Kalle Kaarli; Vladimir Kuchmei

Order affine completeness of lattices with Boolean congruence lattices

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 4, 1049–1065

Persistent URL: http://dml.cz/dmlcz/128227

Czechoslovak Mathematical Journal, 57 (132) (2007), 1049–1065

ORDER AFFINE COMPLETENESS OF LATTICES WITH BOOLEAN CONGRUENCE LATTICES

Kalle Kaarli, Vladimir Kuchmei, Tartu

(Received July 25, 2005)

Abstract. This paper grew out from attempts to determine which modular lattices of finite height are locally order affine complete. A surprising discovery was that one can go quite far without assuming the modularity itself. The only thing which matters is that the congruence lattice is finite Boolean. The local order affine completeness problem of such lattices $\mathbf{L}$ easily reduces to the case when $\mathbf{L}$ is a subdirect product of two simple lattices $\mathbf{L}_1$ and $\mathbf{L}_2$. Our main result claims that such a lattice is locally order affine complete iff $\mathbf{L}_1$ and $\mathbf{L}_2$ are tolerance trivial and one of the following three cases occurs:

1) $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$,
2) $\mathbf{L}$ is a maximal sublattice of the direct product,
3) $\mathbf{L}$ is the intersection of two maximal sublattices, one containing $\langle 0, 1\rangle$ and the other $\langle 1, 0\rangle$.

Keywords: order affine completeness, congruences of lattices, tolerances of lattices

MSC 2000: 06B10, 08A40

1. Introduction

A lattice $\mathbf{L}$ is called (locally) order affine complete if every congruence and order preserving function on $\mathbf{L}$ is a (local) polynomial function. The present paper is a result of an attempt to describe locally order affine complete modular lattices of finite height. Since modular subdirectly irreducible lattices of finite height are known to be simple, every modular lattice $\mathbf{L}$ of finite height is a subdirect product of simple lattices of finite height and therefore its congruence lattice is finite Boolean. Let $\mathbf{L} \leqslant_{\mathrm{sd}} \mathbf{L}_1 \times \ldots \times \mathbf{L}_n$ where the subdirect factors $\mathbf{L}_i$ are simple and let $\mathbf{L}_{ij}$ be 2-fold coordinate projections of $\mathbf{L}$, $i, j = 1, \ldots, n$, $i \neq j$. Now, since lattices admit a majority term, the well-known Baker-Pixley Lemma says that the lattice $\mathbf{L}$ is

Research supported by the Estonian Science Foundation grant number 5368.

determined by the $\mathbf{L}_{ij}$. Moreover, it is quite easy to show that $\mathbf{L}$ is locally order affine complete iff so are all $\mathbf{L}_{ij}$ (for finite $\mathbf{L}$ this result is due to R. Wille [9]). Therefore the question reduces to the case of a subdirect product of two simple modular lattices of finite height. Quite surprisingly, it turned out that at this stage of work the modularity was not relevant at all. Our main result describes which subdirect products of two simple lattices of finite height are locally order affine complete. Recall that a lattice is called *tolerance trivial* (*tolerance simple*) if it has no other tolerances than the congruence relations (if it is tolerance trivial and simple, that is, it has only two nontrivial tolerance relations). A subdirect product $\mathbf{L} \leqslant_{\mathrm{sd}} \mathbf{L}_1 \times \mathbf{L}_2$ is called *nontrivial* if none of the two canonical projections is an isomorphism. Assuming that the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are bounded, we denote by L^{01} the subuniverse of $\mathbf{L}_1 \times \mathbf{L}_2$ generated by L and $\langle 0, 1\rangle$. The subuniverse L^{10} is defined similarly. It is known ([4]) that $L = L^{01} \cap L^{10}$ for every $\mathbf{L} \leqslant_{\mathrm{sd}} \mathbf{L}_1 \times \mathbf{L}_2$. This implies that every maximal sublattice of $\mathbf{L}_1 \times \mathbf{L}_2$ contains either $\langle 0, 1\rangle$ or $\langle 1, 0\rangle$.

Now we state the main result of the present paper.

Theorem 1.1. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be simple lattices of finite height and let $\mathbf{L}$ be a nontrivial subdirect product in $\mathbf{L}_1 \times \mathbf{L}_2$. Then $\mathbf{L}$ is locally order affine complete if and only if $\mathbf{L}_1$ and $\mathbf{L}_2$ are tolerance trivial and one of the following three cases occurs:*

(1) $\mathbf{L} = \mathbf{L}_1 \times \mathbf{L}_2$;

(2) $\mathbf{L}$ *is a maximal sublattice of* $\mathbf{L}_1 \times \mathbf{L}_2$;

(3) $\mathbf{L}$ *is the intersection of two maximal sublattices of* $\mathbf{L}_1 \times \mathbf{L}_2$, *one containing* $\langle 0, 1\rangle$ *and the other* $\langle 1, 0\rangle$.

We wish to emphasize that the assumption that $\mathbf{L}$ must be the subdirect product of $\mathbf{L}_1$ and $\mathbf{L}_2$ is relevant. One can easily find two maximal sublattices of $\mathbf{L}_1 \times \mathbf{L}_2$, one containing $\langle 0, 1\rangle$ and the other $\langle 1, 0\rangle$ such that their intersection is not subdirect in $\mathbf{L}_1 \times \mathbf{L}_2$. Such intersection need not be locally order affine complete.

2. Order affine completeness and tolerances

A *tolerance* of a lattice $\mathbf{L}$ is a subuniverse of $\mathbf{L}^2$ which is reflexive and symmetric as a binary relation. The set $\operatorname{Tol}\mathbf{L}$ of all tolerances of a lattice $\mathbf{L}$ is a lattice under set theoretic inclusion. Clearly the meet operation of this lattice is the usual intersection. The join operation of the tolerance lattice will be denoted by $\sqcup$. Throughout the paper, we denote the identity relation and the all relation on a set A by Δ_A and ∇_A, respectively. The following lemma gives an elementary but important property of tolerances of lattices that will be used throughout the paper, usually without reference.

Lemma 2.1 ([1]). *Let* $\mathbf{L}$ *be a lattice and* $T \in \operatorname{Tol}\mathbf{L}$. *Then* $\langle x, y\rangle \in T$ *if and only if* $[x \wedge y, x \vee y]^2 \subset T$.

The order functionally complete (in other words, order polynomially complete) lattices were first studied by D. Schweigert [7]. M. Kindermann proved in [6] that a finite lattice is order functionally complete iff it is tolerance simple. R. Wille's characterization of finite order affine complete lattices [9] was given in terms of decreasing join endomorphisms. However, there is a natural 1-1 correspondence between decreasing join endomorphisms and tolerances. Therefore one can say that Wille gave his characterization in terms of tolerances.

When writing the book [5], the first author observed that practically everything that had been earlier proved about order functional or order affine completeness of finite lattices, remained true in the case of lattices of finite height, just after adding the adjective "locally". So for example, aforementioned Kindermann's result can be translated to a result "A lattice $\mathbf{L}$ of finite height is locally order functionally complete iff it is tolerance simple".

In order to characterize locally order affine complete lattices of finite height in terms of tolerance relations, the notion of symmetrized product was introduced in [5]. The aim was to modify the usual relational product so that the product of two tolerance relations of a lattice would be a tolerance again.

Let S and T be arbitrary binary relations on a lattice $\mathbf{L}$. Then the *symmetrized product* of S and T is the binary relation $S * T$ of $\mathbf{L}$ defined by

$$S * T = \{\langle a, c\rangle \in L^2 \colon \ \langle a \vee c, a \wedge c\rangle \in S \circ T\}.$$

The next lemma lists the basic properties of the symmetrized product operation on $\operatorname{Tol}\mathbf{L}$. They all are easy to prove.

Lemma 2.2. *The following are true for any lattice* $\mathbf{L}$*:*

(1) *the symmetrized product operation is associative on* $\operatorname{Tol}\mathbf{L}$*;*
(2) *given two tolerances* S *and* T *of* $\mathbf{L}$, *the symmetrized product* $S * T$ *is a tolerance of* $\mathbf{L}$ *containing both* S *and* T*;*
(3) *if* S *is a congruence of a lattice* $\mathbf{L}$ *then* $S * S = S$*;*
(4) *the symmetrized product operation on* $\operatorname{Tol}\mathbf{L}$ *distributes over the intersection operation, that is, if* $S, T, U \in \operatorname{Tol}\mathbf{L}$ *then*

$$S * (T \cap U) = (S * T) \cap (S * U) \quad \text{and} \quad (S \cap T) * U = (S * U) \cap (T * U).$$

We shall call a tolerance of a lattice $\mathbf{L}$ *congruence generated* if it can be obtained from congruences of $\mathbf{L}$ using the operations of symmetrized product and intersection.

It follows from Lemma 2.2 that every congruence generated tolerance of a lattice $\mathbf{L}$ can be represented as the meet of symmetrized products of congruences of $\mathbf{L}$. Now we state a result from [5, Theorem 5.3.8] which is our main tool in the present paper.

Theorem 2.3. *A lattice $\mathbf{L}$ of finite height is locally order affine complete iff every finitely generated tolerance T of $\mathbf{L}$ is congruence generated.*

In [5] we raised the question (Problem 5.3.31) whether the class of all locally order affine complete lattices is closed with respect to homomorphic images. Now we show that in the special case of lattices of finite height the affirmative answer easily follows from Theorem 2.3.

Theorem 2.4. *Let $\mathbf{L}$ be a locally order affine complete lattice of finite height. Then every homomorphic image of $\mathbf{L}$ is locally order affine complete, too.*

Proof. Let $\varphi\colon \mathbf{L} \to \mathbf{L}'$ be a surjective homomorphism of lattices and let $T' \in \operatorname{Tol}\mathbf{L}'$ be the tolerance relation of $\mathbf{L}'$ generated by a finite subset $X' \subset L' \times L'$. We pick in $L \times L$ a finite subset X such that $\varphi(X) = X'$ and consider the tolerance relation T of $\mathbf{L}$ generated by X. Since $\mathbf{L}$ is locally order affine complete, the tolerance T is congruence generated, that is, there are congruences ϱ_{ij} of $\mathbf{L}$ such that

$$T = \bigcap_{i=1}^{m} \varrho_{i1} * \ldots * \varrho_{i,n_i}. \tag{1}$$

Since φ is surjective, all $\varrho'_{ij} = \varphi(\varrho_{ij})$ are congruences of $\mathbf{L}'$ and obviously $\varphi(T) = T'$. Now (1) easily implies

$$T' = \bigcap_{i=1}^{m} \varrho'_{i1} * \ldots * \varrho'_{i,n_i},$$

that is, T is congruence generated. $\square$

The analog of Theorem 2.3 for finite lattices was proved by R. Wille in [9]. The next result which was stated without proof in [5] has also its counterpart in [9].

Theorem 2.5. *Let $\mathbf{L}$ be a subdirect product of lattices $\mathbf{L}_1, \ldots, \mathbf{L}_n$ of finite height. The lattice $\mathbf{L}$ is locally order affine complete iff all the 2-fold coordinate projections $\mathbf{L}_{ij}$ of $\mathbf{L}$ are locally order affine complete.*

Proof. The necessity part of the theorem directly follows from Theorem 2.4. We prove sufficiency by induction on n. Suppose that $n \geqslant 3$ and the claim holds if the number of subdirect factors is less than n. Let f be a k-ary congruence and order preserving function on $\mathbf{L}$. Since f is congruence preserving, f can be identified with the n-tuple $\langle f_1, \dots, f_n \rangle$ of coordinate functions. This means that, given any $a^1, \dots, a^k \in L$ where $a^i = \langle a^i_1, \dots, a^i_n \rangle$, $a^i_j \in L_j$, $i = 1, \dots, k$, we have

$$f(a^1, \dots, a^k) = \langle f_1(a^1_1, \dots, a^k_1), \dots, f_n(a^1_n, \dots, a^k_n) \rangle.$$

Clearly every f_i is a congruence and order preserving function on $\mathbf{L}_i$.

Let X be a finite subset of L. Then there are finite subsets $X_j \subset L_j$, $j = 1, \dots, n$, such that $X \subset X_1 \times \ldots \times X_n$. We assume that the sets X_i are minimal possible; hence X projects onto every X_i, $i = 1, \dots, n$.

We first show that there exist polynomial functions p_i of $\mathbf{L}_i$ such that p_i interpolates f_i at X_i, $i = 2, \dots, n$. Let $\pi\colon\ L_1 \times \ldots \times L_n \to L_2 \times \ldots \times L_n$ be the natural projection map and $L' = \pi(L)$. Then clearly $L'_{ij} = L_{ij}$ for every $2 \leqslant i < j \leqslant n$. Since $f' = \langle f_2, \dots, f_n \rangle$ is a congruence and order preserving function on $\mathbf{L}'$, by induction hypothesis there exists a polynomial function p' of $\mathbf{L}'$ which interpolates f' at $\pi(X)$. Obviously the function p' is induced by a suitable polynomial function p of $\mathbf{L}$. This means, in other words, that p_i, the i'th coordinate function of p, interpolates f_i at X_i, for $i = 2, 3, \dots, n$.

Similarly, there exist polynomial functions q and r of $\mathbf{L}$ such that:

1) q_i, the i'th coordinate function of q, interpolates f_i at X_i for $i = 1, 3, 4, \dots, n$;
2) r_i, the i'th coordinate function of r, interpolates f_i at X_i for $i = 1, 2, 4, 5, \dots, n$.

To conclude it remains to take the majority term $m(x, y, z)$ and notice that the polynomial function $m(p, q, r)$ interpolates f at X. $\square$

Using the results stated above, we are now able to reduce the study of modular locally order affine complete lattices of finite height to the investigation of subdirect products of two tolerance simple lattices. As we shall see soon, in the latter case there is a satisfactory answer even without the modularity assumption.

Theorem 2.6. *A modular lattice* $\mathbf{L}$ *of finite height is locally order affine complete iff it is subdirect in* $\mathbf{L}_1 \times \ldots \times \mathbf{L}_n$ *where all* $\mathbf{L}_i$ *are tolerance simple and all* $\mathbf{L}_{ij}$ *are locally order affine complete.*

P r o o f. As we have mentioned already, there exist simple lattices $\mathbf{L}_1, \dots, \mathbf{L}_n$ such that

$$\mathbf{L} \leqslant_{\mathrm{sd}} \mathbf{L}_1 \times \ldots \times \mathbf{L}_n.$$

Now the neccessity part of the theorem follows from Theorem 2.4 and the sufficiency part is a direct consequence of Theorem 2.5. $\square$

3. Subdirect products of two lattices and their tolerances

We first state without proof an elementary but useful lemma.

Lemma 3.1. *Let a lattice* $\mathbf{L}$ *be a subdirect product of lattices* $\mathbf{L}_1$ *and* $\mathbf{L}_2$*, and let* $\langle a,b\rangle, \langle a,c\rangle \in L$ *where* $b \leqslant c$*. Then* L *also contains all pairs* $\langle a,y\rangle$ *where* $b \leqslant y \leqslant c$*.*

Throughout this section we assume that $\mathbf{L}_1$ and $\mathbf{L}_2$ are lattices of finite height, $\mathbf{L} \leqslant_{\mathrm{sd}} \mathbf{L}_1 \times \mathbf{L}_2$, $\pi_i\colon \mathbf{L} \to \mathbf{L}_i$ is a projection map and $\varrho_i = \operatorname{Ker} \pi_i$, $i = 1,2$.

Always, every element $x \in L$ is represented as the pair $x = \langle x^1, x^2\rangle$ where $x^i \in L_i$, $i = 1,2$. It will turn out that in the lattice $\mathbf{Tol}\,\mathbf{L}$ a very special role is played by the join $\varrho_1 \sqcup \varrho_2$. Therefore we introduce a special symbol for that tolerance: let $\Sigma_{\mathbf{L}} = \varrho_1 \sqcup \varrho_2$. It follows from [3, Lemma 3.8], that $\Sigma_{\mathbf{L}} = (\varrho_1 \circ \varrho_2) \cap (\varrho_2 \circ \varrho_1)$. We now give a few more characterizations of $\Sigma_{\mathbf{L}}$.

Lemma 3.2. *The tolerance* $\Sigma_{\mathbf{L}}$ *can be characterized as follows; in particular, it is congruence generated:*

$$\Sigma_{\mathbf{L}} = (\varrho_1 * \varrho_2) \cap (\varrho_2 * \varrho_1) = \{\langle x,y\rangle \in L^2\colon\ \langle y^1, x^2\rangle, \langle x^1, y^2\rangle \in L\}.$$

Proof. By Lemma 2.2, we have $\Sigma_{\mathbf{L}} \subset (\varrho_1 * \varrho_2) \cap (\varrho_2 * \varrho_1)$. Let $\langle x,y\rangle \in (\varrho_1 * \varrho_2) \cap (\varrho_2 * \varrho_1)$. Then, in particular, $\langle x \vee y, x \wedge y\rangle \in \varrho_1 \circ \varrho_2$ which implies $\langle x^1 \vee y^1, x^2 \wedge y^2\rangle \in L$. Since $\mathbf{L}$ is a sublattice, we also have $\langle x^1 \vee y^1, x^2 \vee y^2\rangle \in L$. Now, because of $x^2 \wedge y^2 \leqslant y^2 \leqslant x^2 \vee y^2$, Lemma 3.1 yields $\langle x^1 \vee y^1, y^2\rangle \in L$. Using a similar argument one easily gets $\langle x^1, x^2 \vee y^2\rangle \in L$. Consequently

$$\langle x^1, y^2\rangle = \langle x^1 \vee y^1, y^2\rangle \wedge \langle x^1, x^2 \vee y^2\rangle \in L.$$

The proof of $\langle y^1, x^2\rangle \in L$ is similar. We have proved the inclusions

$$\Sigma_{\mathbf{L}} \subset (\varrho_1 * \varrho_2) \cap (\varrho_2 * \varrho_1) \subset \{\langle x,y\rangle \in L^2\colon\ \langle y^1, x^2\rangle, \langle x^1, y^2\rangle \in L\}.$$

So it remains to prove that $\{\langle x,y\rangle \in L^2\colon\ \langle y^1, x^2\rangle, \langle x^1, y^2\rangle \in L\}$ is contained in $\Sigma_{\mathbf{L}}$. Let $x, y \in L$ be such that $\langle y^1, x^2\rangle, \langle x^1, y^2\rangle \in L$. By definition of $\Sigma_{\mathbf{L}}$ we have

$$\langle\langle x^1, x^2\rangle, \langle x^1, y^2\rangle\rangle,\ \langle\langle x^1, x^2\rangle, \langle y^1, x^2\rangle\rangle \in \Sigma_{\mathbf{L}},$$

hence also

$$\langle x, x \vee y\rangle = \langle\langle x^1, x^2\rangle, \langle x^1 \vee y^1, x^2 \vee y^2\rangle\rangle \in \Sigma_{\mathbf{L}}.$$

Similarly one can prove $\langle x \vee y, y\rangle \in \Sigma_{\mathbf{L}}$ but then

$$\langle x,y\rangle = \langle x, x \vee y\rangle \wedge \langle x \vee y, y\rangle \in \Sigma_{\mathbf{L}}.$$

□

Corollary 3.2.1. *If $\langle x, z\rangle \in \Sigma_{\mathbf{L}}$, $y \in L_1 \times L_2$ and $x \leqslant y \leqslant z$ then $y \in L$.*

P r o o f. By Lemma 3.2 we have $\langle x^1, z^2\rangle, \langle z^1, x^2\rangle \in L$. Then using Lemma 3.1 we first get $\langle x^1, y^2\rangle, \langle y^1, x^2\rangle \in L$ and finally

$$y = \langle x^1, y^2\rangle \vee \langle y^1, x^2\rangle \in L.$$

□

Definition 1. Given arbitrary sets A_1 and A_2 and $a = \langle a^1, a^2\rangle \in A_1 \times A_2$, the subset $(\{a^1\} \times A_2) \cup (A_1 \times \{a^2\})$ of $A_1 \times A_2$ is called an a-cross.

In the next section we shall show that every nontrivial subdirect product of tolerance simple lattices contains a cross. Now we derive several conclusions from the assumption that L contains a cross. We start with an elementary lemma which will be needed in the sequel.

Lemma 3.3. *If L contains a u-cross then L also contains all elements of $L_1 \times L_2$ comparable with u.*

P r o o f. Let $x \in L_1 \times L_2$ be such that $x \leqslant u$. Then $\langle x_1, u^2\rangle, \langle u^1, x^2\rangle \in L$, hence also

$$x = \langle x^1, u^2\rangle \wedge \langle u^1, x^2\rangle \in L.$$

The case $x \geqslant u$ can be handled similarly. □

Lemma 3.4. *Assume that L contains a u-cross and let T be a tolerance of $\mathbf{L}$ such that $\Sigma_{\mathbf{L}} \leqslant T$. Then, given any $\langle x, y\rangle \in T$, we have $\langle x \wedge u, u \vee y\rangle \in T$.*

P r o o f. By assumption, $\langle u^1, y^2\rangle, \langle y^1, u^2\rangle \in L$, hence

$$\langle\langle u^1, u^2\rangle, \langle u^1, y^2\rangle\rangle \in \varrho_1 \leqslant \Sigma_{\mathbf{L}} \leqslant T$$

and

$$\langle\langle u^1, u^2\rangle, \langle y^1, u^2\rangle\rangle \in \varrho_2 \leqslant \Sigma_{\mathbf{L}} \leqslant T,$$

which implies $\langle u, u \vee y\rangle \in T$. Similarly, using $\langle u^1, x^2\rangle, \langle x^1, u^2\rangle \in L$, we get $\langle x, u \vee x\rangle \in T$, which together with $\langle x, y\rangle \in T$ gives $\langle x, u \vee x \vee y\rangle \in T$ and by Lemma 2.1, $\langle x, u \vee y\rangle \in T$. Now

$$\langle x \wedge u, u \vee y\rangle = \langle x, u \vee y\rangle \wedge \langle u, u \vee y\rangle \in T.$$

This completes the proof. □

Now we are going to show that if $\mathbf{L}$ is bounded and contains a cross then there is a very strong connection between tolerances of $\mathbf{L}$ and subuniverses of $\mathbf{L}_1 \times \mathbf{L}_2$ containing $\mathbf{L}$. For this purpose we introduce two mappings:

$$\Phi\colon \uparrow\Sigma_{\mathbf{L}} \to \uparrow L, \quad \Psi\colon \uparrow L \to \uparrow\Sigma_{\mathbf{L}}$$

where

$$\Phi(T) = L_T = \{y \in L_1 \times L_2 \colon \exists\, x, z \in L,\ \langle x, z\rangle \in T,\ x \leqslant y \leqslant z\}$$

and

$$\Psi(K) = (\Sigma_{\mathbf{K}})|_L$$

for an arbitrary $T \in \operatorname{Tol}\mathbf{L}$ and a subuniverse K of $\mathbf{L}_1 \times \mathbf{L}_2$ containing L. Note that here $\uparrow\Sigma_{\mathbf{L}}$ is the principal filter of $\mathbf{Tol\,L}$ generated by $\Sigma_{\mathbf{L}}$ and $\uparrow L$ is the principal filter of the subuniverse lattice of $\mathbf{L}_1 \times \mathbf{L}_2$ generated by L. It is straightforward to check that L_T is a subuniverse of $\mathbf{L}_1 \times \mathbf{L}_2$ and the reflexivity of T implies $L \subset L_T$.

Theorem 3.5. *Let $\mathbf{L}$ be a subdirect product of bounded lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ and assume that L contains a cross. Then the mappings Φ and Ψ defined above realize an isomorphism between the lattices $\uparrow\Sigma_{\mathbf{L}}$ and $\uparrow L$. Moreover, $\Phi^{-1} = \Psi$.*

Proof. Assume that L contains a u-cross. Clearly both Φ and Ψ are order preserving. Therefore the theorem will be proved when we show that $\Phi^{-1} = \Psi$. We first show that $\Psi(\Phi(T)) = T$ for any tolerance $T \in \uparrow\Sigma_{\mathbf{L}}$. Let $\langle x, y\rangle \in T$. Then $x, y \in L \subset L_T$. On the other hand, $\langle x\wedge y, x\vee y\rangle \in T$ and $x\wedge y \leqslant \langle x^1, y^2\rangle \leqslant x\vee y$, which gives $\langle x^1, y^2\rangle \in L_T$. Since one gets similarly $\langle y^1, x^2\rangle \in L_T$, we have $\langle x, y\rangle \in \Sigma_{\mathbf{L}_T}$. Thus we have proved the inclusion $T \subset \Sigma_{\mathbf{L}_T}$ which clearly implies $T \subset (\Sigma_{\mathbf{L}_T})|_L$, that is, $T \subset \Psi(\Phi(T))$.

Now we prove that also $\Psi(\Phi(T)) \subset T$. Assume that

$$\langle x, y\rangle \in \Psi(\Phi(T)) = (\Sigma_{\mathbf{L}_T})|_L.$$

By Lemma 2.1 we may assume, without loss of generality, that $x \leqslant y$. Applying Lemma 3.4 with $(\Sigma_{\mathbf{L}_T})|_L$ in the role of T, we get $\langle u \wedge x, u \vee y\rangle \in (\Sigma_{\mathbf{L}_T})|_L$. Then by definition of L_T we have

$$\langle u^1 \wedge x^1, u^2 \vee y^2\rangle, \langle u^1 \vee y^1, u^2 \wedge x^2\rangle \in L_T.$$

Hence there exist $a, b, c, d \in L$ such that $\langle a, b\rangle, \langle c, d\rangle \in T$ and

$$a \leqslant \langle u^1 \wedge x^1, u^2 \vee y^2\rangle \leqslant b, \quad c \leqslant \langle u^1 \vee y^1, u^2 \wedge x^2\rangle \leqslant d.$$

Note that by Lemma 3.4 we may assume that $a \leqslant u \leqslant b$ and $c \leqslant u \leqslant d$.

Since $\langle\langle u^1, 0\rangle, \langle u^1, 1\rangle\rangle \in \varrho_1 \leqslant \Sigma_{\mathbf{L}} \leqslant T$, we get

$$\langle\langle a^1, 0\rangle, \langle u^1, b^2\rangle\rangle = \langle\langle a^1, a^2\rangle, \langle b^1, b^2\rangle\rangle \wedge \langle\langle u^1, 0\rangle, \langle u^1, 1\rangle\rangle \in T\,.$$

Now the inequalities

$$\langle a^1, 0\rangle \leqslant \langle u^1 \wedge x^1, 0\rangle \leqslant \langle u^1, u^2 \vee y^2\rangle \leqslant \langle u^1, b^2\rangle$$

imply

$$\langle\langle u^1 \wedge x^1, 0\rangle, \langle u^1, u^2 \vee y^2\rangle\rangle \in T. \tag{2}$$

By a similar argument, starting from $\langle\langle 0, u^2\rangle, \langle 1, u^2\rangle\rangle \in \varrho_2$, one gets

$$\langle\langle 0, u^2 \wedge x^2\rangle, \langle u^1 \vee y^1, u^2\rangle\rangle \in T. \tag{3}$$

The formulas (2) and (3) yield $\langle u \wedge x, u \vee y\rangle \in T$. Since $u \wedge x \leqslant x \leqslant y \leqslant u \vee y$, we also have $\langle x, y\rangle \in T$.

Next we show that $\Phi(\Psi(K)) = K$ for any $K \in {\uparrow}L$. If $x \in K$ then $\langle x \wedge u, u \vee x\rangle \in \Sigma_{\mathbf{K}}$ by Lemma 3.4 . On the other hand, by Lemma 3.3, $x \wedge u, u \vee x \in L$. Since $x \wedge u \leqslant x \leqslant u \vee x$, we have $x \in L_{(\Sigma_{\mathbf{K}})|_L} = \Phi(\Psi(K))$. This proves $K \subset \Phi(\Psi(K))$. For the converse, let $x \in \Phi(\Psi(K))$. Then there exist $a, b \in L$ such that $a \leqslant x \leqslant b$ and $\langle a, b\rangle \in \Sigma_{\mathbf{K}}$. It remains to apply Corollary 3.2.1. $\square$

4. Proof of the main result

In this section we focus on the problem when a nontrivial subdirect product of two tolerance simple lattices of finite height is locally order affine complete. We first prove that these subdirect products contain crosses. Actually this result is an analog of one obtained in [2] for functionally complete algebras.

Lemma 4.1. *Let a lattice* $\mathbf{L}$ *of finite height be a nontrivial subdirect product of two tolerance simple lattices* $\mathbf{L}_1$ *and* $\mathbf{L}_2$. *Then* L *contains a cross.*

P r o o f. Nontriviality of a subdirect product means that none of the two projection maps $\pi_i\colon L \to L_i$ is injective. In particular, there exist $\langle a, b\rangle, \langle a, c\rangle \in L$ such that $b \neq c$. Since then also $\langle a, b \wedge c\rangle, \langle a, b \vee c\rangle \in L$, we may assume that $b < c$. Since $\mathbf{L}$ is of finite height, so is $\mathbf{L}_2$, in particular, $0, 1 \in L_2$. We consider the mapping $f\colon L_2 \to L_2$ such that $f(x) = 0$ for $x \leqslant b$ and $f(x) = 1$ otherwise. Obviously f is order preserving and by simplicity of $\mathbf{L}_2$ it is also congruence preserving. By

Theorem 2.3, $\mathbf{L}_2$ is locally order functionally complete. Hence there exists a unary polynomial function p_2 of $\mathbf{L}_2$ such that $p_2(b) = f(b) = 0$ and $p_2(c) = f(c) = 1$. Let $p_2(x) = t(x, d_1, \dots, d_n)$ where t is a lattice term and $d_1, \dots, d_n \in L_2$. Since $\mathbf{L}$ is subdirect in $\mathbf{L}_1 \times \mathbf{L}_2$, there exist $a_1, \dots, a_n \in L_1$ such that $\langle a_i, d_i \rangle \in L$, $i = 1, \dots, n$. Now consider the polynomial function $p(x) = t(x, \langle a_1, d_1 \rangle, \dots, \langle a_n, d_n \rangle)$ of the lattice $\mathbf{L}$. Denoting $a' = t(a, a_1, \dots, a_n)$, we see that

$$\begin{aligned}
\langle a', 0 \rangle &= \langle t(a, a_1, \dots, a_n), t(b, d_1, \dots, d_n) \rangle \\
&= t(\langle a, b \rangle, \langle a_1, d_1 \rangle, \dots, \langle a_n, d_n \rangle) \\
&= p(\langle a, b \rangle) \in L
\end{aligned}$$

and similarly $\langle a', 1 \rangle = p(\langle a, c \rangle) \in L$. Now Lemma 3.1 implies $\{a'\} \times L_2 \subset L$. Changing the roles of $\mathbf{L}_1$ and $\mathbf{L}_2$, one immediately gets that L also contains a subset $L_1 \times \{b'\}$, for a suitable $b' \in L_2$. This completes the proof. □

In Introduction we introduced the lattices $\mathbf{L}^{01}$ and $\mathbf{L}^{10}$ determined by a given sublattice $\mathbf{L}$ of the direct product of two bounded lattices. The following lemma is a special case of [4, Lemma 4.7]. It gives elementwise characterizations of $\mathbf{L}^{01}$ and $\mathbf{L}^{10}$ and shows that $\mathbf{L}$ equals their intersection, provided $\mathbf{L}$ is subdirect in $\mathbf{L}_1 \times \mathbf{L}_2$.

Lemma 4.2. *Let $\mathbf{L}$ be a subdirect product of bounded lattices $\mathbf{L}_1$ and $\mathbf{L}_2$. Then:*
(1) $L^{01} = \{x \in L_1 \times L_2 \colon \exists y \in L,\ x^1 \leqslant y^1,\ x^2 \geqslant y^2\}$;
(2) $L^{10} = \{x \in L_1 \times L_2 \colon \exists y \in L,\ x^1 \geqslant y^1,\ x^2 \leqslant y^2\}$;
(3) $L = L^{01} \cap L^{10}$.

The following lemmas show that in the situation we are interested in, all tolerances of $\mathbf{L}$ can be easily described and, in particular, $\varrho_1 * \varrho_2$ and $\varrho_2 * \varrho_1$ are exactly the tolerances corresponding to the subuniverses L^{01} and L^{10}, respectively.

Lemma 4.3. *Let $\mathbf{L}$ be a subdirect product of bounded lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ and assume that $\mathbf{L}$ contains a cross. Then $L^{01} = L_{\varrho_1 * \varrho_2}$ and $L^{10} = L_{\varrho_2 * \varrho_1}$.*

Proof. Clearly, it suffices to prove the first of the two equalities. Take an arbitrary element $x \in L_{\varrho_1 * \varrho_2}$. Then there exist $y, z \in L$ such that $y \leqslant x \leqslant z$ and $\langle y, z \rangle \in \varrho_1 * \varrho_2$. This means that $\langle z, y \rangle \in \varrho_1 \circ \varrho_2$, hence $\langle z^1, y^2 \rangle \in L$. Now Lemma 3.1 implies $\langle z^1, x^2 \rangle, \langle x^1, y^2 \rangle \in L$. Consequently,

$$x = (\langle 0, 1 \rangle \wedge \langle z^1, x^2 \rangle) \vee \langle x^1, y^2 \rangle \in L^{01}.$$

This proves the inclusion $L_{\varrho_1 * \varrho_2} \subset L^{01}$. Note that so far we have not used the assumption that L contains a cross.

In order to prove the opposite inclusion assume that L contains a u-cross. Then $\langle u^1,1\rangle,\langle 0,u^2\rangle \in L$ and $\langle\langle u^1,1\rangle,\langle 0,u^2\rangle\rangle \in \varrho_1 * \varrho_2$. Hence $\langle 0,u^2\rangle \leqslant \langle 0,1\rangle \leqslant \langle u^1,1\rangle$ implies $\langle 0,1\rangle \in L_{\varrho_1*\varrho_2}$. This proves $L^{01} \subset L_{\varrho_1*\varrho_2}$. □

Lemma 4.4. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be tolerance simple lattices and let $\mathbf{L} \leqslant_{\mathrm{sd}} \mathbf{L}_1 \times \mathbf{L}_2$. Then for any $T \in \operatorname{Tol}\mathbf{L}$, either $T \in \{\Delta_L, \varrho_1, \varrho_2\}$ or $T \geqslant \Sigma_{\mathbf{L}}$.*

Proof. Let $T \in \operatorname{Tol}\mathbf{L} \setminus \{\Delta_L, \varrho_1, \varrho_2\}$. We denote $T_i = \pi_i(T)$, $i=1,2$. The surjectivity of π_i implies that $T_i \in \operatorname{Tol}\mathbf{L}_i$, $i=1,2$. Since $\mathbf{L}_1$ is tolerance simple, we have $T_1 \in \{\Delta_{L_1}, \nabla_{L_1}\}$.

Assume that $T_1 = \nabla_{L_1}$ and let us show that then $\varrho_2 \leqslant T$. Let $\langle x,y\rangle \in \varrho_2$. Since $T_1 = \nabla_{L_1}$, there exist $z,w \in L$ such that $z^1 = x^1$, $w^1 = y^1$ and $\langle z,w\rangle \in T$. Then also $\langle (z\vee x)\wedge y, (w\vee x)\wedge y\rangle \in T$, which due to the equalities $x^2=y^2$, $z^1=x^1$ and $w^1=y^1$ gives

$$\langle\langle x^1\wedge y^1, x^2\rangle, \langle y^1,y^2\rangle\rangle \in T. \tag{4}$$

Since $\langle x^1\wedge y^1, x^2\rangle \leqslant x \leqslant y$, (4) yields $\langle x,y\rangle \in T$. Together with $T \neq \varrho_2$ this gives $T \nleqslant \varrho_2$, which easily implies $T_2 \neq \Delta_{L_2}$. Since $\mathbf{L}_2$ is tolerance simple, the only possibility is $T_2 = \nabla_{L_2}$. As above, the latter yields $\varrho_1 \leqslant T$ which together with $\varrho_2 \leqslant T$ gives $\Sigma_{\mathbf{L}} \leqslant T$. □

Lemma 4.5. *Let $\mathbf{L}_1$ and $\mathbf{L}_2$ be tolerance simple lattices of finite height and let $\mathbf{L}$ be a subdirect product in $\mathbf{L}_1 \times \mathbf{L}_2$ containing a cross. Then*

$$\varrho_1 * \varrho_2 * \varrho_1 = \varrho_2 * \varrho_1 * \varrho_2 = (\varrho_1 * \varrho_2) \sqcup (\varrho_2 * \varrho_1) = \nabla_L.$$

Proof. Assume that L contains a u-cross. Since

$$\langle 0,u^2\rangle, \langle 1,u^2\rangle, \langle u^1,0\rangle, \langle u^1,1\rangle \in L,$$

we have

$$\langle\langle 0,0\rangle,\langle 1,1\rangle\rangle \in (\varrho_1 * \varrho_2 * \varrho_1) \cap (\varrho_2 * \varrho_1 * \varrho_2).$$

Thus $\varrho_1 * \varrho_2 * \varrho_1 = \varrho_2 * \varrho_1 * \varrho_2 = \nabla_L$.

To prove that $(\varrho_1 * \varrho_2) \sqcup (\varrho_2 * \varrho_1) = \nabla_L$, we use Theorem 3.5 and show that in the filter $\uparrow L$ the equality

$$L_{\varrho_1*\varrho_2} \vee L_{\varrho_2*\varrho_1} = L_1 \times L_2 \tag{5}$$

holds. By Lemma 4.3 we have $\langle 0,1\rangle,\langle 1,0\rangle \in L_{\varrho_1*\varrho_2} \vee L_{\varrho_2*\varrho_1}$ and obviously $\langle 0,0\rangle,\langle 1,1\rangle \in L_{\varrho_1*\varrho_2} \vee L_{\varrho_2*\varrho_1}$. Hence the equality (5) follows from Lemma 3.1. □

Now we are able to prove our main result.

P r o o f of Theorem 1.1. Assume that $\mathbf{L}$ is locally order affine complete. Then by Theorem 2.4 the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are locally order affine complete, too, and due to their simplicity they are actually locally order functionally complete. Thus, by Theorem 5.3.40 of [5], the lattices $\mathbf{L}_1$ and $\mathbf{L}_2$ are tolerance simple.

Furthermore, by Theorem 2.3 all finitely generated tolerances of $\mathbf{L}$ must be congruence generated, hence by Lemma 4.5 the finitely generated tolerances of $\mathbf{L}$ are contained in the set

$$F = \{\Delta_L, \varrho_1, \varrho_2, \Sigma_{\mathbf{L}}, \varrho_1 * \varrho_2, \varrho_2 * \varrho_1, \nabla_L\}.$$

Since F is a subuniverse of the lattice $\mathbf{Tol}\,\mathbf{L}$ and every tolerance of $\mathbf{L}$ is a join of finitely generated tolerances of $\mathbf{L}$, we conclude that $\mathrm{Tol}\,\mathbf{L} = F$ and

$$\uparrow\Sigma_{\mathbf{L}} = \{\Sigma_{\mathbf{L}}, \varrho_1 * \varrho_2, \varrho_2 * \varrho_1, \nabla_L\}. \tag{6}$$

Hence Theorem 3.5 and Lemma 4.3 imply

$$\uparrow L = \{L, L^{01}, L^{10}, L_1 \times L_2\}. \tag{7}$$

This means that both L^{01} and L^{10} either coincide with $L_1 \times L_2$ or are maximal subuniverses of $\mathbf{L}_1 \times \mathbf{L}_2$. Thus the necessity part of the theorem follows from Lemma 4.2.

For the proof of sufficiency assume that the lattices $\mathbf{L}_i$, $i = 1, 2$, are tolerance simple. Then by Lemma 4.1 $\mathbf{L}$ contains a cross. By Theorem 2.3 we have to prove that every finitely generated tolerance T of $\mathbf{L}$ is congruence generated. By Lemma 4.4 we may restrict ourselves to the case $T \geqslant \Sigma_{\mathbf{L}}$. Clearly we shall be done if we prove (6), which by Theorem 3.5 and Lemma 4.3 is equivalent to (7).

Now obviously the two cases ($L = L_1 \times L_2$ or L is a maximal subuniverse of $\mathbf{L}_1 \times \mathbf{L}_2$) are trivial. For the remaining case assume that L is the intersection of two maximal subuniverses K_1 and K_2 of $\mathbf{L}_1 \times \mathbf{L}_2$, the first containing $\langle 0, 1\rangle$ and the other $\langle 1, 0\rangle$. As the first step, we show that $K_1 = L^{01}$ and $K_2 = L^{10}$. Obviously it suffices to prove only one of these equalities.

Clearly, $L^{01} \subset K_1$. Take an arbitrary $x \in K_1$. Since $\mathbf{L}$ is subdirect in $\mathbf{L}_1 \times \mathbf{L}_2$, there exists $y \in L$ such that $x^2 = y^2$. Since $L \subset K_1$, we have $\langle x^1 \vee y^1, x^2\rangle \in K_1$. On the other hand, by Lemma 3.1 $\langle 1, 0\rangle, \langle 1, 1\rangle \in K_2$ implies $\langle 1, x^2\rangle \in K_2$ and similarly $\langle y^1, y^2\rangle, \langle 1, y^2\rangle \in K_2$ implies $\langle x^1 \vee y^1, y^2\rangle \in K_2$. Hence $\langle x^1 \vee y^1, x^2\rangle \in K_1 \cap K_2 = L$ and $x \in L^{01}$ directly follows from Lemma 4.2.

Now let K be a subuniverse of $\mathbf{L}_1 \times \mathbf{L}_2$ containing L. Then obviously $L^{01} \subset K^{01}$, $L^{10} \subset K^{10}$ and the maximality of L^{01} and L^{10} yields

$$K^{01} \in \{L^{01}, L_1 \times L_2\}, \quad K^{10} \in \{L^{10}, L_1 \times L_2\}.$$

Finally, Lemma 4.2 implies

$$K \in \{L, L^{01}, L^{10}, L_1 \times L_2\}.$$

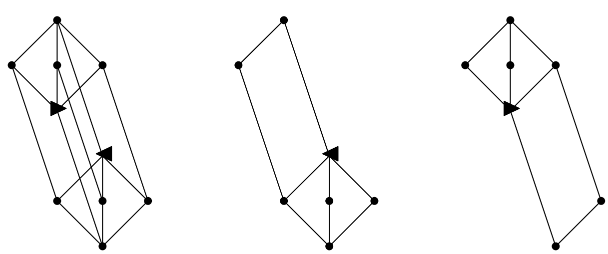
$\square$

5. Examples

Example 1. We first consider the simplest situation when one of the two subdirect factors is the two-element lattice $\mathbf{D}_2$. Let $\mathbf{L}$ be a subdirect product of $\mathbf{D}_2$ and an arbitrary tolerance simple lattice of finite height $\mathbf{M}$. This subdirect product can be trivial only if $\mathbf{M} \simeq \mathbf{D}_2$ and in this case it is obviously locally order affine complete. If $\mathbf{L}$ is not a trivial subdirect product in $\mathbf{D}_2 \times \mathbf{M}$ then it can be locally order affine complete only if it contains a cross. However, it follows from $|D_2| = 2$ that if L contains a cross then it certainly contains one of the "corners" $\langle 0, 1\rangle$ or $\langle 1, 0\rangle$. Now it is easy to see that a sublattice of $\mathbf{D}_2 \times \mathbf{M}$ is maximal iff it has one of the forms $\{\langle x, y\rangle \colon\ y = 1 \text{ or } x \leqslant c\}$ or $\{\langle x, y\rangle \colon\ y = 0 \text{ or } x \geqslant a\}$ where a and c are an atom and a coatom of $\mathbf{M}$, respectively. We conclude that a nontrivial subdirect product of $\mathbf{D}_2$ and $\mathbf{M}$ is locally order affine complete iff it is either the full direct product or has one of the two forms described above.

Let now $\mathbf{M} = \mathbf{M}_3$, the five-element simple modular lattice. In Fig. 1 we have exhibited the lattice $\mathbf{D}_2 \times \mathbf{M}_3$ and two of its maximal sublattices, one containing $\langle 0, 1\rangle$ and the other $\langle 1, 0\rangle$. One can easily check that $\mathbf{K}_1$ and $\mathbf{K}_2$ are up to isomorphism the only maximal sublattices of $\mathbf{D}_2 \times \mathbf{M}_3$. Hence, the lattices given in Fig. 1 are up to isomorphism all order affine complete lattices that are subdirect products of $\mathbf{D}$ and $\mathbf{M}_3$.

In the sequel, in all lattice diagrams, the symbols ◀ and ▶ denote the elements $\langle 0, 1\rangle$ and $\langle 1, 0\rangle$, respectively.



$\mathbf{D} \times \mathbf{M}_3$ $\mathbf{K}_1$ $\mathbf{K}_2$

Figure 1. Subdirect products in $\mathbf{D} \times \mathbf{M}_3$.

Example 2. Now we describe all maximal sublattices of $\mathbf{M}_3 \times \mathbf{M}_3$ given in Fig. 2.

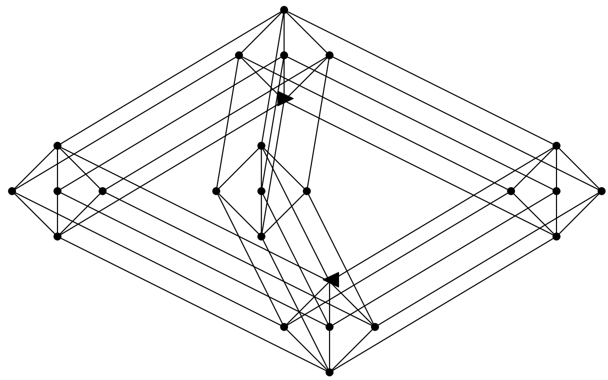


Figure 2. $\mathbf{M}_3 \times \mathbf{M}_3$.

There are two different types of maximal sublattices $\mathbf{K}$ in $\mathbf{M}_3 \times \mathbf{M}_3$. In one of these cases the sublattice $\mathbf{K}$ contains only one cross ($\langle 0,1\rangle$-cross or $\langle 1,0\rangle$-cross). One of such sublattices $\mathbf{K}_3$ is exhibited in Figs. 3 and 7.

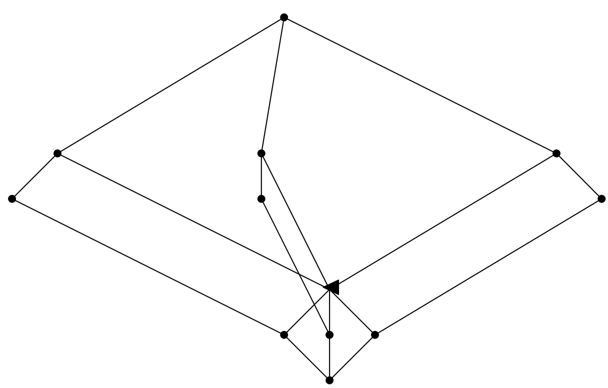


Figure 3. Maximal sublattice $\mathbf{K}_3$ of $\mathbf{M}_3 \times \mathbf{M}_3$.

In the second case the sublattice contains more than one cross. In Figs. 4 and 5 we exhibit two of such maximal sublattices $\mathbf{K}_4$ and $\mathbf{K}_5$, one containing $\langle 0,1\rangle$ and the other $\langle 1,0\rangle$. Note that actually these lattices are isomorphic.

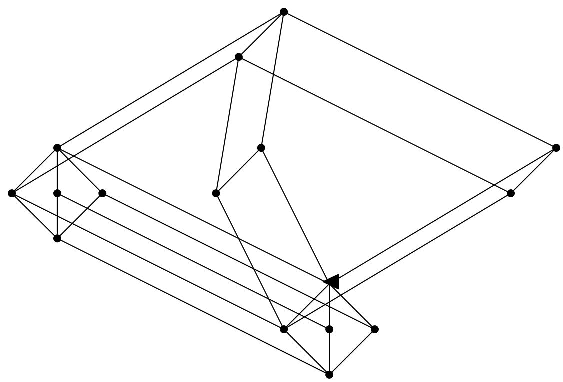


Figure 4. Maximal sublattice $\mathbf{K}_4$ in $\mathbf{M}_3 \times \mathbf{M}_3$.

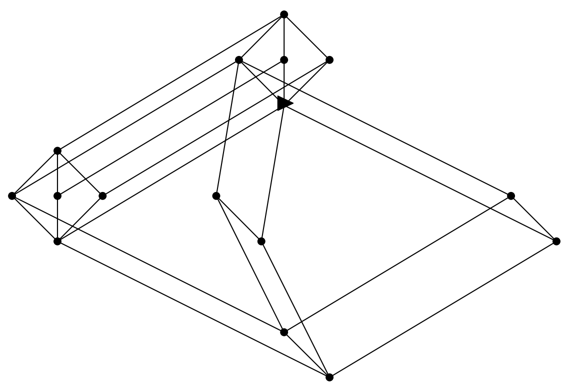


Figure 5. Maximal sublattice $\mathbf{K}_5$ in $\mathbf{M}_3 \times \mathbf{M}_3$.

In Figs. 6 and 7 we exhibit the intersection of sublattices $\mathbf{K}_4$ and $\mathbf{K}_5$. Since it contains a cross (whose elements are denoted by ▪), it is order affine complete.

Example 3. We conclude the paper by a series of examples of locally order affine complete lattices which generalize, in a sense, the lattice $\mathbf{K}_3$. One can easily observe that the set K_3 is exactly the order relation of $\mathbf{M}_3 \times \mathbf{M}_3$, that is, $\langle x, y\rangle \in K_3$ iff $x \leqslant y$. Thus the order relation of $\mathbf{M}_3$ is a maximal subuniverse of $\mathbf{M}_3 \times \mathbf{M}_3$. We are going to show that the same is true in the case of all atomic lattices $\mathbf{M}$ satisfying the following condition $(*)$: for every two distinct atoms p and q of $\mathbf{M}$ there exists an atom r of $\mathbf{M}$ such that $r \leqslant p \vee q$ but $r \notin \{p, q\}$. Clearly all modular simple geometric lattices satisfy this condition.

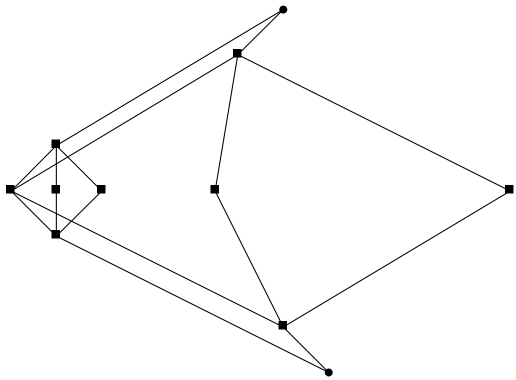
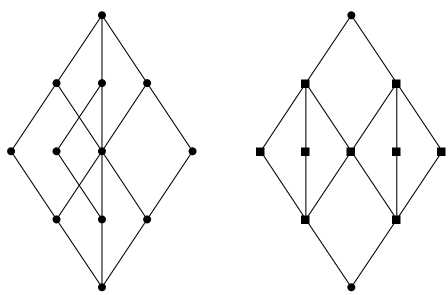


Figure 6. Intersection of $\mathbf{K}_4$ and $\mathbf{K}_5$.



$\mathbf{K}_3$ $\qquad$ $\mathbf{K}_4 \cap \mathbf{K}_5$

Figure 7.

Let $\mathbf{M}$ be an atomistic lattice satisfying the condition $(*)$ and let K be its order relation $\leqslant$. Obviously K is subdirect in $M \times M$. Assume that L is a subuniverse of $\mathbf{M} \times \mathbf{M}$ such that $K \subset L \subset M \times M$ and $K \neq L$. Since K contains Δ_M, it easily follows that L contains a pair $\langle a, b \rangle$ such that $a > b$. Since $\mathbf{M}$ is atomistic, there exists an atom p of $\mathbf{M}$ such that $p \leqslant a$ but $p \nleqslant b$. By Lemma 3.1, $\langle a, b \rangle, \langle b, b \rangle \in L$ implies $\langle b \vee p, b \rangle \in L$, hence also $\langle p, 0 \rangle = \langle b \vee p, b \rangle \wedge \langle p, p \rangle \in L$. Now let q be an arbitrary atom of $\mathbf{M}$, $p \neq q$, and take an atom r of $\mathbf{M}$ such that $r \leqslant p \vee q$ but $r \notin \{p, q\}$. Then $\langle p, 0 \rangle \in L$ implies $\langle p \vee q, q \rangle \in L$ and consequently $\langle q, 0 \rangle = \langle p \vee q, q \rangle \wedge \langle r, r \rangle \in L$. We have proved that L contains all pairs $\langle q, 0 \rangle$ where q is an atom of $\mathbf{M}$. Since $\mathbf{M}$ is atomistic, it must also contain the pair $\langle 1, 0 \rangle$. Now $\langle 0, 1 \rangle, \langle 1, 0 \rangle \in L$ easily implies $L = M \times M$.

References

[1] *I. Chajda, B. Zelinka*: Tolerance relation on lattices. Čas. Pěst. Mat. *99* (1974), 394–399. zbl
[2] *K. Kaarli*: On varieties generated by functionally complete algebras. Algebra Univers. *29* (1992), 495–502. zbl
[3] *G. Czédli, E. Horváth, and S. Radeleczki*: On tolerance lattices of algebras in congruence modular lattices. Acta Math. Hung. *100* (2003), 9–17. zbl
[4] *P. Hegedűs, P. P. Pálfy*: Finite modular congruence lattices. Algebra Univers. *54* (2005), 105–120. zbl
[5] *K. Kaarli, A. F. Pixley*: Polynomial Completeness in Algebraic Systems. Chapman & Hall/CRC, Boca Raton, 2000. zbl
[6] *M. Kindermann*: Über die Äquivalenz von Ordnungspolynomvollständigkeit und Toleranzeinfachheit endlicher Verbände. Contributions to General Algebra (Proc. Klagenfurt Conf. 1978). Verlag J. Heyn, Klagenfurt, 1979, pp. 145–149. zbl
[7] *D. Schweigert*: Über die endliche, ordnungspolynomvollständige Verbände. Monatsh. Math. *78* (1974), 68–76. zbl
[8] *R. Wille*: Eine Characterisierung endlicher, ordnungspolynomvollständiger Verbände. Arch. Math. (Basel) *28* (1977), 557–560. zbl
[9] *R. Wille*: Über endliche, ordnungaffinvollständige Verbände. Math. Z. *155* (1977), 103–107. zbl

Author's address: Kalle Kaarli, Vladimir Kuchmei, Institute of Pure Mathematics, University of Tartu, 50090 Tartu, Estonia, e-mails `kaarli@math.ut.ee`, `kucmei@math.ut.ee`.