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A CLASS OF INTEGRAL OPERATORS ON MIXED NORM SPACES IN THE UNIT BALL

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Abstract. This article provided some sufficient or necessary conditions for a class of integral operators to be bounded on mixed norm spaces in the unit ball.

Keywords: integral operator, mixed norm space, boundedness

MSC 2000: 47B35, 30H05

1. Introduction

Let $B = \{z \in \mathbb{C}^n \colon |z| < 1\}$ be the open unit ball in the complex vector space $\mathbb{C}^n$, $S = \{z \in \mathbb{C}^n \colon |z| = 1\}$ be its boundary. Let $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ be points in $\mathbb{C}^n$ and

$$\langle z, w\rangle = z_1\overline{w}_1 + \ldots + z_n\overline{w}_n.$$

Let $\mathrm{d}v$ denote the normalized Lebesgue area measure on the unit ball B such that $v(B) = 1$, and $\mathrm{d}\sigma$ be the normalized rotation invariant measure on the boundary S of B such that $\sigma(S) = 1$. The weighted Lebesgue measure $\mathrm{d}v_\alpha$ ($\alpha > -1$) is defined by $\mathrm{d}v_\alpha(z) = c_\alpha(1-|z|^2)^\alpha \mathrm{d}v(z)$, where c_α is a normalizing constant such that $v_\alpha(B) = 1$. Using polar coordinates, we can easily obtain that (see [13])

$$c_\alpha = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)}.$$

A positive continuous function φ on $[0,1)$ is normal, if there exist $0 < s < t$, $0 \leqslant r_0 < 1$ such that

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(i) $\varphi(r)/(1-r)^s$ is nonincreasing for $r_0 \leqslant r < 1$ and $\lim_{r\to 1} \varphi(r)/(1-r)^s = 0$;

(ii) $\varphi(r)/(1-r)^t$ is nondecreasing for $r_0 \leqslant r < 1$ and $\lim_{r\to 1} \varphi(r)/(1-r)^t = \infty$.

For $0 < p < \infty$, $0 < q \leqslant \infty$, and a normal function φ, let $L_{p,q}(\varphi)$ denote the space of measurable complex functions on B with

$$\|f\|_{p,q,\varphi} = \left\{ \int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r) M_q^p(r,f)\,\mathrm{d}r \right\}^{1/p} < \infty,$$

where

$$M_q(r,f) = \left\{ \int_S |f(r\xi)|^q \,\mathrm{d}\sigma(\xi) \right\}^{1/q}, \quad 0 < q < \infty.$$

For $1 \leqslant p < \infty$, equipped with the norm $\|\cdot\|_{p,q,\varphi}$, $L_{p,q}(\varphi)$ is a Banach space. When $0 < p < 1$, $\|f\|_{p,q,\varphi}$ is a quasinorm on $L_{p,q}(\varphi)$, $L_{p,q}(\varphi)$ is just a Fréchet space.

If $0 < p < \infty$, $\alpha > -1$, let $L^p(B, \mathrm{d}v_\alpha)$ denote the space of measurable complex functions on B with

$$\int_B |f(z)|^p \,\mathrm{d}v_\alpha(z) = c_\alpha \int_B |f(z)|^p (1-|z|^2)^\alpha \,\mathrm{d}v(z) < \infty.$$

Then from the integral formula in polar coordinates, we have

$$\int_B |f(z)|^p (1-|z|^2)^\alpha \,\mathrm{d}v(z) = 2n \int_0^1 r^{2n-1}(1-r^2)^\alpha M_p^p(r,f)\,\mathrm{d}r.$$

From the above equality, we see that

$$L^p(B, \mathrm{d}v_\alpha) = L_{p,p}((1-r^2)^{(\alpha+1)/p}).$$

The integral operators $T = T_{a,b,c}$ and $S = S_{a,b,c}$ which were introduced by Kurens and Zhu are defined by

$$Tf(z) = (1-|z|^2)^a \int_B \frac{(1-|w|^2)^b}{(1-\langle z,w\rangle)^c} f(w)\,\mathrm{d}v(w),$$

and

$$Sf(z) = (1-|z|^2)^a \int_B \frac{(1-|w|^2)^b}{|1-\langle z,w\rangle|^c} f(w)\,\mathrm{d}v(w).$$

Here a, b and c are real parameters. When $a = 0$, $c = n+1+b$, a sufficient and necessary condition for the boundedness of $T_{0,b,n+1+b}$ on $L^p(B, \mathrm{d}v)$ was obtained by Forelli and Rudin in [3]. In particular, when $c = n+1+a+b$, $T_{a,b,c}$ is holomorphic whenever it is defined and is in some sense similar to the Bergman projection. The boundedness of this operator and some interesting related problems on various spaces were investigated in [2], [4], [5], [7], [8], [9], [10], [12]. In [6], Kurens and Zhu determined exactly when the operators T and S are bounded on $L^p(B, \mathrm{d}v_\alpha)$.

In paper, we study this integral operators T and S on mixed norm spaces of the unit ball. The techniques borrowed from [7], [8] have been modified to make it more efficient. The following four Theorems are our main results.

Theorem 1. *Let φ be a normal function with constants s, t as in the definition of normal function. If $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, $b+1 > t > s > -a$, $c = n+1+a+b$, then $S\colon L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is bounded.*

Theorem 2. *Let φ be a normal function with constants s, t as in the definition of normal function. If $1 \leqslant p < \infty$, $1 \leqslant q < \infty$, $b+1 > t > s > -a$, $n+a+s+t < c < n+1+a+b$, then $S\colon L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is bounded.*

Theorem 3. *Let φ be a normal function with constants s, t as in the definition of normal function. If $0 < p < 1$, $1 < q < \infty$, $c-n-a > t > s > -a$, $c \leqslant n+1+a+b$, then $S\colon L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is bounded.*

Theorem 4. *Let φ be a normal function with constants s, t as in the definition of normal function, $1 \leqslant p < \infty$, $1 \leqslant q < \infty$. If $T\colon L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is bounded and $c > n$, then $a > -t$, $b > s-1$.*

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $C^{-1}B \leqslant A \leqslant CB$.

2. Proofs of main results

In this section, we will prove the main results in this paper. In order to prove the main results, we need some auxiliary results which are incorporated in the following lemmas.

Lemma 1 ([11]). *If $0 < \varrho < 1$, $s_1 > s_2 > 0$, then*

$$\int_0^1 \frac{(1-r)^{s_2-1}}{(1-r\varrho)^{s_1}}\,\mathrm{d}r \leqslant C\frac{1}{(1-\varrho)^{s_1-s_2}}.$$

Lemma 2 ([13]). *Suppose c is real. Then the integral*

$$I_c(z)=\int_S \frac{\mathrm{d}\sigma(\zeta)}{|1-\langle z,\zeta\rangle|^{n+c}}$$

have the following asymptotic properties.

(i) *If $c<0$, then $I_c(z)$ is bounded in B.*

(ii) *If $c=0$, then $I_c(z)\asymp \log(1-|z|^2)^{-1}$ as $|z|\to 1^-$.*

(iii) *If $c>0$, then $I_c(z)\asymp(1-|z|^2)^{-c}$ as $|z|\to 1^-$.*

The following two lemmas can be found in [8].

Lemma 3. *Let φ be a normal function. If $m_1+m_2>t>s>m_1$, then*

$$\int_0^1 \frac{\varphi^p(\varrho)\mathrm{d}\varrho}{(1-\varrho)^{pm_1+1}(1-r\varrho)^{pm_2}}\leqslant C\frac{\varphi^p(r)}{(1-r)^{p(m_1+m_2)}}\quad (0\leqslant r<1,\ \ p>0).$$

Lemma 4. *Let $1<p<\infty$, $1\leqslant q<\infty$, $\frac{1}{p}+\frac{1}{p'}=1$, $\frac{1}{q}+\frac{1}{q'}=1$, φ be a normal function. Then $(L_{p,q}(\varphi))^*=L_{p',q'}(\varphi^{p/p'})$. The pairing is given by*

$$(f,g)=\frac{1}{2n}\int_B f(z)\overline{g(z)}(1-|z|)^{-1}\varphi^p(|z|)\,\mathrm{d}v(z).$$

More precisely, $T\in(L_{p,q}(\varphi))^*$ if and only if there is a unique function $g\in L_{p',q'}(\varphi^{p/p'})$ such that for any $f\in L_{p,q}(\varphi)$, $Tf=(f,g)$ and $\|T\|=\|g\|_{p',q',\varphi^{p/p'}}$.

Lemma 5. *If $c>n$, $1\leqslant q<\infty$, then*

$$M_q(\varrho,Tf)\leqslant C(1-\varrho^2)^a\int_0^1\frac{r^{2n-1}(1-r^2)^b}{(1-r\varrho)^{c-n}}M_q(r,f)\,\mathrm{d}r, \tag{1}$$

and

$$M_q(\varrho,Sf)\leqslant C(1-\varrho^2)^a\int_0^1\frac{r^{2n-1}(1-r^2)^b}{(1-r\varrho)^{c-n}}M_q(r,f)\,\mathrm{d}r. \tag{2}$$

Proof. We only prove the inequality (1), since the proof of (2) is similar. By the definition of T and the integral formula in polar coordinates, we get

$$Tf(z)=2n(1-|z|^2)^a\int_0^1 r^{2n-1}\int_S\frac{(1-r^2)^bf(r\zeta)}{(1-\langle z,r\zeta\rangle)^c}\,\mathrm{d}\sigma(\zeta).$$

Put $z = \varrho\xi$, $\xi \in S$, it follows that

$$(3) \qquad |Tf(\varrho\xi)| \leqslant C(1-\varrho^2)^a \int_0^1 r^{2n-1}(1-r^2)^b \, \mathrm{d}r \int_S \frac{|f(r\zeta)|}{|1-\langle \varrho\xi, r\zeta\rangle|^c} \, \mathrm{d}\sigma(\zeta).$$
$$= C(1-\varrho^2)^a \int_0^1 r^{2n-1}(1-r^2)^b f^*(r,\varrho\xi) \, \mathrm{d}r.$$

Here

$$(4) \qquad f^*(r,\varrho\xi) = \int_S \frac{|f(r\zeta)|}{|1-\langle \varrho\xi, r\zeta\rangle|^c} \, \mathrm{d}\sigma(\zeta).$$

If $q = 1$, the result follows from (3) and (4) directly. If $1 < q < \infty$, by Hölder's inequality and Lemma 2 and $c > n$, we get

$$f^*(r,\varrho\xi) \leqslant \left\{\int_S \frac{|f(r\zeta)|^q}{|1-\langle \varrho\xi, r\zeta\rangle|^c} \, \mathrm{d}\sigma(\zeta)\right\}^{1/q} \left\{\int_S \frac{1}{|1-\langle \varrho\xi, r\zeta\rangle|^c} \, \mathrm{d}\sigma(\zeta)\right\}^{1/q'}$$
$$\leqslant C \frac{1}{(1-r\varrho)^{(c-n)/q'}} \left\{\int_S \frac{|f(r\zeta)|^q}{|1-\langle \varrho\xi, r\zeta\rangle|^c} \, \mathrm{d}\sigma(\zeta)\right\}^{1/q}.$$

Since $c > n$, by using Fubini's Theorem and Minkowski's inequality we get

$$M_q(\varrho, Tf) = \left\{\int_S |Tf(\varrho\xi)|^q \, \mathrm{d}\sigma(\xi)\right\}^{1/q}$$
$$\leqslant C\left\{\int_S (1-\varrho^2)^{aq} \left[\int_0^1 r^{2n-1}(1-r^2)^b f^*(r,\varrho\xi) \, \mathrm{d}r\right]^q \mathrm{d}\sigma(\xi)\right\}^{1/q}$$
$$\leqslant C(1-\varrho^2)^a \left\{\int_S \left[\int_0^1 \frac{r^{2n-1}(1-r^2)^b}{(1-r\varrho)^{(c-n)/q'}} \left\{\int_S \frac{|f(r\zeta)|^q}{|1-\langle \varrho\xi, r\zeta\rangle|^c} \, \mathrm{d}\sigma(\zeta)\right\}^{1/q} \mathrm{d}r\right]^q \mathrm{d}\sigma(\xi)\right\}^{1/q}$$
$$= C(1-\varrho^2)^a \int_0^1 r^{2n-1}(1-r^2)^b \frac{1}{(1-r\varrho)^{(c-n)/q'}} \left\{\int_S \left[\int_S \frac{|f(r\zeta)|^q \, \mathrm{d}\sigma(\zeta)}{|1-\langle \varrho\xi, r\zeta\rangle|^c}\right] \mathrm{d}\sigma(\xi)\right\}^{1/q} \mathrm{d}r$$
$$\leqslant C(1-\varrho^2)^a \int_0^1 \frac{r^{2n-1}(1-r^2)^b}{(1-r\varrho)^{c-n}} M_q(r,f) \, \mathrm{d}r.$$

□

Lemma 6. *If $c > n$, $0 < p < 1$, $1 < q < \infty$, then*

$$(5) \qquad M_q^p(\varrho, Tf) \leqslant C(1-\varrho^2)^{pa} \int_0^1 \frac{r^{p(2n-1)}(1-r^2)^{p(b+1)-1}}{(1-r\varrho)^{p(c-n)}} M_q^p(r,f) \, \mathrm{d}r,$$

and

$$(6) \qquad M_q^p(\varrho, Sf) \leqslant C(1-\varrho^2)^{pa} \int_0^1 \frac{r^{p(2n-1)}(1-r^2)^{p(b+1)-1}}{(1-r\varrho)^{p(c-n)}} M_q^p(r,f) \, \mathrm{d}r.$$

P r o o f. We only prove the inequality (5), the proof of (6) is similar. Write $s_k = 1 - 1/2^k$, then $0 = s_0 < s_1 < s_2 < \ldots < 1$ forms a partition of the interval $[0,1)$. It is obvious that

$$s_k - s_{k-1} = 1 - s_k = 2(1 - s_{k+1}).$$

By Lemma 5, the elementary inequality

$$(a+b)^p \leqslant \left\{ \begin{matrix} a^p + b^p, & p \in (0,1), \\ 2^{p-1}(a^p + b^p), & p \geqslant 1 \end{matrix} \right\}, \quad a > 0, \ b > 0,$$

and the monotonicity of the integral means $M_q^p(r,f)$ with respect to r, we obtain

$$\begin{aligned} M_q^p(\varrho, Tf) &\leqslant C(1-\varrho^2)^{pa} \left(\int_0^1 \frac{r^{2n-1}(1-r)^b}{(1-r\varrho)^{c-n}} M_q(r,f) \,\mathrm{d}r \right)^p \\ &= C(1-\varrho^2)^{pa} \left(\sum_{k=1}^{\infty} \int_{s_{k-1}}^{s_k} \frac{r^{2n-1}(1-r)^b}{(1-r\varrho)^{c-n}} M_q(r,f) \,\mathrm{d}r \right)^p \\ &\leqslant C(1-\varrho^2)^{pa} \left(\sum_{k=1}^{\infty} \frac{s_k^{2n-1}(1-s_{k-1})^{b+1}}{(1-s_k\varrho)^{c-n}} M_q(s_k,f) \,\mathrm{d}r \right)^p \\ &\leqslant C(1-\varrho^2)^{pa} \sum_{k=1}^{\infty} \frac{s_k^{p(2n-1)}(1-s_{k-1})^{p(b+1)}}{(1-s_k\varrho)^{p(c-n)}} M_q^p(s_k,f) \,\mathrm{d}r \\ &\leqslant C(1-\varrho^2)^{pa} \int_0^1 \frac{r^{p(2n-1)}(1-r)^{p(b+1)-1}}{(1-r\varrho)^{p(c-n)}} M_q^p(r,f) \,\mathrm{d}r. \end{aligned}$$

This completes the proof of the lemma. □

Now, we can prove our main results.

P r o o f of Theorem 1. Let $p = 1$, $b + 1 > t > s > -a$, $c = n + 1 + a + b$. Then by Lemma 3 and Lemma 5 we have

$$\begin{aligned} \|Sf\|_{1,q,\varphi} &= \int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi(\varrho) M_q(\varrho, Sf) \,\mathrm{d}\varrho \\ &\leqslant C \int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi(\varrho)(1-\varrho^2)^a \int_0^1 \frac{r^{2n-1}(1-r^2)^b}{(1-r\varrho)^{c-n}} M_q(r,f) \,\mathrm{d}r \,\mathrm{d}\varrho \\ &= C \int_0^1 r^{2n-1}(1-r^2)^b M_q(r,f) \,\mathrm{d}r \int_0^1 \frac{\varrho^{2n-1}(1-\varrho)^{-1}\varphi(\varrho)(1-\varrho^2)^a}{(1-r\varrho)^{c-n}} \,\mathrm{d}\varrho \\ &\leqslant C \int_0^1 r^{2n-1}(1-r^2)^b M_q(r,f) \,\mathrm{d}r \int_0^1 \frac{(1-\varrho)^{a-1}\varphi(\varrho)}{(1-r\varrho)^{c-n}} \,\mathrm{d}\varrho \\ &\leqslant C \int_0^1 r^{2n-1}(1-r^2)^{b-c+n+a} \varphi(r) M_q(r,f) \,\mathrm{d}r \\ &\leqslant C \int_0^1 r^{2n-1}(1-r)^{-1}\varphi(r) M_q(r,f) \,\mathrm{d}r = C\|f\|_{1,q,\varphi}. \end{aligned}$$

Then the result follows from the above inequality.

If $p > 1$, we write $b+1 = b_1 + b_2 = b_3 + b_4$, where

(i) $b_i > 0$, $i = 1, 2, 3, 4$;

(ii) $s + b_1 > b_3 > b_1$;

(iii) $b_3 + a > b_1$;

(iv) $b_2 > t$.

Then, by Lemma 1 we obtain

$$\begin{aligned} M_q(\varrho, Sf) &\leqslant C(1-\varrho^2)^a \int_0^1 \frac{r^{2n-1}(1-r^2)^b}{(1-r\varrho)^{c-n}} M_q(r,f)\,\mathrm{d}r \\ &\leqslant C(1-\varrho^2)^a \left\{\int_0^1 \frac{r^{(2n-1)p}(1-r^2)^{pb_2-1}}{(1-r\varrho)^{pb_4}} M_q^p(r,f)\,\mathrm{d}r\right\}^{1/p} \\ &\quad\times \left\{\int_0^1 \frac{(1-r^2)^{p'b_1-1}}{(1-r\varrho)^{p'(b_3+a)}}\,\mathrm{d}r\right\}^{1/p'} \\ &\leqslant \frac{C}{(1-\varrho)^{b_3-b_1}} \left\{\int_0^1 \frac{r^{(2n-1)p}(1-r^2)^{pb_2-1}}{(1-r\varrho)^{pb_4}} M_q^p(r,f)\,\mathrm{d}r\right\}^{1/p}. \end{aligned}$$

Therefore, by Lemma 3, we have

$$\begin{aligned} &\|Sf\|_{p,q,\varphi}^p = \int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi^p(\varrho) M_q^p(\varrho, Sf)\,\mathrm{d}\varrho \\ &\leqslant C\int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi^p(\varrho)\frac{1}{(1-\varrho)^{p(b_3-b_1)}} \int_0^1 \frac{r^{(2n-1)p}(1-r^2)^{pb_2-1}}{(1-r\varrho)^{pb_4}} M_q^p(r,f)\,\mathrm{d}r\,\mathrm{d}\varrho \\ &= C\int_0^1 r^{(2n-1)p}(1-r^2)^{pb_2-1} M_q^p(r,f)\,\mathrm{d}r \int_0^1 \frac{\varphi^p(\varrho)}{(1-\varrho)^{p(b_3-b_1)+1}(1-r\varrho)^{pb_4}}\,\mathrm{d}\varrho \\ &\leqslant C\int_0^1 r^{2n-1}(1-r)^{-1} M_q^p(r,f)\,\mathrm{d}r \leqslant C\|f\|_{p,q,\varphi}^p. \end{aligned}$$

Hence $S\colon L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is a bounded operator. □

P r o o f of Theorem 2. When $p = 1$, the proof is similar to the proof of Theorem 1. Now consider $1 < p < \infty$, we write $b+1 = b_1 + b_2$, $c - n = b_3 + b_4$, which satisfies

(i) $b_i > 0$, $i = 1, 2, 3, 4$;

(ii) $s + b_1 > b_3 > b_1$;

(iii) $c - n - a - b_1 > s + t > b_3 - b_1 - a$.

Then by Lemma 1 we get

$$M_q(\varrho, Sf) \leqslant C(1-\varrho^2)^a \left\{ \int_0^1 \frac{r^{(2n-1)p}(1-r^2)^{pb_2-1}}{(1-r\varrho)^{pb_4}} M_q^p(r,f) dr \right\}^{1/p}$$

$$\times \left\{ \int_0^1 \frac{(1-r)^{p'b_1-1}}{(1-r\varrho)^{p'b_3}} \,\mathrm{d}r \right\}^{1/p'}$$

$$\leqslant C \frac{1}{(1-\varrho)^{b_3-b_1-a}} \left\{ \int_0^1 \frac{r^{(2n-1)p}(1-r)^{pb_2-1}}{(1-r\varrho)^{pb_4}} M_q^p(r,f)\,\mathrm{d}r \right\}^{1/p}.$$

Hence by Fubini's Theorem and Lemma 3

$$\|Sf\|_{p,q,\varphi}^p = \int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi^p(\varrho) M_q^p(\varrho, Sf)\,\mathrm{d}\varrho$$

$$\leqslant C \int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi^p(\varrho) \frac{1}{(1-\varrho)^{p(b_3-b_1-a)}} \int_0^1 \frac{r^{(2n-1)p}(1-r)^{pb_2-1}}{(1-r\varrho)^{pb_4}} M_q^p(r,f)\,\mathrm{d}r\,\mathrm{d}\varrho$$

$$= C \int_0^1 r^{(2n-1)p}(1-r)^{pb_2-1} M_q^p(r,f) \int_0^1 \frac{\varphi^p(\varrho)}{(1-\varrho)^{p(b_3-b_1-a)+1}(1-r\varrho)^{pb_4}}\,\mathrm{d}\varrho$$

$$\leqslant C \int_0^1 r^{2n-1}(1-r)^{pb_2-1} M_q^p(r,f) \frac{\varphi^p(r)}{(1-r)^{p(b_3+b_4-b_1-a)}}\,\mathrm{d}r$$

$$= C \int_0^1 r^{2n-1}(1-r)^{-1}(1-r)^{p(n+b+a+1-c)} M_q^p(r,f)\varphi^p(r)\,\mathrm{d}r$$

$$\leqslant C \int_0^1 r^{2n-1}(1-r)^{-1} M_q^p(r,f)\varphi^p(r)\,\mathrm{d}r \leqslant C\|f\|_{p,q,\varphi}^p.$$

Therefore $S\colon L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is a bounded operator. □

Proof of Theorem 3. For any $f \in L_{p,q}(\varphi)$, by $c-n-a > t > s > -a$, $c \leqslant n+1+a+b$, Lemma 3, Lemma 6 and Fubini's Theorem, we get

$$\|Sf\|_{p,q,\varphi}^p \leqslant \int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi^p(\varrho) M_q^p(\varrho, Sf)\,\mathrm{d}\varrho$$

$$\leqslant C \int_0^1 (1-\varrho)^{pa-1}\varphi^p(\varrho) \int_0^1 \frac{r^{p(2n-1)}(1-r)^{p(b+1)-1}}{(1-r\varrho)^{p(c-n)}} M_q^p(r,f)\,\mathrm{d}r\,\mathrm{d}\varrho$$

$$= C \int_0^1 r^{p(2n-1)}(1-r)^{p(b+1)-1} M_q^p(r,f) \left(\int_0^1 \frac{\varphi^p(\varrho)}{(1-\varrho)^{1-pa}(1-r\varrho)^{p(c-n)}}\,\mathrm{d}\varrho \right) \mathrm{d}r$$

$$\leqslant C \int_0^1 r^{p(2n-1)}(1-r)^{p(b+1)-1} M_q^p(r,f) \frac{\varphi^p(r)}{(1-r)^{p(c-n-a)}}\,\mathrm{d}r$$

$$= C \int_0^1 r^{p(2n-1)}(1-r)^{-1}(1-r)^{p(b+1-c+n+a)}\varphi^p(r) M_q^p(r,f)\,\mathrm{d}r$$

$$\leqslant C \int_0^1 r^{p(2n-1)}(1-r)^{-1}\varphi^p(r) M_q^p(r,f)\,\mathrm{d}r.$$

By using the change of variables $r = \varrho^{1/p}$, we can easily get the following inequalities (or see [7])

$$(1-\varrho^{1/p})^{-1} < (1-\varrho)^{-1}, \quad M_q^p(\varrho^{1/p}, f) < M_q^p(\varrho, f), \quad \varphi^p(\varrho^{1/p}) \leqslant C\varphi^p(\varrho).$$

Therefore we obtain

$$\|Sf\|_{p,q,\varphi}^p \leqslant C\int_0^1 \varrho^{2n-1}(1-\varrho)^{-1}\varphi^p(\varrho)M_q^p(\varrho, f)\,\mathrm{d}\varrho = C\|f\|_{p,q,\varphi}^p$$

as desired. □

Remark. It is clear that

$$|Tf(z)| \leqslant S(|f|)(z).$$

Hence the boundedness of S implies that of T. Therefore the conditions of Theorem 1, Theorem 2 and Theorem 3 actually are sufficient conditions for the integral operator T to be bounded on the mixed space $L_{p,q}(\varphi)$ in the unit ball.

Proof of Theorem 4. Let N be a positive integral large enough such that $f_N(z) = (1-|z|^2)^N \in L_{p,q}(\varphi)$, hence

$$Tf_N(z) = (1-|z|^2)^a \int_B \frac{(1-|w|^2)^{N+b}}{(1-\langle z, w\rangle)^c}\,\mathrm{d}v(w).$$

Write the integrand in terms of its Taylor series, it follows from the orthogonality of $\{w^\alpha\}$ (α multiindex) in B that the above integral is a constant, that is (see [8])

$$Tf_N(z) = C(1-|z|^2)^a.$$

Since $Tf_N(z)$ belongs to $L_{p,q}(\varphi)$, we have

$$\begin{aligned}
\infty > \|Tf_N\|_{p,q,\varphi}^p &= \|C(1-|z|^2)^a\|_{p,q,\varphi}^p \\
&= C\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)M_q^p(r, (1-|z|^2)^a)\,\mathrm{d}r \\
&= C\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)\Big[\Big[\int_S (1-r^2)^{aq}\,\mathrm{d}\sigma(\zeta)\Big]^{1/q}\Big]^p\,\mathrm{d}r \\
&= C\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)(1-r^2)^{ap}\,\mathrm{d}r \\
&\geqslant C\int_0^1 r^{2n-1}(1-r)^{ap-1}\varphi^p(r)\,\mathrm{d}r \\
&\geqslant C\int_\varepsilon^1 (1-r)^{ap-1+pt}\,\mathrm{d}r \quad (r_0 < \varepsilon < 1),
\end{aligned}$$

from which we obtain $a > -t$.

If $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then by Lemma 5, we see that the boundedness of T on $L_{p,q}$ is equivalent to the boundedness of the adjoint operator T^* on $L_{p',q'}(\varphi^{p/p'})$. Moreover,

$$T^* f(z) = \frac{(1-|z|^2)^{b+1}}{\varphi^p(|z|)} \int_B \frac{(1-|w|^2)^{a-1}\varphi^p(|w|)f(w)}{(1-\langle z, w\rangle)^c} \, \mathrm{d}v(w).$$

Since $g_N(z) = (1-|z|^2)^N/\varphi^p(|z|) \in L_{p',q'}(\varphi^{p/p'})$ for a sufficiently large positive number N (see [8]), we get

$$\begin{aligned}
\infty > \|T^* g_N\|^{p'}_{p',q',\varphi^{p/p'}} &\geqslant C \int_0^1 r^{2n-1}(1-r)^{-1}\varphi^{p/p'\cdot p'}(r) M^{p'}_{q'}(r, T^* g_N) \, \mathrm{d}r \\
&= C \int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r) \biggl\{ \biggl[\int_S \biggl| \frac{(1-r^2)^{(b+1)}}{\varphi^p(r)} \biggr|^{q'} \mathrm{d}\sigma(\xi) \biggr]^{1/q'} \biggr\}^{p'} \mathrm{d}r \\
&= C \int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)(1-r)^{p'(b+1)}\varphi^{-pp'}(r) \, \mathrm{d}r \\
&= C \int_0^1 r^{2n-1}(1-r)^{-1}(1-r)^{p'(b+1)}\varphi^{-p'}(r) \, \mathrm{d}r \\
&\geqslant C \int_\varepsilon^1 (1-r)^{p'(b+1)-1-p's} \, \mathrm{d}r \quad (r_0 < \varepsilon < 1).
\end{aligned}$$

From the above we see that $b > s - 1$.

When $p = 1$, with the notation of [1], write

$$\begin{aligned}
\|f\|^p_{p,q,\varphi} &= \biggl\{ \int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r) M^p_q(r, f) \, \mathrm{d}r \biggr\}^{1/p} \\
&= \bigl\| \|f_r\|_{L^q(S,\,\mathrm{d}\sigma)} \bigr\|_{L^p(I,\,\mathrm{d}\mu)},
\end{aligned}$$

where $f_r(z) = f(rz)$, $I = [0, 1]$, $\mathrm{d}\mu = r^{2n-1}(1-r)^{-1}\varphi^p(r) \, \mathrm{d}r$.

Using the above notation and by [1], the norm of $(L_{p,q}(\varphi))^*$ is given by (see [8])

$$\bigl\| \| \cdot \|_{L^{q'}(S,\,\mathrm{d}\sigma)} \bigr\|_{L^\infty(I,\,\mathrm{d}\mu)}.$$

From Lemma 4 we see that the pairing is given by

$$(f, g) = \int_I \biggl[\int_S f(r\zeta)\overline{g(r\zeta)} \, \mathrm{d}\sigma(\zeta) \biggr] \mathrm{d}\mu = \frac{1}{2n} \int_B f(z)\overline{g(z)}(1-|z|)^{-1}\varphi(|z|) \, \mathrm{d}v(z).$$

Hence the adjoint operator T^* is given by

$$T^* f(z) = \frac{(1-|z|^2)^{b+1}}{\varphi(|z|)} \int_B \frac{(1-|w|^2)^{a-1}\varphi(|w|)f(w)}{(1-\langle z, w\rangle)^c} \, \mathrm{d}v(w).$$

Since T^* is bounded on $(L_{1,q}(\varphi))^*$, we can get a function $T^* g_N$ by applying T^* to a bounded function of the form

$$g_N(z) = \frac{(1-|z|^2)^N}{\varphi(|z|)} \in (L_{1,q}(\varphi))^*,$$

where N is a sufficiently large positive number. It follows that

$$\infty > \big\| \|(T^* g_N)_r\|_{L^{q'}(S,\,\mathrm{d}\sigma)} \big\|_{L^\infty(I,\,\mathrm{d}\mu)} = \sup_{0<r<1} C \frac{(1-r^2)^{b+1}}{\varphi(r)}. \tag{7}$$

From (7), by the condition $\lim\limits_{r\to 1} \varphi(r)/(1-r)^s = 0$ we get $b > s-1$, as desired. □

Remark. In fact, we have proved that under the hypothesis of Theorem 4, if $S\colon L_{p,q}(\varphi) \to L_{p,q}(\varphi)$ is bounded and $c > n$, then $a > -t$, $b > s-1$.

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