and $\theta$ is a unit vector such that the symmetric difference of $G$ and the half-space $\{x \in \mathbb{R}^m;\ (x - z)\cdot\theta < 0\}$ has $m$-dimensional density zero at $z$ then $n^G(z) = \theta$ is called the exterior normal of $G$ at $z$ in Federer's sense. If there is no exterior normal of $G$ at $z$ in this sense, we denote by $n^G(z)$ the zero vector in $\mathbb{R}^m$. (The exterior normal of $G$ at $z$ in the ordinary sense is the exterior normal of $G$ at $z$ in Federer's sense.) For $x \in \mathbb{R}^m$ denote

$$v^G(x) = \int_{\partial G} |n^G(y) \cdot \nabla h_x(y)| \,\mathrm{d}\mathscr{H}_{m-1}(y)$$

the cyclic variation of $G$ at $x$. Suppose that the cyclic variation of $G$ is bounded. (This is true for $G$ convex or for $G$ with $\partial G \subset L_1 \cup \ldots \cup L_k$, where $L_i$ are

$(m-1)$-dimensional Ljapunov surfaces, i.e., of class $C^{1+\alpha}$.) If $\mathscr{F} \in \mathscr{E}(\partial G) \cap \mathscr{C}'(\partial G)$ then $N^G \mathscr{U} \mathscr{F} \in \mathscr{E}(\partial G) \cap \mathscr{C}'(\partial G)$ and

$$N^G \mathscr{U} \mathscr{F}(M) = \int_M d_G(x) \, \mathrm{d}\mathscr{F}(x) + \int_{\partial G} \int_{\partial G \cap M} n^G(y) \cdot \nabla h_x(y) \, \mathrm{d}\mathscr{H}_{m-1}(y) \, \mathrm{d}\mathscr{F}(x)$$

for each Borel set $M$ (see [11]). Here

$$d_G(x) = \lim_{r \to 0_+} \frac{\mathscr{H}_m(G \cap \Omega_r(x))}{\mathscr{H}_m(\Omega_r(x))}$$

is the density of $G$ at $x$.

**Theorem 5.7.** *Let $\mathscr{F}$ be a distribution supported on $\partial G$. If there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ then $\mathscr{F} \in \mathscr{E}(\partial G)$ and*

$$\int_{\mathbb{R}^m} \nabla \mathscr{U} \mathscr{F} \cdot \overline{\nabla \varphi} \, \mathrm{d}\mathscr{H}_m = 0$$

*for every $\varphi \in L^1_2(\mathbb{R}^m)$ which is constant on each component of $G$. Namely, if $\mathscr{B} \in \operatorname{Ker} N^G \mathscr{U}$ then $(\mathscr{F}, \mathscr{B})_{\mathscr{E}} = 0$. If $\varphi \in \mathscr{D}$ is constant on each component of $G$ then $\mathscr{F}(\varphi) = 0$.*

Proof. According to Theorem 4.1 there is $\mathscr{G} \in \mathscr{E}(\partial G)$ such that $\mathscr{U}\mathscr{G}$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ and therefore $\mathscr{F} = N^G \mathscr{U} \mathscr{G} = J_G \mathscr{G}$ by Lemma 5.3. Suppose that $\varphi \in L^1_2(\mathbb{R}^m)$ is constant on each component of $G$. According to [3], p. 155 there are $\mathscr{B} \in \mathscr{E}$ and a constant $a$ such that $\varphi = \mathscr{U}\mathscr{B} + a$ a.e. in $\mathbb{R}^m$. Since $\mathscr{U}\mathscr{B}$ is constant on each component of $G$

$$\int_{\mathbb{R}^m} \nabla \mathscr{U} \mathscr{F} \cdot \overline{\nabla \varphi} \, \mathrm{d}\mathscr{H}_m = (\mathscr{F}, \mathscr{B})_{\mathscr{E}} = (J_G \mathscr{G}, \mathscr{B})_{\mathscr{E}} = \int_G \nabla \mathscr{U} \mathscr{G} \cdot \overline{\nabla \mathscr{U} \mathscr{B}} \, \mathrm{d}\mathscr{H}_m = 0.$$

If $\mathscr{B} \in \operatorname{Ker} N^G \mathscr{U}$ then $\mathscr{U}\mathscr{B} \in L^1_2(\mathbb{R}^m)$ is constant on each component of $G$ by Theorem 3.1. Therefore

$$(\mathscr{F}, \mathscr{B})_{\mathscr{E}} = \int_{\mathbb{R}^m} \nabla \mathscr{U} \mathscr{F} \cdot \overline{\nabla \mathscr{U} \mathscr{B}} \, \mathrm{d}\mathscr{H}_m = 0.$$

If $\varphi \in \mathscr{D}$ is constant on each component of $G$ then

$$\mathscr{F}(\varphi) = N^G \mathscr{U} \mathscr{G}(\varphi) = \int_G \nabla \mathscr{U} \mathscr{G} \cdot \nabla \varphi \, \mathrm{d}\mathscr{H}_m = 0.$$

□

## 6. The kernel of $N^G\mathscr{U}$

According to Proposition 5.4 we have $\mathrm{cl}[N^G\mathscr{U}(\mathscr{E}(\partial G))] = [\operatorname{Ker} N^G\mathscr{U}]^\perp$. In this paragraph we shall study the kernel of $N^G\mathscr{U}$.

**Lemma 6.1.** *Let $\mathscr{F} \in \mathscr{E}(\partial G)$. Then $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$ if and only if the function $\mathscr{U}\mathscr{F}$ is constant on each component of $G$. If $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$ and $\partial G$ is compact then $\mathscr{U}\mathscr{F} = 0$ on the unbounded component of $G$.*

Proof. We have $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$ if and only if the function $\mathscr{U}\mathscr{F}$ is constant on each component of $G$ by Theorem 3.1. Suppose now that $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$, $\partial G$ is compact and $H$ is the unbounded component of $G$. Then there is a constant $c$ such that $\mathscr{U}\mathscr{F} = c$ on $H$. According to [12], Theorem 6.4

$$\frac{1}{r^2}\lim_{r\to\infty}\int_{\partial\Omega_r(0)} \mathscr{U}\mathscr{F}(x)\,\mathrm{d}\mathscr{H}_{m-1}(x) = 0.$$

An easy calculation yields

$$\frac{1}{r^2}\lim_{r\to\infty}\int_{\partial\Omega_r(0)} \mathscr{U}\mathscr{F}(x)\,\mathrm{d}\mathscr{H}_{m-1}(x) = \lim_{r\to\infty} cAr^{m-3}$$

where $A$ is the area of the unit sphere in $\mathbb{R}^m$. Therefore $c = 0$. □

**Proposition 6.2.** *Let $H \subset \mathbb{R}^m$ be an open set such that $\mathscr{H}_m((G\setminus H)\cup(H\setminus G)) = 0$. Then $\partial(G\cup H) \subset \partial G$ and $\operatorname{Ker} N^G\mathscr{U} = \operatorname{Ker} N^H\mathscr{U} = \operatorname{Ker} N^{G\cup H}\mathscr{U}$. If $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$ then $\mathscr{U}\mathscr{F}$ is constant on each component of $G\cup H$.*

Proof. Let $x \in \partial(G\cup H)$. Then $x \notin G\cup H$ because $G\cup H$ is open. Fix $\varepsilon > 0$. Then there is $y \in G\cup H$ such that $|x-y| < \varepsilon$. If $y \in H$ there is $\delta \in (0,\varepsilon)$ such that $\Omega_\delta(y) \subset H$. Since $\mathscr{H}_m(H\setminus G) = 0$ there is $z \in G$ such that $|z-y| < \delta$. Therefore there is $z \in G$ such that $|x-z| < 2\varepsilon$. Thus $x \in \partial G$.

Let $\mathscr{F} \in \operatorname{Ker} N^{G\cup H}\mathscr{U}$. Then $\mathscr{F} \in \mathscr{E}(\partial(G\cup H)) \subset \mathscr{E}(\partial G)$ and $\mathscr{U}\mathscr{F}$ is constant on each component of $G\cup H$ by Lemma 6.1. Since $\mathscr{U}\mathscr{F}$ is constant on each component of $G$ Lemma 6.1 yields that $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$.

Let now $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$. Since $\mathscr{U}\mathscr{F}$ is constant on each component of $G$ by Lemma 6.1 we obtain $\nabla\mathscr{U} = 0$ in $G$. Since $\mathscr{H}_m((G\cup H)\setminus G) = \mathscr{H}_m(H\setminus G) = 0$ the vector function $\nabla\mathscr{U} = 0$ $\mathscr{H}_m$-a.e. in $G\cup H$. Let $V$ be a component of $G\cup H$. According to [13], Lemma on page 11 there is a constant $c$ such that $\mathscr{U}\mathscr{F} = c$ $\mathscr{H}_m$-a.e. in $V$. Using (4.3) we obtain $\mathscr{U} = c$ in $V$. Since $\mathscr{U}\mathscr{F}$ is harmonic in $G\cup H$ and $\Delta\mathscr{U}\mathscr{F} = -\mathscr{F}$ by [3], p. 158 we deduce that $\operatorname{spt}\mathscr{F} \subset \partial G\setminus H \subset \partial(G\cup H)$. Since $\mathscr{F} \in \mathscr{E}(\partial(G\cup H))$ and $\mathscr{U}\mathscr{F}$ is constant on each component of $G\cup H$, Lemma 6.1 yields that $\mathscr{F} \in \operatorname{Ker} N^{G\cup H}\mathscr{U}$.

Thus $\operatorname{Ker} N^G\mathscr{U} = \operatorname{Ker} N^{G\cup H}\mathscr{U}$. Similarly, $\operatorname{Ker} N^H\mathscr{U} = \operatorname{Ker} N^{G\cup H}\mathscr{U}$. □

**Corollary 6.3.** *Denote by $H$ the interior of* $\operatorname{cl} G$. *Suppose* $\mathscr{H}_m(H\setminus G)=0$. *Then* $\operatorname{Ker} N^G\mathscr{U}=\operatorname{Ker} N^H\mathscr{U}$. *If* $\mathscr{F}\in\operatorname{Ker} N^G\mathscr{U}$ *then* $\mathscr{U}\mathscr{F}$ *is constant on each component of* $H$.

**Proposition 6.4.** *Let $H$ be a bounded open subset of $G$ such that* $\operatorname{cl} H\cap\operatorname{cl}(G\setminus H)=\emptyset$. *Then there is* $\mathscr{F}_H\in\mathscr{E}(\partial G)\cap C'(\partial G)$ *such that* $\mathscr{U}\mathscr{F}_H=1$ *on $H$ and* $\mathscr{U}\mathscr{F}_H=0$ *on* $G\setminus H$.

Proof. Fix $\varphi\in\mathscr{D}$ such that $\varphi=1$ on $\operatorname{cl} H$ and $\varphi=0$ on $\operatorname{cl}(G\setminus H)$. Then $\varphi=-\mathscr{U}\Delta\varphi$ and $\Delta\varphi\in\mathscr{E}$ by [12], p. 100 and [12], Theorem 6.4. Denote by $\mathscr{F}$ the orthogonal projection of $(-\Delta\varphi)$ to $\mathscr{E}(\operatorname{cl} G)$. Since $(-\Delta\varphi)\in C'(\mathbb{R}^m)$, [3], p. 143 yields that $\mathscr{F}\in C'(\partial G)$. Moreover, $\mathscr{U}\mathscr{F}=\mathscr{U}(-\Delta\varphi)=\varphi$ on $G$ by [3], Chapitre I, Théorème 4. Since $\mathscr{U}\mathscr{F}$ is harmonic in $G$ and $\Delta\mathscr{U}\mathscr{F}=-\mathscr{F}$ by [3], p. 158 we deduce that $\operatorname{spt}\mathscr{F}\subset\mathbb{R}^m\setminus G$. Therefore $\mathscr{F}\in\mathscr{E}(\partial G)$ and $\mathscr{U}\mathscr{F}=1$ on $H$ and $\mathscr{U}\mathscr{F}=0$ on $G\setminus H$. $\square$

**Definition 6.5.** A set $E\subset\mathbb{R}^m$ is called a $(1,2)$-thin at a point $x\in\mathbb{R}^m$ if

$$\int_0^1 C_{1,2}(E\cap\Omega_r(x))r^{1-m}\,\mathrm{d}r<\infty.$$

If $E$ is not $(1,2)$-thin at $x$, it is said to be $(1,2)$-thick there. A set $E\subset\mathbb{R}^m$ is called a $(1,2)$-fine neighbourhood of $x\in\mathbb{R}^m$ if $x\in E$ and $\mathbb{R}^m\setminus E$ is $(1,2)$-thin at $x$. The collection of $(1,2)$-fine neighbourhoods gives the so-called $(1,2)$-fine topology.

**Remark 6.6.** If $G$ is $(1,2)$-thin at $x$ then

$$\int_0^1 \operatorname{cap}(E\cap\Omega_r(x))r^{1-m}\,\mathrm{d}r<\infty.$$

Using [2], Corollary 7.2.4 we get $d_G(x)=0$ (the density of $G$ at $x$).

**Proposition 6.7.** *Let* $C_{1,2}(\{x\in\partial G;\ G$ *is* (1,2)*-thin at* $x\})=0$. *If* $\mathscr{F}\in\operatorname{Ker} N^G\mathscr{U}$, $\mathscr{U}\mathscr{F}=0$ *in $G$ then* $\mathscr{F}=0$.

Proof. Let $\mathscr{F}\in\operatorname{Ker} N^G\mathscr{U}$, $\mathscr{U}\mathscr{F}=0$ in $G$. Since $\mathscr{U}\mathscr{F}$ is $(1,2)$-quasicontinous there is a set $N$ with $C_{1,2}(N)=0$ such that $\mathscr{U}\mathscr{F}$ is $(1,2)$-finely continous at each points of $\mathbb{R}^m\setminus N$ (see [1], Theorem 6.4.5). Denote $M=\{x\in\partial G;\ G$ is (1,2)-thin at $x\}$. Fix $x\in\partial G\setminus(M\cup N)$. Since $G$ is $(1,2)$-thick at $x$, every $(1,2)$-fine neighbourhood of $x$ meets $G$. Thus $\mathscr{U}\mathscr{F}(x)=0$, because $\mathscr{U}\mathscr{F}(x)=0$ in $G$. Therefore $\mathscr{U}\mathscr{F}=0$ on $\partial G\setminus(M\cup N)$.

According to [3], p. 143 there are measures $\mu_n\in\mathscr{E}(\partial G)\cap C'(\partial G)$ such that $\mu_n\to\mathscr{F}$ in $\mathscr{E}$ as $n\to\infty$. Since $0\leqslant\operatorname{cap}(M\cup N)\leqslant C_{1,2}(M\cup N)=0$ and $\mu_n$ do not charge sets of zero Newtonian capacity (see [12], Chapter II, Theorem 2.2), [12], Theorem 6.2 yields

$$(\mathscr{F},\mathscr{F})_{\mathscr{E}} = \lim_{n\to\infty} (\mathscr{F},\mu_n)_{\mathscr{E}} = \lim_{n\to\infty} 4\pi^2 \int_{\partial G} \mathscr{U}\mathscr{F} \,\mathrm{d}\overline{\mu_n} = 0,$$

because $\mathscr{U}\mathscr{F} = 0$ on $\partial G\setminus(N\cup M)$. The fact that $\|\mathscr{F}\|_{\mathscr{E}} = 0$ implies that $\mathscr{F} = 0$. $\square$

**Remark 6.8.** According to [1], Theorem 11.5.5 and [1], Theorem 11.5.4 there is an unbounded domain $G \subset \mathbb{R}^m$ with compact boundary such that $C_{1,2}(\{x \in \partial G;\ G \text{ is } (1,2)\text{-thin at } x\}) > 0$.

**Proposition 6.9.** *Suppose that $G \subset \mathbb{R}^m$ is an unbounded open set with compact boundary such that $C_{1,2}(\{x \in \partial G;\ G$ is $(1,2)$-thin at $x\}) > 0$. Then there is $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$, $\mathscr{F} \neq 0$ such that $\mathscr{U}\mathscr{F} = 0$ in $G$.*

Proof. According to [1], Theorem 11.5.4 there is a $(1,2)$-quasicontinuous function $u \in W_2^1(\mathbb{R}^m)$ such that $u = 0$ in $G$ and $C_{1,2}(\{x \in \partial G;\ u(x) \neq 0\}) > 0$. According to [1], Corollary 9.1.8 there is a $(1,2)$-quasicontinuous function $v \in W_2^1(\mathbb{R}^m)$ such that $v$ is harmonic in $\mathbb{R}^m \setminus \operatorname{cl} G$ and $v = u$ $(1,2)$-q.e. in $\operatorname{cl} G$. According to [12], Theorem 6.4 there is $\mathscr{F} \in \mathscr{E}$ such that $v = \mathscr{U}\mathscr{F}$ a.e. Since $v$, $\mathscr{U}\mathscr{F}$ are $(1,2)$-quasicontinuous, $v = \mathscr{U}\mathscr{F}$ $(1,2)$-q.e. by [1], Theorem 6.1.4. Using (4.3) we get $\mathscr{U}\mathscr{F} = 0$ in $G$ and $\mathscr{U}\mathscr{F} = v$ in $\mathbb{R}^m \setminus G$. Since $\mathscr{U}\mathscr{F}$ is harmonic in $\mathbb{R}^m \setminus \partial G$ and $\Delta\mathscr{U}\mathscr{F} = -\mathscr{F}$ by [3], p. 158 we conclude that $\mathscr{F} \in \mathscr{E}(\partial G)$. Since $\mathscr{U}\mathscr{F} = 0$ in $G$, Lemma 6.1 yields $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$. But $\mathscr{F} \neq 0$ because $C_{1,2}(\{x \in \partial G;\ \mathscr{U}\mathscr{F}(x) \neq 0\}) > 0$. $\square$

**Corollary 6.10.** *Let $H \subset \mathbb{R}^m$ be an open set such that $G \subset H$ and $\mathscr{H}_m(H\setminus G) = 0$. Let $C_{1,2}(\{x \in \partial H; H$ is $(1,2)$-thin at $x\}) = 0$. Suppose that $\partial H$ is compact and $H$ has finitely many components $H_1,\dots,H_n$. Suppose that $\operatorname{cl} H_i \cap \operatorname{cl} H_j = \emptyset$ for $i \neq j$. Denote by $H_1,\dots,H_k$ all bounded components of $H$. Then there are $\mathscr{F}_1,\dots,\mathscr{F}_k \in \mathscr{E}(\partial H)\cap C'(\partial H)$ such that $\mathscr{U}\mathscr{F}_j = 1$ on $H_j$ and $\mathscr{U}\mathscr{F}_j = 0$ on $H\setminus H_j$. The real measures $\mathscr{F}_1,\dots,\mathscr{F}_k$ form a basis of $\operatorname{Ker} N^G\mathscr{U}$.*

Proof. According to Proposition 6.4 there are $\mathscr{F}_1,\dots,\mathscr{F}_k \in \mathscr{E}(\partial H)\cap C'(\partial H)$ such that $\mathscr{U}\mathscr{F}_j = 1$ on $H_j$ and $\mathscr{U}\mathscr{F}_j = 0$ on $H \setminus H_j$. Lemma 6.1 shows that $\mathscr{F}_1,\dots,\mathscr{F}_k \in \operatorname{Ker} N^H\mathscr{U}$. It is easily seen that $\mathscr{F}_1,\dots,\mathscr{F}_k$ are linearly independent. Let now $\mathscr{F} \in \operatorname{Ker} N^H\mathscr{U}$. By Lemma 6.1 there are constants $c_1,\dots,c_k$ such that $\mathscr{U}\mathscr{F} = c_i$ on $H_i$, $i = 1,\dots,k$, and $\mathscr{U}\mathscr{F} = 0$ on $H\setminus(H_1\cup\dots\cup H_k)$. Since $\mathscr{U}(\mathscr{F} - \sum c_i\mathscr{F}_i) = 0$ in $H$, Proposition 6.7 gives $\mathscr{F} - \sum c_i\mathscr{F}_i = 0$. Hence $\mathscr{F}_1,\dots,\mathscr{F}_k$ form a basis of $N^H\mathscr{U}$. Since $N^G\mathscr{U} = N^H\mathscr{U}$ by Proposition 6.2 we get that $\mathscr{F}_1,\dots,\mathscr{F}_k$ is a basis of $N^G\mathscr{U}$. $\square$

## 7. Solution of the problem

As was shown in Theorem 4.1, solving the Neumann problem for an open set $G$ with the boundary condition $\mathscr{B}$ is equivalent to solving the equation $N^G\mathscr{U}\mathscr{F} = \mathscr{B}$. For simple domains we are able to calculate $N^G\mathscr{U}\mathscr{F}$ and to solve the equation $N^G\mathscr{U}\mathscr{F} = \mathscr{B}$. In Example 7.1 we show that $N^G\mathscr{U} = \frac{1}{2}I$ if $G$ is a halfspace. In the classical theory when $G$ has smooth boundary and the boundary conditions are Hölder continuous functions, the corresponding integral operator has the form $\frac{1}{2}I+K$ where $K$ is a compact operator. This is not true for $N^G\mathscr{U}$. (In Example 7.2 we consider $G$ such that $\mathscr{H}_m(\mathbb{R}^m \setminus G) = 0$. For such a $G$ we have $N^G\mathscr{U} = I$.) The theory, when the boundary conditions are real measures, leads to an integral operator $T$. Under the assumption that the essential spectral radius of $(T - \frac{1}{2}I)$ is smaller than $\frac{1}{2}$, a necessary and sufficient condition for the solvability of the equation $T\nu = \mu$ is given. (The essential spectral radius of an operator $S$ is defined as $\sup\{|\lambda|;\ S - \lambda I$ is not Fredholm$\}$.) If $G$ is moreover a simply connected bounded domain then $\nu = \sum[-2(T-\frac{1}{2}I)]^n\mu$ is a solution of the equation $T\nu = \mu$ (see [16]). If $G$ is an open set such that $\mathscr{H}_m(\mathbb{R}^m\setminus G) = 0$ then the essential radius of $(N^G\mathscr{U} - \frac{1}{2}I)$ is equal to $\frac{1}{2}$ and the sum $\sum[-2(N^G\mathscr{U} - \frac{1}{2}I)]^n\mathscr{F}$ diverges for each nonzero $\mathscr{F} \in \mathscr{E}(\partial G)$ in spite of the fact that the operator $N^G\mathscr{U}(= I)$ is invertible and we are able to calculate its inverse. If the Neumann problem is studied on Lipschitz domains and with boundary conditions from $L_2$ then the necessary and sufficient condition for the solvability of the problem is a consequence of the fact that the corresponding integral operator $T$ is Fredholm. It is evident that $N^G\mathscr{U}$ is not Fredholm in general. (Put $G = \bigcup\{\Omega_1(4n);\ n \in N\}$. Then the dimension of $\operatorname{Ker} N^G\mathscr{U}$ is infinite by Proposition 6.4 and therefore the operator $N^G\mathscr{U}$ is not Fredholm.) So, we shall study the solvability of the Neumann problem under the milder condition that $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is a closed subspace of $\mathscr{E}(\partial G)$. Under this condition we shall show that $\mathscr{B} = \sum(N^G\mathscr{U} - I)^n\mathscr{F}$ is a solution of the equation $N^G\mathscr{U}\mathscr{B} = \mathscr{F}$ if this equation has a solution.

**Example 7.1.** Denote $x' = (x_1, \ldots, x_{m-1})$. Suppose that $G = \{(x', x_m);\ x_m > 0\}$. It is evident that $\mathscr{U}\mathscr{F}(x', x_m) = \mathscr{U}\mathscr{F}(x', -x_m)$ for each $\mathscr{F} \in \mathscr{E}(\partial G)$ and $(x', x_m) \in \mathbb{R}^m$. If $\mathscr{B}, \mathscr{F} \in \mathscr{E}(\partial G)$ then we get, using the substitution $(y', y_m) \mapsto (y', -y_m)$,

$$\begin{aligned}(\mathscr{B}, \mathscr{F})_{\mathscr{E}} &= \int_G \nabla\mathscr{U}\mathscr{B} \cdot \overline{\nabla\mathscr{U}\mathscr{F}} \,\mathrm{d}\mathscr{H}_m + \int_{\mathbb{R}^m\setminus \mathrm{cl}\, G} \nabla\mathscr{U}\mathscr{B} \cdot \overline{\nabla\mathscr{U}\mathscr{F}} \,\mathrm{d}\mathscr{H}_m \\ &= 2\int_G \nabla\mathscr{U}\mathscr{B} \cdot \overline{\nabla\mathscr{U}\mathscr{F}} \,\mathrm{d}\mathscr{H}_m = 2(\mathscr{B}, J_G\mathscr{F})_{\mathscr{E}},\end{aligned}$$

because $\mathscr{H}_m(\partial G) = 0$. If $\mathscr{F} \in \mathscr{E}(\partial G)$, $\mathscr{B} \in \mathscr{E}(\partial G)^\perp \subset \mathscr{E}$ then

$$(\mathscr{B}, \mathscr{F})_{\mathscr{E}} = 0 = 2(\mathscr{B}, J_G\mathscr{F})_{\mathscr{E}},$$

because $J_G\mathscr{F} \in \mathscr{E}(\partial G)$ by Lemma 5.3. Lemma 5.3 now gives $N^G\mathscr{U}\mathscr{F} = J_G\mathscr{F} = \frac{1}{2}\mathscr{F}$ for each $\mathscr{F} \in \mathscr{E}(\partial G)$ and therefore $(N^G\mathscr{U})^{-1} = 2I$. If $\mathscr{F} \in \mathscr{E}(\partial G)$ then $2\mathscr{U}\mathscr{F} + c$, where $c$ is a complex constant, is the general form of a solution of the weak Neumann problem in $G$ with the boundary condition $\mathscr{F}$ by Theorem 4.1 and Theorem 3.1.

**Example 7.2.** Suppose that $G \subset \mathbb{R}^m$ is an open set such that $\mathscr{H}_m(\mathbb{R}^m \setminus G) = 0$. If $\mathscr{F} \in \mathscr{E}(\partial G)$, $\mathscr{B} \in \mathscr{E}$ then

$$(\mathscr{B}, \mathscr{F})_{\mathscr{E}} = \int_G \nabla\mathscr{U}\mathscr{B} \cdot \overline{\nabla\mathscr{U}\mathscr{F}} \, \mathrm{d}\mathscr{H}_m = (\mathscr{B}, J_G\mathscr{F})_{\mathscr{E}}.$$

According to Lemma 5.3 we have $N^G\mathscr{U}\mathscr{F} = J_G\mathscr{F} = \mathscr{F}$. If $\mathscr{F} \in \mathscr{E}(\partial G)$ then $\mathscr{U}\mathscr{F}+c$, where $c$ is a complex constant, is the general form of a solution of the weak Neumann problem in $G$ with the boundary condition $\mathscr{F}$ (see Theorem 4.1 and Theorem 3.1).

**Definition 7.3.** Let $X$ be a Banach space and let $T$, $S$ be bounded linear operators on $X$. The operator $S$ is called a Drazin inverse of $T$, written $S = T_d$, if

$$TS = ST, \quad S = STS, \quad T^k = T^kST,$$

for some nonnegative integer $k$. The least nonnegative integer $k$ for which these equations hold is called the Drazin index of $T$.

**Remark 7.4.** According to [9], Lemma 2.4 the Drazin inverse of an operator is unique. Moreover, the operator $T$ is invertible if and only if there is a Drazin inverse of $T$ and the Drazin index of $T$ is equal to 0. In this case $T_d = T^{-1}$.

**Proposition 7.5.** *The following statements are equivalent:*

1) *There is a Drazin inverse of* $N^G\mathscr{U}$.
2) $0$ *is not an accumulation point of* $\sigma(N^G\mathscr{U})$, *the spectrum of* $N^G\mathscr{U}$.
3) $N^G\mathscr{U}(\mathscr{E}(\partial G))$ *is closed.*
4) $N^G\mathscr{U}(\mathscr{E}(\partial G)) = (\operatorname{Ker} N^G\mathscr{U})^{\perp}$.
5) *If* $P$ *is the projection of* $\mathscr{E}(\partial G)$ *to* $\operatorname{cl}(N^G\mathscr{U}(\mathscr{E}(\partial G)))$ *along* $\operatorname{Ker} N^G\mathscr{U}$ *then the operator* $(I - N^G\mathscr{U})P$ *is contractive, i.e.,* $\|(I - N^G\mathscr{U})P\| < 1$.
6) *There is a Drazin inverse of* $N^G\mathscr{U}$, *the Drazin index of* $N^G\mathscr{U}$ *is at most* 1 *and*

$$(N^G\mathscr{U})_d = \sum_{j=0}^{\infty}(I - N^G\mathscr{U})^j P, \tag{7.1}$$

*where* $P$ *is the projection of* $\mathscr{E}(\partial G)$ *to* $\operatorname{cl}(N^G\mathscr{U}(\mathscr{E}(\partial G)))$ *along* $\operatorname{Ker} N^G\mathscr{U}$.

P r o o f. $1 \Rightarrow 2$: If there is a Drazin inverse of $N^G\mathscr{U}$ then there is the so-called generalized Drazin inverse of $N^G\mathscr{U}$ (see [9]). Therefore 0 is not an accumulation point of $\sigma(N^G\mathscr{U})$ by [9], Theorem 4.2.

$2 \Rightarrow 3$: Denote by $T$ the restriction of $N^G\mathscr{U}$ to $\operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G))$. Then $\sigma(T) \subset \sigma(N^G\mathscr{U})$ because the space $\mathscr{E}(\partial G)$ is the direct sum of $\operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G))$ and $\operatorname{Ker} N^G\mathscr{U}$ by Proposition 5.4. This gives that 0 is not an accumulation point of $\sigma(T)$. Since $N^G\mathscr{U}$ is a selfadjoint operator by Proposition 5.4, the operator $T$ is selfadjoint and therefore hyponormal (see [19], § 11.5). Since every isolated point of the spectrum of a hyponormal operator is an eigenvalue by [22], Theorem 2 and $\operatorname{Ker} N^G\mathscr{U} \cap \operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G)) = \{0\}$ by Proposition 5.4 we deduce that $0 \notin \sigma(T)$. Hence $\operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G)) = N^G\mathscr{U}(\mathscr{E}(\partial G))$.

$3 \Rightarrow 4$: Since $N^G\mathscr{U}(\mathscr{E}(\partial G))^\perp = \operatorname{Ker} N^G\mathscr{U}$ by Proposition 5.4 and the subspace $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is closed we conclude that $N^G\mathscr{U}(\mathscr{E}(\partial G)) = (\operatorname{Ker} N^G\mathscr{U})^\perp$.

$4 \Rightarrow 5$: Denote by $T$ the restriction of $N^G\mathscr{U}$ to $X = N^G\mathscr{U}(\mathscr{E}(\partial G))$. Since $N^G\mathscr{U}$ is a selfadjoint operator and $0 \leqslant N^G\mathscr{U} \leqslant I$ by Proposition 5.4 the operator $T$ is selfadjoint and $0 \leqslant T \leqslant I$, $0 \leqslant I - T \leqslant I$. Therefore $\sigma(I - T) \subset [0, 1]$ by [25], Chapter XI, § 8, Theorem 2. Since $N^G\mathscr{U}(\mathscr{E}(\partial G)) = (\operatorname{Ker} N^G\mathscr{U})^\perp$, the operator $T$ is injective and $T(X) = X$. Therefore $0 \notin \sigma(T)$ by [25], Chapter II, § 6. Since $\sigma(I - T) \subset [0, 1)$, the spectral radius of $T$ is smaller than 1 (see [25], Chapter VIII, § 2, Theorem 4). Since the operator $I - T$ is selfadjoint it is hyponormal. Since the norm of a hyponormal operator is equal to its spectral radius (see [22], Theorem 1) we have $\|T\| < 1$. Since $\|P\| \leqslant 1$ by [25], Chapter III, Theorem 3 we obtain $\|(I - N^G\mathscr{U})P\| = \|TP\| \leqslant \|T\|\|P\| < 1$ using [25], Chapter I, Proposition 2.

$5 \Rightarrow 6$: The series

$$S = \sum_{j=0}^{\infty} (I - N^G\mathscr{U})^j P$$

converges. If $\mathscr{F} \in \operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G))$ then $N^G\mathscr{U} S\mathscr{F} = \mathscr{F} = SN^G\mathscr{U}\mathscr{F}$. If $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$ then $N^G\mathscr{U} S\mathscr{F} = 0 = SN^G\mathscr{U}\mathscr{F}$. Since $\mathscr{E}(\partial G)$ is the direct sum of $\operatorname{Ker} N^G\mathscr{U}$ and $\operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G))$ by Proposition 5.4 we obtain that $SN^G\mathscr{U} = N^G\mathscr{U} S$. If $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$ then $SN^G\mathscr{U} S\mathscr{F} = 0 = S\mathscr{F}$ and $N^G\mathscr{U} SN^G\mathscr{U}\mathscr{F} = 0 = N^G\mathscr{U}\mathscr{F}$. If $\mathscr{F} \in \operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G))$ then $SN^G\mathscr{U} S\mathscr{F} = S\mathscr{F}$ and $N^G\mathscr{U} SN^G\mathscr{U}\mathscr{F} = N^G\mathscr{U}\mathscr{F}$. Since $\mathscr{E}(\partial G)$ is the direct sum of $\operatorname{Ker} N^G\mathscr{U}$ and $\operatorname{cl} N^G\mathscr{U}(\mathscr{E}(\partial G))$ by Proposition 5.4 we have $SN^G\mathscr{U} S = S$ and $N^G\mathscr{U} SN^G\mathscr{U} = N^G\mathscr{U}$. Hence the Drazin index of $N^G\mathscr{U}$ is at most 1 and $(N^G\mathscr{U})_d = S$.

$6 \Rightarrow 1$: This implication is trivial. □

**Theorem 7.6.** *Suppose that $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is closed. Let $\mathscr{F} \in \mathscr{E}(\partial G)$. Then there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if and only if $\mathscr{F} \in (\operatorname{Ker} N^G\mathscr{U})^\perp$. If $(N^G\mathscr{U})_d$ is given* *by* (7.1) *and*

$$\mathscr{B} = (N^G\mathscr{U})_d\mathscr{F} = \sum_{j=0}^{\infty}(I - N^G\mathscr{U})^j\mathscr{F}, \tag{7.2}$$

*then* $\mathscr{U}\mathscr{B}$ *is a solution of this problem. Moreover, there is a positive constant* $M$ *dependent only on* $G$ *such that*

$$\sqrt{\int_G |\nabla u|^2 \, \mathrm{d}\mathscr{H}_m} \leqslant M\|\mathscr{F}\|_{\mathscr{E}}$$

*for each solution* $u$ *of the weak Neumann problem for the Laplace equation in* $G$ *with the boundary condition* $\mathscr{F}$*. If* $G$ *is connected then* $\mathscr{U}\mathscr{B} + c$*, where* $c$ *is a constant, is the general form of a solution of the weak Neumann problem for the Laplace equation in* $G$ *with the boundary condition* $\mathscr{F}$*.*

Proof. Suppose that there is a solution $u$ of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. According to Theorem 4.1 there are $\mathscr{B} \in \mathscr{E}(\partial G)$ and a constant $a$ such that $u = \mathscr{U}\mathscr{B} + a$ in $G$. Thus $N^G\mathscr{U}\mathscr{B} = \mathscr{F} \in (\operatorname{Ker} N^G\mathscr{U})^{\perp}$ by Proposition 7.5.

Suppose now that $\mathscr{F} \in (\operatorname{Ker} N^G\mathscr{U})^{\perp}$. Proposition 7.5 yields that the series (7.1) converges and $\mathscr{F} \in N^G\mathscr{U}(\mathscr{E}(\partial G))$. Put $\mathscr{B} = (N^G\mathscr{U})_d\mathscr{F}$. An easy calculation yields that $\mathscr{B}$ is given by (7.2). If $c$ is a constant then $\mathscr{U}\mathscr{B} + c \in L^1_2(\mathbb{R}^m)$ is a harmonic function in $G$ by Lemma 5.3. Since

$$N^G(\mathscr{U}\mathscr{B}+c) = N^G\mathscr{U}\sum_{j=0}^{\infty}(I-N^G\mathscr{U})^j\mathscr{F} = [I-(I-N^G\mathscr{U})]\sum_{j=0}^{\infty}(I-N^G\mathscr{U})^j\mathscr{F} = \mathscr{F},$$

$\mathscr{U}\mathscr{B} + c$ is a solution of the weak Neumann problem in $G$ with the boundary condition $\mathscr{F}$. Put $M = \|(N^G\mathscr{U})_d\|$. If $u$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ then $u - \mathscr{U}\mathscr{B}$ is constant on each component of $G$ by Theorem 3.1. Therefore

$$\sqrt{\int_G |\nabla u|^2 \, \mathrm{d}\mathscr{H}_m} = \sqrt{\int_G |\nabla \mathscr{U}\mathscr{B}|^2 \, \mathrm{d}\mathscr{H}_m} \leqslant \|\mathscr{B}\|_{\mathscr{E}} \leqslant M\|\mathscr{F}\|_{\mathscr{E}}.$$

□

**Theorem 7.7.** *Let $N^G\mathscr{U}(\mathscr{E}(\partial G))$ be closed. Suppose that there is an open set $H \subset \mathbb{R}^m$ such that $G \subset H$, $\mathscr{H}_m(H \setminus G) = 0$ and $C_{1,2}(\{x \in \partial H;\ H$ is $(1,2)$-thin at $x\}) = 0$. Suppose that $\partial H$ is compact and $H$ has finitely many components $H_1, \ldots, H_n$. Suppose that $\operatorname{cl} H_i \cap \operatorname{cl} H_j = \emptyset$ for $i \neq j$. Denote by $H_1, \ldots, H_k$ all bounded components of $H$. Fix $\varphi_1, \ldots, \varphi_k \in \mathscr{D}$ so that $\varphi_j = 1$ on $H_j$ and $\varphi_j = 0$ on $H \setminus H_j$, $j = 1, \ldots, k$. Let $\mathscr{F} \in \mathscr{E}(\partial G)$. Then there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if and only if $\mathscr{F}(\varphi_j) = 0$ for $j = 1, \ldots, k$. (We can write $\mathscr{F}(\partial H_j) = 0$ for $j = 1, \ldots, k$.) If $\mathscr{B}$ is given by (7.2) then*

$$\mathscr{U}\mathscr{B} + \sum_{j=1}^{n} c_j \chi_{H_j}, \quad c_1, \ldots, c_n \in C, \tag{7.3}$$

*is the general form of a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. (Here $\chi_M$ denotes the characteristic function of a set $M$.)*

P r o o f. If there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ then $\mathscr{F}(\varphi_j) = 0$ for $j = 1, \ldots, k$ by Theorem 5.7.

Suppose now that $\mathscr{F}(\varphi_j) = 0$ for $j = 1, \ldots, k$. We have $\varphi_j = -\mathscr{U}\Delta\varphi_j$ and $\Delta\varphi_j \in \mathscr{E}$ for $j = 1, \ldots, k$ by [12], p. 100 and [12], Theorem 6.2. Denote by $\mathscr{F}_j$ the orthogonal projection of $-\Delta\varphi_j$ onto $\mathscr{E}(\operatorname{cl} H)$. Then $\mathscr{U}\mathscr{F}_j = \mathscr{U}(-\Delta\varphi_j) = \varphi_j$ on $H$ by [3], Chapitre I, Théorème 4. Since $\Delta\mathscr{U}\mathscr{F}_j = -\mathscr{F}_j$ by [3], p. 158 and $\Delta\mathscr{U}\mathscr{F}_j = \Delta\varphi_j = 0$ on $H$ we obtain $\mathscr{F}_j \in \mathscr{E}(\partial H) \subset \mathscr{E}(\partial G)$. Since $\mathscr{U}\mathscr{F}_j = 1$ on $H_j$ and $\mathscr{U}\mathscr{F}_j = 0$ on $H \setminus H_j$ the distributions $\mathscr{F}_1, \ldots, \mathscr{F}_k$ form a linearly independent subset of $\operatorname{Ker} N^G\mathscr{U}$. Since the dimension of $\operatorname{Ker} N^G\mathscr{U}$ is equal to $k$ by Corollary 6.10, the distributions $\mathscr{F}_1, \ldots, \mathscr{F}_k$ form a basis of $\operatorname{Ker} N^G\mathscr{U}$. Since $\mathscr{F}_j$ is the orthogonal projection of $-\Delta\varphi_j$ onto $\mathscr{E}(\operatorname{cl} H)$ and $\mathscr{F} \in \mathscr{E}(\operatorname{cl} H)$, Lemma 5.1 yields

$$(\mathscr{F}, \mathscr{F}_j)_{\mathscr{E}} = (\mathscr{F}, -\Delta\varphi_j)_{\mathscr{E}} = \mathscr{F}(\varphi_j) = 0, \quad j = 1, \ldots, k.$$

Since $\mathscr{F}_1, \ldots, \mathscr{F}_k$ is a basis of $\operatorname{Ker} N^G\mathscr{U}$ we deduce that $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}^{\perp}$. If $\mathscr{B}$ is given by (7.2) then the function $\mathscr{U}\mathscr{B}$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ by Theorem 7.6. If $c_1, \ldots, c_n \in C$ then $\mathscr{U}\mathscr{B} + \sum_{j=1}^{n} c_j \chi_{H_j}$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. Let now $u \in L^1_2(\mathbb{R}^m)$ be a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. We can choose $v \in L^1_2(\mathbb{R}^m)$ such that $u(x) = v(x)$ for $\mathscr{H}_m$

almost all $x \in \mathbb{R}^m$ and

$$v(x) = \lim_{r \to 0_+} \frac{1}{\mathscr{H}_m(\Omega_r(x))} \int_{\Omega_r(x)} v(y) \, \mathrm{d}\mathscr{H}_m(y) \tag{7.4}$$

whenever the limit exists (see [1], Theorem 1.2.3, [1], Theorem 6.2.1 and [1], Corollary 5.1.14). Then $v = u$ in $G$, because $u$ is continuous in $G$. Thus $v$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ and $v - \mathscr{U}\mathscr{B}$ is a solution of the weak Neumann problem in $G$ for the Laplace equation with zero boundary condition. Theorem 3.1 yields that $\nabla(v - \mathscr{U}\mathscr{B}) = 0$ in $G$. Since $\mathscr{H}_m(H \setminus G) = 0$, the vector function $\nabla(v - \mathscr{U}\mathscr{B}) = 0$ a.e. in $H$. Using this fact, (7.4) and (4.3), [13], Lemma on page 11 yields that there are $c_1, \ldots, c_n$ such that $v - \mathscr{U}\mathscr{B} = \sum_{j=1}^{n} c_j \chi_{H_j}$ in $H$. This gives that $u$ has the form (7.3) in $G$. □

## 8. Extendible open sets

**Definition 8.1.** An open set $G \subset \mathbb{R}^m$ is said to be $W_2^1$-*extendible* if there is a bounded linear operator $T\colon W_2^1(G) \to W_2^1(\mathbb{R}^m)$ such that $Tu = u$ on $G$ for each $u \in W_2^1(G)$.

**Definition 8.2.** A domain $G$ is said to be an $(\varepsilon, \delta)$ domain, $\varepsilon > 0$, $0 < \delta \leqslant \infty$, if, whenever $x, y \in G$ and $|x - y| < \delta$, then there is a rectifiable arc $\gamma \subset G$ with length $l(\gamma)$ joining $x$ and $y$ and satisfying

$$l(\gamma) < \frac{|x-y|}{\varepsilon},$$

$$\operatorname{dist}(z, \partial G) \geqslant \frac{\varepsilon |x - z| \cdot |y - z|}{|x - y|} \quad \text{for all } z \in \gamma.$$

**Remark 8.3.** If $G$ is an $(\varepsilon, \delta)$ domain then it is $W_2^1$-extendible (see [8], Theorem 1). S. Jerison and C. E. Kenig studied in [7] the so called nontangentially accessible domains. As was noticed by P. W. Jones in [8], p. 73, these domains are precisely $(\varepsilon, \infty)$ domains. Note that Lipschitz domains and polyhedral domains are nontangentially accessible domains. If $G$ is an $(\varepsilon, \delta)$ domain then $\mathscr{H}_m(\partial G) = 0$ (see [8], Lemma 2.3). The boundary of an $(\varepsilon, \delta)$ domain can be highly nonrectifiable and no regularity condition on the boundary can be inferred from the $(\varepsilon, \delta)$ property. In general, $(\varepsilon, \delta)$ domains are not sets of finite perimeter.

**Lemma 8.4.** *Let $G$ be bounded. Then the operator $T\colon \mathscr{F} \mapsto \mathscr{U}\mathscr{F}$ is a bounded linear operator from $\mathscr{E}(\operatorname{cl} G)$ to $W_2^1(G)$. If $G$ is $W_2^1$-extendible then $T(\mathscr{E}(\operatorname{cl} G)) = W_2^1(G)$ and there is a positive constant $C$ such that*

$$\|\mathscr{F}\|_{\mathscr{E}} \leqslant C\|\mathscr{U}\mathscr{F}\|_{W_2^1(G)}$$

*for each $\mathscr{F} \in \mathscr{E}(\operatorname{cl} G) \cap [\operatorname{Ker} T]^{\perp}$.*

P r o o f. Because $\mathscr{U}\mathscr{F} \in L_2^1(\mathbb{R}^m)$ for each $\mathscr{F} \in \mathscr{E}(\operatorname{cl} G)$, the operator $T$ is a linear operator from $\mathscr{E}(\operatorname{cl} G)$ to $W_2^1(G)$. We show that $T$ is a closed operator. Suppose that $\mathscr{F}_n \to \mathscr{F}$ in $\mathscr{E}(\operatorname{cl} G)$ and $\mathscr{U}\mathscr{F}_n \to g$ in $W_2^1(G)$. According to [3], Chap. II, § 2 and [12], Theorem 3.13 we can choose a subsequence $\mathscr{F}_{n(j)}$ such that $\mathscr{U}\mathscr{F}_{n(j)} \to \mathscr{U}\mathscr{F}$ $\mathscr{H}_m$-a.e. and $\mathscr{U}\mathscr{F}_{n(j)} \to g$ $\mathscr{H}_m$-a.e. in $G$. Since $\mathscr{U}\mathscr{F} = g$ $\mathscr{H}_m$-a.e., we deduce that $T\mathscr{F} = g$ in $W^{1,2}(G)$. Since $T$ is a closed linear operator from the Banach space $\mathscr{E}(\operatorname{cl} G)$ to the Banach space $W_2^1(G)$ which is defined on the whole space $\mathscr{E}(\operatorname{cl} G)$, it is a bounded operator by the closed graph theorem (see [25], Chapter II, § 6, Theorem 1).

Suppose now that $G$ is $W_2^1$-extendible and $g \in W_2^1(G)$. Since $G$ is $W_2^1$-extendible we can suppose that $g \in W_2^1(\mathbb{R}^m)$. Since $G$ is bounded we can suppose that $g$ has compact support. According to [12], Chapter VI, Theorem 6.4 there is $\mathscr{B} \in \mathscr{E}$ such that $g = \mathscr{U}\mathscr{B}$. Denote by $\mathscr{F}$ the orthogonal projection of $\mathscr{B}$ to $\mathscr{E}(\operatorname{cl} G)$. Then $\mathscr{U}\mathscr{F} = \mathscr{U}\mathscr{B} = g$ on $G$ by [3], Chapitre I, Théorème 4. Therefore $T(\mathscr{E}(\operatorname{cl} G)) = W_2^1(G)$. Since $T$ is a bounded injective linear operator from the Banach space $\mathscr{F} \in \mathscr{E}(\operatorname{cl} G) \cap [\operatorname{Ker} T]^{\perp}$ onto the Banach space $W_2^1(G)$ it is continuously invertible, i.e., there is a positive constant $C$ such that

$$\|\mathscr{F}\|_{\mathscr{E}} \leqslant C\|T\mathscr{F}\|_{W_2^1(G)}$$

for each $\mathscr{F} \in \mathscr{E}(\operatorname{cl} G) \cap [\operatorname{Ker} T]^{\perp}$ (see [25], Chapter II, § 5). □

**Lemma 8.5.** *Let $G$ be a bounded $W_2^1$-extendible open set. Then the identity operator $I$ is a compact linear operator from $W_2^1(G)$ to $L_2(G)$.*

P r o o f. Fix $R > 0$ such that $G \subset \Omega_R(0)$. Since $G$ is $W_2^1$-extendible there is a bounded linear operator $T$ from $W_2^1(G)$ to $W_2^1(\Omega_R(0))$ such that $Tu = u$ on $G$ for each $u \in W_2^1(G)$. Since $\Omega_R(0)$ is a bounded $W_2^1$-extendible domain, the identity operator $\tilde{I}$ from $W_2^1(\Omega_R(0))$ to $L_2(G)$ is a compact operator (see [13], § 1.10, Theorem 3). Since $I = \tilde{I}T$ is the composition of a compact operator and a bounded operator, it is a compact operator (see [25], Chapter X, § 2). □

**Lemma 8.6.** *Let $G$ be a bounded $W_2^1$-extendible open set. Then $G$ has finitely many components and each of these components is a $W_2^1$-extendible domain.*

Proof. Put $T_1(u) = [u, \nabla u]$. Then $T_1$ is a bounded linear operator from $W_2^1(G)$ to $[L_2(G)]^{m+1}$. Since $\|T_1 u\| = \|u\|$ for each $u \in W_2^1(G)$ and $W_2^1(G)$ is a Banach space, we obtain that $T_1(W_2^1(G))$ is a closed subspace of $[L_2(G)]^{m+1}$. Denote $T_2([u, \nabla u]) = [u, 0, \ldots, 0]$ for $[u, \nabla u] \in T_1(W_2^1(G))$. Then $T_2$ is a compact linear operator from $T_1(W_2^1(G))$ to $[L_2(G)]^{m+1}$ by Lemma 8.5. Denote by $T_3$ the orthogonal projection of $[L_2(G)]^{m+1}$ onto $T_1(W_2^1(G))$. Then $T_2T_3$ is a compact linear operator on $[L_2(G)]^{m+1}$ as a composition of a compact linear operator and a bounded linear operator (see [25], Chapter X, § 2). Since $T_2T_3$ is a compact operator, the dimension of $\operatorname{Ker}(I - T_2T_3)$ is finite (see [25], Chapter X, § 4, Theorem 2). If $H$ is a component of $G$ then $(\chi_H, \nabla \chi_H) \in \operatorname{Ker}(I - T_2T_3)$. Since the dimension of $\operatorname{Ker}(I - T_2T_3)$ is finite $G$ has finitely many components. Let $T_4$ be a continuous linear extension operator from $W_2^1(G)$ to $W_2^1(\mathbb{R}^m)$. If $H$ is a component of $G$ define $T_5 u = u$ on $H$, $T_5 u = 0$ on $G \setminus H$ for $u \in W_2^1(H)$. Then $T_5$ is a continuous linear operator from $W_2^1(H)$ to $W_2^1(G)$ and $T_4T_5$ is a continuous linear extension operator from $W_2^1(H)$ to $W_2^1(\mathbb{R}^m)$. □

**Lemma 8.7.** *Let $G$ be a bounded $W_2^1$-extendible open set, $P$ be a seminorm on $W_2^1(G)$ such that $u \in W_2^1(G)$, $\nabla u = 0$ and $P(u) = 0$ imply $u \equiv 0$. Then the norm*

$$|u|_P = |P(u)| + \sqrt{\int_G |\nabla u|^2 \, \mathrm{d}\mathscr{H}_m}$$

*is equivalent to the norm in $W_2^1(G)$.*

Proof. Clearly, $|\cdot|_P$ is a norm. $G$ has finitely many components $G_1, \ldots, G_n$ by Lemma 8.6. According to [13], § 3.1.1, Lemma there is a positive constant $c_1$ such that

$$\begin{aligned} \frac{1}{c_1}\left[\left|\int_{G_j} u \, \mathrm{d}\mathscr{H}_m\right| + \sqrt{\int_{G_j} |\nabla u|^2 \, \mathrm{d}\mathscr{H}_m}\right] &\leqslant \sqrt{\int_{G_j} [|u|^2 + |\nabla u|^2] \, \mathrm{d}\mathscr{H}_m} \\ &\leqslant c_1 \left[\left|\int_{G_j} u \, \mathrm{d}\mathscr{H}_m\right| + \sqrt{\int_{G_j} |\nabla u|^2 \, \mathrm{d}\mathscr{H}_m}\right] \end{aligned}$$

for each $u \in W_2^1(G_j)$, $j = 1, \ldots, n$. Therefore

$$\begin{aligned} \frac{1}{nc_1}\left[\left|\int_{G_1} u \, \mathrm{d}\mathscr{H}_m\right| + \ldots + \left|\int_{G_n} u \, \mathrm{d}\mathscr{H}_m\right| + \sqrt{\int_G |\nabla u|^2 \, \mathrm{d}\mathscr{H}_m}\right] &\leqslant \sqrt{\int_G [|u|^2 + |\nabla u|^2] \, \mathrm{d}\mathscr{H}_m} \\ \leqslant nc_1 \left[\left|\int_{G_1} u \, \mathrm{d}\mathscr{H}_m\right| + \ldots + \left|\int_{G_n} u \, \mathrm{d}\mathscr{H}_m\right| + \sqrt{\int_G |\nabla u|^2 \, \mathrm{d}\mathscr{H}_m}\right] \end{aligned}$$

for each $u \in W_2^1(G)$. Denote

$$\|u\| = |P(u)| + \left|\int_{G_1} u \,\mathrm{d}\mathscr{H}_m\right| + \ldots + \left|\int_{G_n} u \,\mathrm{d}\mathscr{H}_m\right| + \sqrt{\int_G |\nabla u|^2 \,\mathrm{d}\mathscr{H}_m}$$

which is a norm on $W_2^1(G)$. Since

$$\sqrt{\int_G [|u|^2 + |\nabla u|^2] \,\mathrm{d}\mathscr{H}_m} \leqslant nc_1\|u\|$$

for each $u \in W_2^1(G)$, the identity operator is a continuous operator from $W_2^1(G)$ equipped with the norm $\|\cdot\|$ to $W_2^1(G)$. According to [19], Theorem 3.8 the inverse operator is continuous. Thus there is a strictly positive constant $c_2$ such that

$$c_2\|u\| \leqslant \sqrt{\int_G [|u|^2 + |\nabla u|^2] \,\mathrm{d}\mathscr{H}_m}$$

for each $u \in W_2^1(G)$. Denote

$$Y = \left\{u \in W_2^1(G);\ \int_{G_j} u \,\mathrm{d}\mathscr{H}_m = 0 \ \text{ for } \ j = 1, \ldots, n\right\}.$$

Then $Y$ is a closed subspace of $W_2^1(G)$ and

$$c_2|u|_P = c_2\|u\| \leqslant \sqrt{\int_G [|u|^2 + |\nabla u|^2] \,\mathrm{d}\mathscr{H}_m} \leqslant nc_1\|u\| = nc_1|u|_P$$

for each $u \in Y$. According to [25], Chapter I, § 10 there is a Banach space $X$ with norm $|\cdot|_X$ such that $W_2^1(G) \subset X$ and $|\cdot|_P = |\cdot|_X$ on $W_2^1(G)$. Then $Y$ is a closed subspace of $X$. Since $W_2^1(G)$ is the sum of the closed space $Y$ and a finite-dimensional space it is a closed subspace of $X$. Thus $W_2^1(G)$ equipped with the norm $|\cdot|_P$ is a Banach space. Since the identity operator is a continuous operator from $W_2^1(G)$ equipped with the norm $\|\cdot\|$ to $W_2^1(G)$ equipped with the norm $|\cdot|_P$, the inverse of this operator is continuous by [19], Theorem 3.8. Hence there is a positive constant $c_3$ such that $\|u\| \leqslant c_3|u|_P$ for each $u \in W_2^1(G)$. This gives

$$c_2|u|_P \leqslant \sqrt{\int_G [|u|^2 + |\nabla u|^2] \,\mathrm{d}\mathscr{H}_m} \leqslant nc_1c_3|u|_P$$

for each $u \in W_2^1(G)$. □

**Theorem 8.8.** *If $G$ is a bounded $W_2^1$-extendible open set then $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is closed.*

Proof. The operator $T\colon \mathscr{F} \mapsto \mathscr{U}\mathscr{F}$ is a bounded linear operator from $\mathscr{E}(\operatorname{cl} G)$ to $W_2^1(G)$ by Lemma 8.4. Denote $Y = \mathscr{E}(\operatorname{cl} G) \cap [\operatorname{Ker} T]^\perp$. Denote by $\tilde{T}$ the restriction of $T$ to $Y$. Then $\tilde{T}$ is a bounded continuously invertible operator from $Y$ onto $W_2^1(G)$ by Lemma 8.4. According to Lemma 8.6 the set $G$ has finitely many components $G_1, \ldots, G_n$. Denote

$$\mathscr{F}_j = \tilde{T}^{-1}\chi_{G_j}, \quad j = 1, \ldots, n.$$

Define on $W_2^1(G)$ the seminorm

$$P(u) = \sum_{j=1}^{n} |(\tilde{T}^{-1}u, \mathscr{F}_j)_{\mathscr{E}}|.$$

According to Lemma 8.7 there is a strictly positive constant $c$ such that

$$c \cdot \sqrt{\int_G [|u|^2 + |\nabla u|^2] \,\mathrm{d}\mathscr{H}_m} \leqslant P(u) + \sqrt{\int_G |\nabla u|^2 \,\mathrm{d}\mathscr{H}_m}.$$

We now show that $\operatorname{Ker} T \subset \operatorname{Ker} N^G\mathscr{U}$. Let $\mathscr{F} \in \operatorname{Ker} T$. Since $\Delta\mathscr{U}\mathscr{F} = 0$ in $G$ and $\Delta\mathscr{U}\mathscr{F} = -\mathscr{F}$ by [3], Chapitre 1, Théorème 4 we deduce that $\mathscr{F} \in \mathscr{E}(\partial G)$ and thus $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}$.

Let now $\mathscr{F} \in \mathscr{E}(\partial G) \cap [\operatorname{Ker} N^G\mathscr{U}]^\perp$, $\|\mathscr{F}\|_{\mathscr{E}} = 1$. Since $\operatorname{Ker} T \subset \operatorname{Ker} N^G\mathscr{U}$ we have $\mathscr{F} \in Y$. Since $\mathscr{F} \in [\operatorname{Ker} N^G\mathscr{U}]^\perp$ and $\mathscr{F}_1, \ldots, \mathscr{F}_n \in [\operatorname{Ker} N^G\mathscr{U}]$ we obtain $P(\mathscr{U}\mathscr{F}) = 0$. Therefore

$$\begin{aligned} 1 \geqslant (\mathscr{F}, N^G\mathscr{U}\mathscr{F})_{\mathscr{E}} &= \int_G |\nabla\mathscr{U}\mathscr{F}|^2 \,\mathrm{d}\mathscr{H}_m \\ &\geqslant c^2 \int_G [|\mathscr{U}\mathscr{F}|^2 + |\nabla\mathscr{U}\mathscr{F}|^2] \,\mathrm{d}\mathscr{H}_m \geqslant c^2\|\tilde{T}^{-1}\|^{-2}. \end{aligned}$$

Since the restriction of $N^G\mathscr{U}$ to $\mathscr{E}(\partial G) \cap [\operatorname{Ker} N^G\mathscr{U}]^\perp$ is a selfadjoint operator by Proposition 5.4, its spectrum is a subset of the interval $[c^2\|\tilde{T}^{-1}\|^{-2}, 1]$ (see [6], Satz 70.8). Let $0 < |\lambda| < c^2\|\tilde{T}^{-1}\|^{-2}$. Since $N^G\mathscr{U} - \lambda I$ is a continuously invertible operator on $\mathscr{E}(\partial G) \cap [\operatorname{Ker} N^G\mathscr{U}]^\perp$ and on $\operatorname{Ker} N^G\mathscr{U}$ it is continuously invertible on $\mathscr{E}(\partial G)$. This gives that 0 is not an accumulation point of the spectrum of $N^G\mathscr{U}$ and thus $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is closed by Proposition 7.5. □

**Theorem 9.1.** *Let $G$, $H$ be open subsets of $\mathbb{R}^m$ such that $\mathscr{H}_m((G\setminus H)\cup(H\setminus G)) = 0$. Then $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is closed if and only if $N^H\mathscr{U}(\mathscr{E}(\partial H))$ is closed.*

Proof. Suppose that $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is closed. Denote $V = G \cup H$. Then $V$ is open, $G \subset V$ and $\mathscr{H}_m(V \setminus G) = 0$. We show that $N^V\mathscr{U}(\mathscr{E}(\partial V))$ is closed. Let $\mathscr{F} \in \mathscr{E}(\partial V) \cap [\operatorname{Ker} N^V\mathscr{U}]^\perp$. Then $\mathscr{F} \in \mathscr{E}(\partial G) \cap [\operatorname{Ker} N^G\mathscr{U}]^\perp$ by Proposition 6.2. Since $N^G\mathscr{U}(\mathscr{E}(\partial G))$ is closed Proposition 7.5 gives that there is $\mathscr{G} \in \mathscr{E}(\partial G)$ such that $N^G\mathscr{U}\mathscr{G} = \mathscr{F}$. Since $\mathscr{H}_m(V \setminus G) = 0$ we have $\mathscr{G} - \mathscr{F} = \mathscr{G} - N^G\mathscr{U}\mathscr{G} = J_{\mathbb{R}^m\setminus G}\mathscr{G} = J_{\mathbb{R}^m\setminus V} \in \mathscr{E}(\mathbb{R}^m \setminus V)$ by Lemma 5.2 and Lemma 5.3. Since $\mathscr{G} - \mathscr{F} \in \mathscr{E}(\partial G) \cap \mathscr{E}(\mathbb{R}^m \setminus V)$ we deduce that $\mathscr{G} - \mathscr{F} \in \mathscr{E}(\partial V)$. Since $\mathscr{F} \in \mathscr{E}(\partial V)$ we have $\mathscr{G} \in \mathscr{E}(\partial V)$. Thus $\mathscr{E}(\partial V) \cap [\operatorname{Ker} N^V\mathscr{U}]^\perp \subset N^V\mathscr{U}(\mathscr{E}(\partial V))$. Proposition 5.4 gives $\mathscr{E}(\partial V) \cap [\operatorname{Ker} N^V\mathscr{U}]^\perp = N^V\mathscr{U}(\mathscr{E}(\partial V))$. Therefore $N^V\mathscr{U}(\mathscr{E}(\partial V))$ is closed by Proposition 7.5.

Now we show that $N^H\mathscr{U}(\mathscr{E}(\partial H))$ is closed. Let $\mathscr{F} \in \mathscr{E}(\partial H)\cap[\operatorname{Ker} N^H\mathscr{U}]^\perp$. Then $N^H\mathscr{U}\mathscr{F} \in \mathscr{E}(\partial H)\cap[\operatorname{Ker} N^H\mathscr{U}]^\perp$ by Proposition 5.4. Since $\mathscr{H}_m(V\setminus H) = 0$ it follows from Lemma 5.3 that $\mathscr{F} - N^H\mathscr{U}\mathscr{F} = J_{\mathbb{R}^m\setminus H}\mathscr{F} = J_{\mathbb{R}^m\setminus V}\mathscr{F}$. Thus $\mathscr{F} - N^H\mathscr{U}\mathscr{F} \in \mathscr{E}(\mathbb{R}^m \setminus V)$ by Lemma 5.2. This forces $\mathscr{F} - N^H\mathscr{U}\mathscr{F} \in \mathscr{E}(\partial V)$. Since $\operatorname{Ker} N^H\mathscr{U} = \operatorname{Ker} N^V\mathscr{U}$ by Proposition 6.2, we have $\mathscr{F} - N^H\mathscr{U}\mathscr{F} \in \mathscr{E}(\partial V) \cap [\operatorname{Ker} N^V\mathscr{U}]^\perp$. Since $N^V\mathscr{U}(\mathscr{E}(\partial V))$ is closed Proposition 7.5 yields that there is $\mathscr{G} \in \mathscr{E}(\partial V)$ such that $N^V\mathscr{U}\mathscr{G} = \mathscr{F} - N^H\mathscr{U}\mathscr{F}$. Since $\partial V \subset \partial H$ by Proposition 6.2 we have $\mathscr{G} \in \mathscr{E}(\partial H)$. Since $\mathscr{H}_m(V \setminus H) = 0$ we deduce $N^H\mathscr{U}\mathscr{G} = N^V\mathscr{U}\mathscr{G} = \mathscr{F} - N^H\mathscr{U}\mathscr{F}$. The result is $N^H\mathscr{U}(\mathscr{G}+\mathscr{F}) = \mathscr{F}$. Hence $\mathscr{E}(\partial H)\cap[\operatorname{Ker} N^H\mathscr{U}]^\perp \subset N^H\mathscr{U}(\mathscr{E}(\partial H))$. Proposition 5.4 gives $\mathscr{E}(\partial H) \cap [\operatorname{Ker} N^H\mathscr{U}]^\perp = N^H\mathscr{U}(\mathscr{E}(\partial H))$. Thus $N^H\mathscr{U}(\mathscr{E}(\partial H))$ is closed by Proposition 7.5. □

**Lemma 9.2.** *Suppose that there is a $W_2^1$-extendible open set $H$ such that $H \subset G$, $\mathscr{H}_m(G \setminus H) = 0$. Then $G$ is a $W_2^1$-extendible open set.*

Proof. Let $f \in W_2^1(G)$. Then $f \in W_2^1(H)$ and there is $F \in W_2^1(\mathbb{R}^m)$ such that $F = f$ in $H$. Denote $\tilde{F} = F$ in $\mathbb{R}^m \setminus G$, $\tilde{F} = f$ in $G$. Then $\tilde{F} \in W_2^1(\mathbb{R}^m)$, because $\tilde{F} = F$ in $\mathbb{R}^m \setminus (G \setminus H)$ and $\mathscr{H}_m(G \setminus H) = 0$. □

**Theorem 9.3.** *Suppose that there is a bounded $W_2^1$-extendible open set $H$ such that $\mathscr{H}_m((G \setminus H) \cup (H \setminus G)) = 0$. Put $W = G \cup H$. Then $W$ has finitely many components $W_1, \ldots, W_n$ and there are $\varphi_1, \ldots, \varphi_n \in W_2^1(\mathbb{R}^m)$ with compact support such that $\varphi_j = 1$ in $W_j$, $\varphi_j = 0$ in $W \setminus W_j$, $j = 1, \ldots, n$. Let $\mathscr{F} \in \mathscr{E}(\partial G)$. Then there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if and only if $\mathscr{F} \in (\operatorname{Ker} N^G\mathscr{U})^\perp$. If $\mathscr{B}$ is given by (7.2)* *then*

$$\mathscr{U}\mathscr{B} + \sum_{j=1}^{n} c_j \chi_{W_j}, \quad c_1, \dots, c_n \in C, \tag{9.1}$$

*is the general form of a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. Suppose moreover that $C_{1,2}(\{x \in \partial W;\ W$ is $(1,2)$-thin at $x\}) = 0$. Then there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if and only if*

$$\int_{\mathbb{R}^m} \nabla \mathscr{U}\mathscr{F} \cdot \overline{\nabla \varphi_j} \, \mathrm{d}\mathscr{H}_m = 0 \tag{9.2}$$

*for $j = 1, \dots, n$. If $\operatorname{cl} W_j \cap \operatorname{cl} W_i = \emptyset$ for $i \neq j$ we can take $\varphi_1, \dots, \varphi_n \in \mathscr{D}$ and there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if and only if $\mathscr{F}(\varphi_j) = 0$ for $j = 1, \dots, n$.*

Proof. The set $W$ is $W_2^1$-extendible by Lemma 9.2. Since $W$, $H$ are open sets, $H$ is bounded and $\mathscr{H}_m(W \setminus H) = 0$ the set $W$ is bounded. $W$ has finitely many components $W_1, \dots, W_n$ by Lemma 8.6. Put $\varphi_j = 1$ in $W_j$, $\varphi_j = 0$ in $W \setminus W_j$, $j = 1, \dots, n$. Since $\varphi_j \in W_2^1(W)$ there is an extension of $\varphi_j$ from $W_2^1(\mathbb{R}^m)$.

Since $N^H \mathscr{U}(\mathscr{E}(\partial H))$ is closed by Theorem 8.8, Theorem 9.1 yields that the space $N^G \mathscr{U}(\mathscr{E}(\partial G))$ is closed. According to Theorem 7.6 there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if and only if $\mathscr{F} \in (\operatorname{Ker} N^G \mathscr{U})^{\perp}$. If $\mathscr{B}$ is given by (7.2) then $\mathscr{U}\mathscr{B}$ is a weak solution of this problem. Since $\varphi_1, \dots, \varphi_n \in W_2^1(\mathbb{R}^m)$ and $\varphi_1, \dots, \varphi_n$ are constant on each component of $G$

$$\mathscr{U}\mathscr{B} + \sum_{j=1}^{n} c_j \chi_{W_j} = \mathscr{U}\mathscr{B} + \sum_{j=1}^{n} c_j \varphi_j$$

is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. Let now $u \in L_2^1(\mathbb{R}^m)$ be a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$. Then $v = u - \mathscr{U}\mathscr{B}$ is a solution of the weak Neumann problem for the Laplace equation in $G$ with zero boundary condition. Since $v$ is constant on each component of $G$ by Lemma 3.1, we obtain $\nabla v = 0$ in $G$. Since $\mathscr{H}_m(W \setminus G) = 0$ the vector function $\nabla v = 0$ $\mathscr{H}_m$-a.e. in $W$. According to [13], Lemma on page 11 there are constants $c_1, \dots, c_n$ such that $v = c_j$ $\mathscr{H}_m$-a.e. in $W_j$ for $j = 1, \dots, n$. Since $v$ is continuous in $G$ we obtain $v = c_j$ in $G \cap W_j$. Thus $u$ has the form (9.1).

Suppose that $C_{1,2}(\{x \in \partial W;\ W$ is $(1,2)$-thin at $x\}) = 0$. If there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ then (9.2) holds for $j = 1, \ldots, n$ by Theorem 5.7. Suppose now that (9.2) holds for $j = 1, \ldots, n$. According to [12], Chapter VI, Theorem 6.4 there are $\mathscr{B}_1, \ldots, \mathscr{B}_n \in \mathscr{E}$ such that $\varphi_j = \mathscr{U}\mathscr{B}_j$ almost everywhere for $j = 1, \ldots, n$. Denote by $\mathscr{F}_j$ the orthogonal projection of $\mathscr{B}_j$ onto $\mathscr{E}(\operatorname{cl} W)$. Then $\mathscr{U}\mathscr{F}_j = \mathscr{U}\mathscr{B}_j = \varphi_j$ on $W$ by [3], Chapitre I, Théorème 4. Since $\Delta\mathscr{U}\mathscr{F}_j = -\mathscr{F}_j$ by [3], p. 158 and $\Delta\mathscr{U}\mathscr{F}_j = \Delta\varphi_j = 0$ on $W$ we obtain $\mathscr{F}_j \in \mathscr{E}(\partial W)$. Since $\mathscr{U}\mathscr{F}_j = 1$ on $W_j$ and $\mathscr{U}\mathscr{F}_j = 0$ on $W \setminus W_j$ the distributions $\mathscr{F}_1, \ldots, \mathscr{F}_n$ form a linearly independent subset of $\operatorname{Ker} N^G\mathscr{U}$. Since the dimension of $\operatorname{Ker} N^G\mathscr{U}$ is equal to $n$ by Corollary 6.10, the distributions $\mathscr{F}_1, \ldots, \mathscr{F}_n$ form a basis of $\operatorname{Ker} N^G\mathscr{U}$. Since $\mathscr{F}_j$ is the orthogonal projection of $\mathscr{B}_j$ onto $\mathscr{E}(\operatorname{cl} W)$ and $\mathscr{F} \in \mathscr{E}(\operatorname{cl} W)$ we obtain

$$(\mathscr{F}, \mathscr{F}_j)_{\mathscr{E}} = (\mathscr{F}, \mathscr{B}_j) = \int_{\mathbb{R}^m} \nabla\mathscr{U}\mathscr{F} \cdot \overline{\nabla\varphi_j} \, \mathrm{d}\mathscr{H}_m = 0, \quad j = 1, \ldots, n.$$

Since $\mathscr{F}_1, \ldots, \mathscr{F}_n$ is a basis of $\operatorname{Ker} N^G\mathscr{U}$ we deduce that $\mathscr{F} \in \operatorname{Ker} N^G\mathscr{U}^{\perp}$.

If $\varphi_j \in \mathscr{D}$ then $\varphi_j = -\mathscr{U}\Delta\varphi$ by [12], p. 100. Therefore Lemma 5.1 yields

$$\mathscr{F}(\varphi_j) = -(\mathscr{F}, \Delta\varphi)_{\mathscr{E}} = \int_{\mathbb{R}^m} \nabla\mathscr{U}\mathscr{F} \cdot \overline{\nabla\mathscr{U}(-\Delta\varphi)} \, \mathrm{d}\mathscr{H}_m = \int_{\mathbb{R}^m} \nabla\mathscr{U}\mathscr{F} \cdot \overline{\nabla\varphi} \, \mathrm{d}\mathscr{H}_m.$$

Thus (9.2) is equivalent to $\mathscr{F}(\varphi_j) = 0$. □

**Corollary 9.4.** *Suppose that $G$ is bounded. Suppose that for each $x \in \partial G$ there is a neighbourhood $U$ of $x$ and a choice of a coordinate system such that $U \cap \partial G$ is a subset of the graph of a Lipschitz function. Then $\operatorname{cl} G$ has finitely many components $H_1, \ldots, H_n$. Choose $\varphi_1, \ldots, \varphi_n \in \mathscr{D}$ such that $\varphi_j = 1$ in $H_j$, $\varphi_j = 0$ in $\operatorname{cl} G \setminus H_j$, $j = 1, \ldots, n$. Let $\mathscr{F} \in \mathscr{E}(\partial G)$. Then there is a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$ if and only if $\mathscr{F}(\varphi_j) = 0$ for $j = 1, \ldots, n$. If $\mathscr{B}$ is given by (7.2) then*

$$\mathscr{U}\mathscr{B} + \sum_{j=1}^{n} c_j \chi_{H_j}, \quad c_1, \ldots, c_n \in C,$$

*is the general form of a solution of the weak Neumann problem for the Laplace equation in $G$ with the boundary condition $\mathscr{F}$.*

Proof. Denote by $W$ the set of all $x \in \mathbb{R}^m$ for which there is a neighbourhood $U$ of $x$ such that $\mathscr{H}_m(U \setminus G) = 0$. Then $W$ is a bounded open set, $G \subset W$ and $\mathscr{H}_m(W \setminus G) = 0$. Moreover, for each $x \in \partial W$ there is a neighbourhood $U$ of $x$, a Lipschitz function $f$ on $\mathbb{R}^{m-1}$ and a choice of a coordinate system such that $W \cap U = U \cap \{[z, t];\ z \in \mathbb{R}^{m-1}, t > f(z)\}$. The set $W$ has finitely many components

$W_1, \dots, W_n$ such that $\operatorname{cl} W_i \cap \operatorname{cl} W_j = \emptyset$ for $i \neq j$. Put $H_j = \operatorname{cl} W_j$, $j = 1, \dots, n$. The set $\operatorname{cl} G$ has components $H_1, \dots, H_n$, because $\operatorname{cl} G = \operatorname{cl} W$. Since $W_1, \dots, W_n$ are Lipschitz domains and $\operatorname{cl} W_i \cap \operatorname{cl} W_j = \emptyset$ for $i \neq j$ the open set $W$ is $W_2^1$-extendible. Further, $W$ is $(1,2)$-thick at each point of $\partial W$ by Remark 6.6. The rest follows from Theorem 9.3. □

**Remark 9.5.** The set $W$ from Theorem 9.3 is $W^{1,2}$-extendible by Lemma 9.2. As has recently been shown by P. Koskela and H. Tuominen (see [10]) a $W^{1,2}$-extendible domain is (1,2)-thick at all points of its boundary. Thus the hypothesis "$C_{1,2}(\{x \in \partial W;\ W \text{ is } (1,2)\text{-thin at } x\}) = 0$" can be removed from Theorem 9.3.

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*Author's address*: D. Medková, Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, 115 67 Praha 1, Czech Republic, e-mail: medkova@math.cas.cz.