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A NOTE ON DEGREE-CONTINUOUS GRAPHS

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Abstract. The minimum orders of degree-continuous graphs with prescribed degree sets were investigated by Gimbel and Zhang, Czechoslovak Math. J. 51 (126) (2001), 163–171. The minimum orders were not completely determined in some cases. In this note, the exact values of the minimum orders for these cases are obtained by giving improved upper bounds.

Keywords: order, degree-continuous graph, degree set, Erdös-Gallai theorem, regular graph

MSC 2000: 05C12

A graph G is *degree-continuous* if the degrees of every two adjacent vertices of G differ by at most 1.

For integers $r_1 \leqslant r_2$, let $[r_1, r_2]$ denote the set $\{r_1, r_1+1, r_1+2, \ldots, r_2\}$. We call $[r_1, r_2]$ an *interval of integers*. Let S be an interval of integers. We use $m(S)$ to denote the minimum order of a degree-continuous graph having S as its degree set. Gimbel and Zhang [3] proved the following theorem. There is a misprint in line 9 of page 167 in [3]; $m = \lceil \frac{1}{3}s \rceil$ should be $m = [\frac{1}{3}s]$.

Theorem 1 ([3]). *Let $S = [r, r+s]$ where r is a positive integer and s is a nonnegative integer.*

1. *If $s = 3m$ where m is a nonnegative integer, then*

$$(m+1)\Big(r+1+\frac{3m}{2}\Big) \leqslant m(S) \leqslant 1+(m+1)\Big(r+1+\frac{3m}{2}\Big).$$

Moreover, if r or m is odd, then

$$m(S) = (m+1)\Big(r+1+\frac{3m}{2}\Big).$$

2. *If $s = 3m + 1$ where m is a nonnegative integer, then*

$$m(S) = (m+1)\Big(r + 2 + \frac{3m}{2}\Big).$$

3. *If $s = 3m + 2$ where m is a nonnegative integer, then*

$$1 + (m+1)\Big(r + 3 + \frac{3m}{2}\Big) \leqslant m(S) \leqslant 2 + (m+1)\Big(r + 3 + \frac{3m}{2}\Big).$$

Moreover, if r is even or m is odd, then

$$m(S) = 1 + (m+1)\Big(r + 3 + \frac{3m}{2}\Big).$$

In the above theorem, the exact value of $m(S)$ is not completely decided if $s = 3m$ and r, m are both even or $s = 3m + 2$ and r is odd, m is even. In this note, we will determine $m(S)$ for these cases. We need some lemmas for our discussions. Let us begin with the well known result of Erdös and Gallai.

Proposition 2 (Erdös-Gallai [1], [2]). *Let $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_n$ be nonnegative integers. Then $d_1, d_2, \ldots, d_n$ is a degree sequence if and only if $d_1 + d_2 + \ldots + d_n$ is even and $\sum\limits_{i=1}^{k} d_i \leqslant k(k-1) + \sum\limits_{i=k+1}^{n} \min\{k, d_i\}$ for $1 \leqslant k \leqslant n$.*

The following lemma follows from the above proposition.

Lemma 3. *Suppose that both n and r are odd integers such that $r \leqslant n - 2$. Let $d_1 = r + 1$ and $d_2 = d_3 = \ldots = d_n = r$. Then $d_1, d_2, \ldots, d_n$ is a degree sequence.*

Proof. It is obvious that $d_1 + d_2 + \ldots + d_n$ is even. By Proposition 2, it suffices to show that $\sum\limits_{i=1}^{k} d_i \leqslant k(k-1) + \sum\limits_{i=k+1}^{n} \min\{k, d_i\}$, $k = 1, 2, \ldots, n$. We distinguish three cases: Case 1: $1 \leqslant k \leqslant r$, Case 2: $r + 1 \leqslant k \leqslant n - 1$, Case 3: $k = n$.

Case 1: $1 \leqslant k \leqslant r$. Then

$$\begin{aligned} k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} &= k(k-1) + (n-k)k = k(n-1) \\ &\geqslant k(r+1) \geqslant kr + 1 = \sum_{i=1}^{k} d_i. \end{aligned}$$

Case 2: $r + 1 \leqslant k \leqslant n - 1$. Then

$$\begin{aligned} k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} &\geqslant k(k-1) + \min\{k, d_n\} = k(k-1) + r \\ &\geqslant kr + 1 = \sum_{i=1}^{k} d_i. \end{aligned}$$

Case 3: $k = n$. Then

$$k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\} = n(n-1) \geqslant n(r+1) \geqslant nr + 1$$
$$= \sum_{i=1}^{n} d_i = \sum_{i=1}^{k} d_i.$$

□

Lemma 3 can also be proved by the following direct construction. Suppose that $n = 2n_1 + 1$, $r = 2r_1 + 1$ are odd integers with $r \leqslant n - 2$. Let G be the graph with $V(G) = \{v_i\colon\ i = 1, 2, \ldots, n\}$ and $E(G) = \{v_i v_j\colon\ i = 1, 2, \ldots, n,\ j = i+1, i+2, \ldots, i+r_1 \pmod n\} \cup \{v_i v_{i+n_1}\colon\ i = 1, 2, \ldots, n_1\} \cup \{v_2 v_n\}$. Then G has the degree sequence $r+1, r, r, \ldots, r$. The following two easy lemmas are also needed for our discussions.

Lemma 4. *For positive integers n, r with $r \leqslant n-1$ and nr being even, there exists an r-regular graph of order n.*

Lemma 5. *For positive integers n, r with $r \leqslant n$, there exists an r-regular bipartite graph with each part of cardinality n.*

For graphs $G_1, G_2, \ldots, G_k$, the *P_k-path composition* of $G_1, G_2, \ldots, G_k$ is the graph G with $V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_k)$ and $E(G) = E(G_1) \cup E(G_2) \cup \ldots \cup E(G_k) \cup \{xy\colon\ x \in V(G_i),\ y \in V(G_{i+1}),\ i = 1, 2, \ldots, k-1\}$. The P_k-path composition of $G_1, G_2, \ldots, G_k$ is denoted by $P_k[G_1, G_2, \ldots, G_k]$. It is easy to see that if $H = P_k[G_1, G_2, \ldots, G_k]$, then

$$\deg_H x = \begin{cases} \deg_{G_1} x + |V(G_2)| & \text{if } x \in V(G_1), \\ |V(G_{i-1})| + \deg_{G_i} x + |V(G_{i+1})| & \text{if } x \in V(G_i),\ \ 2 \leqslant i \leqslant k-1, \\ |V(G_{k-1})| + \deg_{G_k} x & \text{if } x \in V(G_k). \end{cases}$$

Now we prove the main result of this note.

Theorem 6. *Let $S = [r, r+s]$ where r is a positive integer and s is a nonnegative integer.*

1. *If $s = 3m$ and r, m are both even, then*

$$m(S) \leqslant (m+1)\Big(r + 1 + \frac{3m}{2}\Big).$$

2. *If $s=3m+2$ and r is odd, m is even, then*

$$m(S) \leqslant 1+(m+1)\Big(r+3+\frac{3m}{2}\Big).$$

P r o o f. 1. Suppose that $s=3m$ and both r and m are even.

It suffices to construct a degree-continuous graph G with degree set $[r, r+3m]$ and $|V(G)|=(m+1)(r+1+\frac{3}{2}m)$. We distinguish three cases:

Case 1: $m=0$, Case 2: $m \geqslant 2$, $r=2$, and Case 3: $m \geqslant 2$, $r \geqslant 4$.

Case 1: $m=0$. Take $G=K_{r+1}$. Then G is a degree-continuous graph with degree set $[r,r]$ and $|V(G)|=r+1$.

Case 2: $m \geqslant 2$, $r=2$. If $m=2$, take $G=P_7[K_1, K_2^c, K_2, K_1, K_3^c, K_5^c, K_4]$. Then G is a degree-continuous graph with degree set $[2,8]$ and $|V(G)|=18$.

If $m \geqslant 4$, take $G=P_{3m+1}[K_1, K_2, K_1, K_2, K_3, K_2, \ldots, K_{m-1}, K_m, K_{m-1}, K_m, K_{m+1}^c, R_{2m}^{m-3}, K_{m+3}]$ where R_{2m}^{m-3} is an $(m-3)$-regular graph of order $2m$ (the existence of R_{2m}^{m-3} is due to Lemma 4). Then G is a degree-continuous graph with degree set $[2, 2+3m]$ and $|V(G)|=(m+1)\big(3+\frac{3}{2}m\big)$.

Case 3: $m \geqslant 2$, $r \geqslant 4$. First let $G_1=K_{r+m-1}$, and let G_2 be a graph with $V(G_2)=\{a_1, a_2, \ldots, a_{r+m-1}\}$ such that $\deg_{G_2}(a_1)=r-2$ and $\deg_{G_2}(a_i)=r-3$ for $2 \leqslant i \leqslant r+m-1$ (the existence of G_2 is due to Lemma 3 since both $r+m-1$ and $r-3$ are odd and $r-3 \leqslant (r+m-1)-2$), and let $G_3=K_{2m+2}$ with $V(G_3)=\{b_1, b_2, \ldots, b_{2m+2}\}$.

Next, let

$$G_1' = P_{3m-1}[K_1, K_r, K_1, K_2, K_{r+1}, K_2, \ldots, K_{m-1}, K_{r+m-2}, K_{m-1}, K_m, G_1].$$

Finally let G be the graph obtained from disjoint copies of G_1', G_2, G_3 by adding some edges between vertices in G_1 and vertices in G_2 so that these edges induces an m-regular bipartite graph with bipartite sets $V(G_1)$, $V(G_2)$ (by Lemma 5, this can be done), and also adding the edges a_ib_j where $1 \leqslant i \leqslant r+m-1$, $1 \leqslant j \leqslant 2m+2$, and $(i,j) \neq (1,1)$.

Then G is a degree-continuous graph with degree set $[r, r+3m]$ and $|V(G)|=(m+1)(r+1+\frac{3}{2}m)$.

2. Suppose that $s=3m+2$ and r is odd, m is even.

It suffices to construct a degree-continuous graph G with degree set $[r, r+3m+2]$ and $|V(G)|=1+(m+1)(r+3+\frac{3}{2}m)$. We distinguish three cases: Case 1: $m=0$, Case 2: $m \geqslant 2$, $r=1$, and Case 3: $m \geqslant 2$, $r \geqslant 3$.

Case 1: $m=0$. If $r=1$, take $G=P_4[K_1, K_1, K_1, K_2]$. Then G is a degree-continuous graph with degree set $[1,3]$ and $|V(G)|=5$.

If $r \geqslant 3$, take $G = P_3[K_1, R_r^{r-3}, K_3]$ where R_r^{r-3} is an $(r-3)$-regular graph of order r. Then G is a degree-continuous graph with degree set $[r, r+2]$ and $|V(G)| = r+4$.

Case 2: $m \geqslant 2$, $r = 1$. Take $G = P_{3m+3}[K_1, K_1, K_1, K_2, K_2, K_2, \ldots, K_m, K_m, K_m, R_{m+1}^{m-2}, K_{m+3}^c, K_{2m+1}]$ where R_{m+1}^{m-2} is an $(m-2)$-regular graph of order $m+1$. Then G is a degree-continuous graph with degree set $[1, 3m+3]$ and $|V(G)| = 1 + (m+1)\big(4 + \frac{3}{2}m\big)$.

Case 3: $m \geqslant 2$, $r \geqslant 3$. Take $G = P_{3m+3}[K_1, K_r, K_1, K_2, K_{r+1}, K_2, \ldots, K_m, K_{r+m-1}, K_m, K_{m+1}, R_{r+m}^{r-3}, K_{2m+3}]$ where R_{r+m}^{r-3} is an $(r-3)$-regular graph of order $r+m$. Then G is a degree-continuous graph with degree set $[r, r+3m+2]$ and $|V(G)| = 1 + (m+1)\big(r + 3 + \frac{3}{2}m\big)$. □

Combining Theorem 1 with Theorem 6, we have the following:

Theorem 7. *Let $S = [r, r+s]$ where r is a positive integer and s is a nonnegative integer.*

1. *If $s = 3m$ where m is a nonnegative integer, then*

$$m(S) = (m+1)\Big(r + 1 + \frac{3m}{2}\Big).$$

2. *If $s = 3m+1$ where m is a nonnegative integer, then*

$$m(S) = (m+1)\Big(r + 2 + \frac{3m}{2}\Big).$$

3. *If $s = 3m+2$ where m is a nonnegative integer, then*

$$m(S) = 1 + (m+1)\Big(r + 3 + \frac{3m}{2}\Big).$$

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