Czechoslovak Mathematical Journal

Zhiming Gao; Yichen Ma; Hong Wei Zhuang

Shape Hessian for generalized Oseen flow by differentiability of a minimax: A Lagrangian approach

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 3, 987–1011

Persistent URL: http://dml.cz/dmlcz/128221

Czechoslovak Mathematical Journal, 57 (132) (2007), 987–1011

SHAPE HESSIAN FOR GENERALIZED OSEEN FLOW BY DIFFERENTIABILITY OF A MINIMAX: A LAGRANGIAN APPROACH

Zhiming Gao, Yichen Ma, Hongwei Zhuang, Shaanxi

(Received September 14, 2005)

Abstract. The goal of this paper is to compute the shape Hessian for a generalized Oseen problem with nonhomogeneous Dirichlet boundary condition by the velocity method. The incompressibility will be treated by penalty approach. The structure of the shape gradient and shape Hessian with respect to the shape of the variable domain for a given cost functional are established by an application of the Lagrangian method with function space embedding technique.

Keywords: shape sensitivity analysis, shape Hessian, Eulerian semiderivative, differentiability of a minimax, Oseen flow

MSC 2000: 76D55, 49Q12, 49K35, 76D07

1. Introduction

In this paper we establish expressions for the shape gradient and Hessian of a general cost functional associated with the penalized Oseen problem by the application of the theorem on differentiability of a saddle point coupling it with the function space embedding technique.

In general, the size of computations for the analysis of a shape sensitivity problem can be quite large. Therefore, it's very important for us to understand the fundamental structure of the shape gradient and shape Hessian in order to simplify the numerical implementation and obtain mathematically meaningful expressions. Moreover, the discrete gradient (or Hessian) in a finite element problem can also be obtained from the continuous gradient (or Hessian) by a suitable choice of the

This work was supported by the National Natural Science Fund of China (No. 10371096) for ZM Gao and YC Ma.

velocity field and we can deal with the parametrized shapes in the same framework (see Delfour et al. [5]).

The use of the theorem on differentiability of the minimax or the saddle point of a Lagrangian with respect to a parameter provides a very efficient and powerful tool to obtain the shape gradient and Hessian without the usual study of the derivative of the state with respect to the variable domain.

To our knowledge, few papers have been concerned with the second variation of a cost functional for linear partial differential equations. N. Fujii [12] used a second order perturbation of the identity along the normal of the boundary for second order elliptic problems in 1986. J. Simon [19] computed the second variation via the first order perturbation of the identity in 1988. A general approach via the velocity method was systematically characterized by Delfour & Zolesio [7], [8], and they computed the shape Hessian for a simple Neumann problem in [7] and a nonhomogeneous Dirichlet problem in [8].

The paper is organized as follows. In Section 2 we recall the definitions of the shape gradient and shape Hessian coupling them with the velocity (or speed) method, and we also give the Hadamard-Zolesio structure theorems.

Section 3 is devoted to the relation between the generalized Oseen problem and the associated penalized problem. We introduce a Lagrangian to avoid the difficulty of the extra boundary constraint (i.e., nonhomogeneous boundary condition), and finally we give a saddle point formulation of the penalized Oseen problem with the use of the above Lagrangian.

In Section 4, first we state a shape optimization problem with penalized Oseen equations as a constraint. We computate the shape gradient of a given cost functional by the theorem on differentiability of a minimax combining it with the function space embedding technique.

The last section is devoted to the computation of the shape Hessian. We give several expressions for the shape Hessian which only involve the state, adjoint state and "the first derivative of the state" without the usual requirement for the second derivative of the state. In addition we also note that the shape Hessian is not symmetric.

Notation. For two tensors $\boldsymbol{A} = (a_{ij})$ and $\boldsymbol{B} = (b_{ij})$, we denote the scalar product $\boldsymbol{A} : \boldsymbol{B} \stackrel{\text{def}}{=} \sum_{i=1}^{N} a_{ij} b_{ij}$.

$(\cdot,\cdot)_D$ denotes the L^2 inner product in $L^2(D)$ (or in $L^2(D)^N$, $L^2(D)^{N\times N}$ if necessary) and by $\|\cdot\|_{l,D}$ and $|\cdot|_{l,D}$ we denote the $H^l(D)$ (or $H^l(D)^N$ if necessary) norm and seminorm, respectively, with the usual convention $H^0(D) = L^2(D)$.

2. Eulerian semiderivatives and structure theorems

In this section we briefly recall the definitions of the shape gradient and shape Hessian based on the velocity method (see J. Céa [3] and J.-P. Zolésio [10], [21]) and the associated Hadamard-Zolesio structure theorem (see [10]).

2.1. Velocity (speed) method. Let $\mathbf{V} \in \mathrm{E}^k \stackrel{\text{def}}{=} C([0,\tau); \mathscr{D}^k(\mathbb{R}^N, \mathbb{R}^N))$, where $\mathscr{D}^k(\mathbb{R}^N, \mathbb{R}^N)$ denotes the space of all k-times continuously differentiable functions with compact support contained in the Euclidean space $\mathbb{R}^N$ and τ is a small positive real number. The velocity field

$$\mathbf{V}(t)(x) = \mathbf{V}(t,x), \qquad x \in \mathbb{R}^N,\ t \geqslant 0$$

belongs to $\mathscr{D}^k(\mathbb{R}^N, \mathbb{R}^N)$ for each t. It can generate a transformation

$$T_t(\mathbf{V})X = x(t,X), \quad t \geqslant 0, \quad X \in \mathbb{R}^N$$

through the dynamical system

$$\begin{cases} \mathrm{d}x/\mathrm{d}t(t,X) = \mathbf{V}(t, x(t,X)), \\ x(0,X) = X \end{cases} \tag{2.1}$$

with the initial value X given. We denote the "transformed domain" $T_t(\mathbf{V})(\Omega)$ at $t \geqslant 0$ by $\Omega_t(\mathbf{V})$.

Furthermore, for sufficiently small $t > 0$, the Jacobian J_t is strictly positive:

$$J_t(x) := \det |\mathrm{D}T_t(x)| = \det \mathrm{D}T_t(x) > 0, \quad (\mathrm{D}T_t)_{ij} = \partial_j T_i \tag{2.2}$$

where $\mathrm{D}T_t(x)$ denotes the Jacobian matrix of the transformation T_t evaluated at a point $x \in \mathbb{R}^N$ associated with the velocity field $\mathbf{V}$. In what follows, we will also use the following notation:

$$\begin{aligned} \mathscr{T}(t) &\stackrel{\text{def}}{=} \mathrm{D}T_t^{-1} = \text{the inverse of the matrix } \mathrm{D}T_t, \\ {}^*\mathscr{T}(t) &= \text{the transpose of the matrix } \mathrm{D}T_t^{-1}. \end{aligned}$$

2.2. Shape gradient.

Definition 2.1. Given a cost functional $J(\Omega)$ defined in some Sobolev spaces,

(i) we say that J has a *Eulerian semiderivative* at Ω in the direction $\mathbf{V}$ if the limit

$$\lim_{t \searrow 0} \frac{J(\Omega_t(\mathbf{V})) - J(\Omega)}{t} \stackrel{\text{def}}{=} \mathrm{d}J(\Omega; \mathbf{V})$$

exists.

(ii) When $\mathrm{d}J(\Omega;\mathbf{V})$ exists for all $\mathbf{V}$ in E^k and the map

$$\mathbf{V} \mapsto \mathrm{d}J(\Omega;\mathbf{V}) : \mathrm{E}^k \to \mathbb{R}$$

is well-defined, linear and continuous, we say that

$$\mathrm{d}J(\Omega;\mathbf{V}) = \mathrm{d}J(\Omega;\mathbf{V}(0))$$

and J is *shape differentiable* at Ω. In the distributional sense we have

$$\mathrm{d}J(\Omega;\mathbf{V}) = \langle \Im, \mathbf{V} \rangle_{\mathscr{D}^k(\mathbb{R}^N,\mathbb{R}^N)' \times \mathscr{D}^k(\mathbb{R}^N,\mathbb{R}^N)}, \tag{2.3}$$

and we say that $\Im$ is the k-th order *shape gradient* of J at Ω.

The following theorem is known as the so-called *Hadamard-Zolesio structure theorem.*

Theorem 2.1. *Assume that the boundary Γ of Ω is of class C^{k+1} for an integer $k \geqslant 0$. Then there exists a scalar distribution g in $\mathscr{D}^k(\Gamma)'$ such that*

$$\mathrm{d}J(\Omega;\mathbf{V}) = \langle \Im, \mathbf{V} \rangle = \langle g, \mathbf{V}_n \rangle_{\mathscr{D}^k(\Gamma)' \times \mathscr{D}^k(\Gamma)}$$

where $\mathbf{V}_n \stackrel{\text{def}}{=} \mathbf{V} \cdot \mathbf{n}$ is the normal component of $\mathbf{V}$ on Γ.

2.3. Shape Hessian.

Let $\mathbf{V}$ and $\mathbf{W}$ be two time-independent vector fields, i.e.,

$$\mathbf{V}, \mathbf{W} \in \mathrm{E}_k \stackrel{\text{def}}{=} \mathscr{D}^k(\mathbb{R}^N, \mathbb{R}^N)$$

do not depend on $t \geqslant 0$. As in Section 2.1, we associate $\mathbf{V}$ and $\mathbf{W}$ with the transformations $T_t(\mathbf{V})$ and $T_t(\mathbf{W})$ and the transformed domains $\Omega_t(\mathbf{V})$ and $\Omega_t(\mathbf{W})$. We have the following definition.

Definition 2.2. Assume that the first Eulerian semi-derivative $\mathrm{d}J(\Omega_t(\mathbf{W});\mathbf{V})$ exists in some neighborhood of $t = 0$.

(i) We say that $J(\Omega)$ has the *second order Eulerian semiderivative* at Ω in the directions $(\mathbf{V}, \mathbf{W})$ if the limit

$$\lim_{t \searrow 0} \frac{\mathrm{d}J(\Omega_t(\mathbf{W});\mathbf{V}) - \mathrm{d}J(\Omega;\mathbf{V})}{t} \tag{2.4}$$

exists. When it exists, it is denoted by $\mathrm{d}^2 J(\Omega;\mathbf{V};\mathbf{W})$.

(ii) When the mapping

$$(\mathbf{V},\mathbf{W}) \to \mathrm{d}^2 J(\Omega;\mathbf{V};\mathbf{W})\colon\ \mathrm{E}_k \times \mathrm{E}_k \to \mathbb{R} \tag{2.5}$$

is well-defined, bilinear and continuous with the Fréchet space topology on E_k, we say that J is *twice shape differentiable* and the map (2.5) is denoted by h.

(iii) We denote by $H(\Omega)$ the vector distribution in $(\mathrm{E}_k \otimes \mathrm{E}_k)'$ associated with h:

$$\mathrm{d}^2 J(\Omega;\mathbf{V};\mathbf{W}) = \langle H(\Omega), \mathbf{V}\otimes\mathbf{W}\rangle = h(\mathbf{V},\mathbf{W}),$$

where $\mathbf{V}\otimes\mathbf{W}$ is the tensor product of $\mathbf{V} = (\mathbf{V}_i)$ and $\mathbf{W} = (\mathbf{W}_j)$ defined as

$$(\mathbf{V}\otimes\mathbf{W})_{ij}(x,y) = \mathbf{V}_i(x)\mathbf{W}_j(y), \qquad 1 \leqslant i,j \leqslant N.$$

$H(\Omega)$ will be called the k-order *shape Hessian* of J at Ω.

Next we give an equivalent form of the Hadamard-Zolesio structure theorem for $\mathrm{d}^2 J(\Omega;\mathbf{V};\mathbf{W})$.

Theorem 2.2. *Let Ω be a domain in $\mathbb{R}^N$ with boundary Γ and assume the functional J is twice shape differentiable at Ω.*

(i) *$H(\Omega)$ has support in $\Gamma\times\Gamma$. Moreover, the support of $H(\Omega)$ is compact when its order is finite.*

(ii) *If $H(\Omega)$ is of finite order $k \geqslant 0$ and the boundary Γ is of class C^{k+1}, then there exists a linear and continuous vector distribution $h(\Gamma\otimes\Gamma)$ on $\mathscr{D}^k(\Gamma,\mathbb{R}^N)\otimes\mathscr{D}^k(\Gamma)$ of order k such that for all $\mathbf{V}$ and $\mathbf{W}$ in E_k,*

$$\mathrm{d}^2 J(\Omega;\mathbf{V};\mathbf{W}) = \langle h(\Gamma\otimes\Gamma), (\gamma_\Gamma \mathbf{V})\otimes((\gamma_\Gamma\mathbf{W})\cdot\mathbf{n})\rangle,$$

where γ_Γ denotes the trace operator on the boundary Γ.

Remark 2.1. We can find that in general, the shape Hessian is not symmetric:

$$\exists\,\mathbf{V}\text{and } \mathbf{W} \text{ in } \mathrm{E}_k,\ \ \mathrm{d}^2 J(\Omega;\mathbf{V};\mathbf{W}) \neq \mathrm{d}^2 J(\Omega;\mathbf{W};\mathbf{V}).$$

3. A saddle point formulation

Let Ω be a fluid domain in $\mathbb{R}^N$ ($N = 2$ or 3), and let its boundary $\Gamma := \partial\Omega$ be smooth. The fluid is described by its velocity $\mathbf{u}$ and pressure π satisfying the generalized Oseen problem:

$$\begin{cases} \sigma\mathbf{u} - \alpha\Delta\mathbf{u} + \mathrm{D}\mathbf{u}\cdot\mathbf{w} + \nabla\pi = \mathbf{f} & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{g} & \text{on } \Gamma \end{cases} \tag{3.1}$$

with σ is a positive real number, α stands for the kinematic viscosity coefficient, and $\mathbf{w}\colon \Omega \to \mathbb{R}^N$ is a vectorial function such that $\operatorname{div}\mathbf{w} = 0$ in Ω.

In order to eliminate the incompressibility, we introduce

$$\begin{cases} \mathscr{L}\mathbf{y} + \mathrm{D}\mathbf{y}\cdot\mathbf{w} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{y} = \mathbf{g} & \text{on } \Gamma \end{cases} \tag{3.2}$$

where the differential operator $\mathscr{L}\colon \mathscr{L}\varphi \overset{\text{def}}{=} \sigma\varphi - \alpha\Delta\varphi - \beta\nabla\operatorname{div}\varphi$ and the penalty parameter $\beta > 0$. An interpretation of this is connected with the infinite-dimensional optimization theory and calculus of variations. If we consider the equation $\operatorname{div}\mathbf{u} = 0$ as a constraint then π appears as the Lagrange multiplier associated with this constraint (see [11]), and it is natural from the point of view of the calculus of variations to introduce the penalized form of the problem: to minimize $1/2(|\mathbf{y}|^2 + \alpha|\mathrm{D}\mathbf{y}|^2 + \beta|\operatorname{div}\mathbf{y}|^2) - (\mathbf{f},\mathbf{y})_\Omega$ among the functions $\mathbf{y}$ the *Euler-Lagrange equation* of this problem is (3.2).

The relation between $\mathbf{u}$ and $\mathbf{y}$ is given in the following theorem.

Theorem 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Assume that $\mathbf{f} \in L^2(\Omega)^N$, $\mathbf{g} \in H^{1/2}(\Gamma)^N$ and $\mathbf{w} \in H^1_{\operatorname{div}}(\mathbb{R}^N) \overset{\text{def}}{=} \{\mathbf{u} \in H^1(\mathbb{R}^N)^N\colon \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega\}$. For $\beta > 0$ fixed, there exists a unique $\mathbf{y} \in H^1(\Omega)^N$ which satisfies (3.2). When $\beta \to +\infty$, then*

$$\mathbf{y} \to \mathbf{u} \qquad \textit{strongly in } H^1(\Omega)^N, \tag{3.3}$$

$$-\beta\operatorname{div}\mathbf{y} \to \pi, \qquad \textit{strongly in } L^2(\Omega), \tag{3.4}$$

where $\mathbf{u}$ and π are defined by (3.1) and moreover

$$(\pi, 1)_\Omega = (\mathbf{g}, \mathbf{n})_\Omega = 0.$$

P r o o f. It is easy to find that the variational form of problem (3.2) is

$$\text{To find}\quad \mathbf{y}\in H^1(\Omega)^N \text{ such that } a(\Omega;\mathbf{y},\psi)=(\mathbf{f},\psi)_\Omega \qquad \forall\psi\in H_0^1(\Omega)^N, \tag{3.5}$$

where the bilinear form is defined by

$$a(\Omega;\mathbf{u},\psi)\stackrel{\text{def}}{=}\int_\Omega [\sigma\langle\mathbf{u},\psi\rangle+\alpha\mathrm{D}\mathbf{u}\colon \mathrm{D}\psi+\beta\operatorname{div}\mathbf{u}\operatorname{div}\psi+\langle\mathrm{D}\mathbf{u}\cdot\mathbf{w},\psi\rangle]\,dx.$$

The existence and uniqueness of $\mathbf{y}$ satisfying (3.5) result from the application of the Lax-Milgram lemma (see Gilbarg & Trudinger[13]).

Now (3.3), (3.1) and (3.2) yield

$$\sigma\hat{\mathbf{u}}-\alpha\Delta\hat{\mathbf{u}}+\mathrm{D}\hat{\mathbf{u}}\cdot\mathbf{w}+\beta\nabla\operatorname{div}\mathbf{y}=-\nabla\pi \tag{3.6}$$

where $\hat{\mathbf{u}}\stackrel{\text{def}}{=}\mathbf{u}-\mathbf{y}$ and on the boundary Γ we have $\hat{\mathbf{u}}=0$.

Thus we obtain

$$a(\Omega;\hat{\mathbf{u}},\varphi)=(\pi,\operatorname{div}\varphi)_\Omega,\qquad \forall\varphi\in H_0^1(\Omega)^N.$$

Now let $\varphi=\hat{\mathbf{u}}$,

$$\begin{aligned}\sigma\|\hat{\mathbf{u}}\|_{0,\Omega}^2&+\alpha\|\mathrm{D}\hat{\mathbf{u}}\|_{0,\Omega}^2+(\mathrm{D}\hat{\mathbf{u}}\cdot\mathbf{w},\hat{\mathbf{u}})_\Omega+\beta\|\operatorname{div}\hat{\mathbf{u}}\|_{0,\Omega}^2\\&=(\pi,\operatorname{div}\hat{\mathbf{u}})_\Omega\leqslant\|\pi\|_{0,\Omega}\|\operatorname{div}\hat{\mathbf{u}}\|_{0,\Omega}\leqslant\frac{\beta}{2}\|\operatorname{div}\hat{\mathbf{u}}\|_{0,\Omega}^2+\frac{1}{2\beta}\|\pi\|_{0,\Omega}^2.\end{aligned}$$

In addition, since $\operatorname{div}\mathbf{w}=0$ in Ω and $\hat{\mathbf{u}}=0$ on Γ,

$$(\mathrm{D}\hat{\mathbf{u}}\cdot\mathbf{w},\hat{\mathbf{u}})_\Omega=-\tfrac{1}{2}(\operatorname{div}\mathbf{w},|\hat{\mathbf{u}}|^2)_\Omega\geqslant 0.$$

Thus we obtain

$$2\sigma\|\hat{\mathbf{u}}\|_{0,\Omega}^2+2\alpha\|\mathrm{D}\hat{\mathbf{u}}\|_{0,\Omega}^2+\beta\|\operatorname{div}\hat{\mathbf{u}}\|_{0,\Omega}^2-(\operatorname{div}\mathbf{w},|\hat{\mathbf{u}}|^2)_\Omega\leqslant\frac{1}{\beta}\|\pi\|_{0,\Omega}^2.$$

This proves that $\mathbf{y}$ converges to $\mathbf{u}$ in the strong topology of $H^1(\Omega)^N$. Consequently, (3.6) shows that

$$\nabla(\beta\operatorname{div}\mathbf{y})\to-\nabla\pi$$

strongly in $H^{-1}(\Omega)^N$ since $\Delta(\mathbf{y}-\mathbf{u})$ strongly converges to zero in $H^{-1}(\Omega)^N$ because of (3.3).

According to the next lemma, there exists a constant $C=C(\Omega)$ depending only on Ω such that

$$\|\pi+\beta\operatorname{div}\mathbf{y}\|_{0,\Omega}\leqslant[|(\pi+\beta\operatorname{div}\mathbf{y},1)_\Omega|+\|\nabla(\pi+\beta\operatorname{div}\mathbf{y})\|_{-1,\Omega}],\ \text{and}$$

since $(\pi,1)_\Omega=(\mathbf{g},\mathbf{n})_\Omega=0$, we obtain $(\pi+\beta\operatorname{div}\mathbf{y},1)_\Omega=0$.

Finally, the convergence (3.4) is obtained. □

Lemma 3.1 [20]. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Then there exists a constant $c = c(\Omega)$ depending only on Ω such that*

$$|w|_{0,\Omega} \leqslant c(\Omega)\Big\{|(w,1)_\Omega| + \sum_{i=1}^{N} \Big\|\frac{\partial w}{\partial x_i}\Big\|_{-1,\Omega}\Big\}, \qquad \forall\, w \in L^2(\Omega).$$

In what follows, we assume that $\mathbf{f} \in H^1(\mathbb{R}^N)^N$ and $\mathbf{g} \in H^{5/2}(\mathbb{R}^N)^N$. We know that the solution $\mathbf{y}$ belongs to $H^1(\Omega)^N$ and even to $H^3(\Omega)^N$ when Γ is of class C^3 by the regularity theorem (see Gilbarg & Trudinger [13]).

Now we can say that $\mathbf{y} \in H^2(\Omega)^N$ solves the weak form,

$$\int_\Omega \langle \mathscr{L}\mathbf{y} + \mathrm{D}\mathbf{y}\cdot\mathbf{w} - \mathbf{f}, \psi\rangle\,\mathrm{d}x - \int_\Gamma \langle \mathbf{y} - \mathbf{g}, \mu\rangle\,\mathrm{d}\Gamma = 0$$

for all $\psi \in H^2(\Omega)^N$ and $\mu \in H^{-1/2}(\Gamma)^N$, since the corresponding Lagrange functional is

$$L(\Omega, \varphi, \psi, \mu) \stackrel{\text{def}}{=} \int_\Omega \langle \mathscr{L}\varphi + \mathrm{D}\varphi\cdot\mathbf{w} - \mathbf{f}, \psi\rangle\,\mathrm{d}x - \int_\Gamma \langle \varphi - \mathbf{g}, \mu\rangle\,\mathrm{d}\Gamma,$$

its saddle point $(\mathbf{y}, \mathbf{p}, \mu)$ being characterized by

State equations.

$$\text{(3.7a)} \qquad \begin{cases} \mathscr{L}\mathbf{y} + \mathrm{D}\mathbf{y}\cdot\mathbf{w} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{y} = \mathbf{g} & \text{on } \Gamma. \end{cases}$$

Adjoint state equations.

$$\text{(3.7b)} \qquad \begin{cases} \mathscr{L}\mathbf{p} - \mathrm{D}\mathbf{p}\cdot\mathbf{w} = 0 & \text{in } \Omega, \\ \mathbf{p} = 0 & \text{on } \Gamma. \end{cases}$$

Lagrangian multiplier.

$$\mu = \mathscr{B}\psi \stackrel{\text{def}}{=} \alpha \mathrm{D}\psi\mathbf{n} + \beta\,\mathrm{div}\,\psi\mathbf{n}.$$

Now we can rewrite the functional $L(\Omega, \varphi, \psi, \mu)$ as a functional of $(\varphi, \psi) \in H^2(\Omega)^N \times H^2(\Omega)^N$:

$$L(\Omega, \varphi, \psi) = \int_\Omega \langle \mathscr{L}\varphi + \mathrm{D}\varphi\cdot\mathbf{w} - \mathbf{f}, \psi\rangle\,\mathrm{d}x - \int_\Gamma \langle \varphi - \mathbf{g}, \mathscr{B}\psi\rangle\,\mathrm{d}\Gamma.$$

Note that since the boundary is sufficiently smooth, the following identities are derived by the Green formula:

$$\int_\Gamma \langle \varphi - \mathbf{g}, \mathrm{D}\psi \mathbf{n}\rangle \,\mathrm{d}\Gamma = \int_\Omega [\langle \varphi - \mathbf{g}, \Delta\psi\rangle + \mathrm{D}(\varphi - \mathbf{g})\colon\ \mathrm{D}\psi]\,\mathrm{d}x;$$
$$\int_\Gamma \langle \varphi - \mathbf{g},\, \operatorname{div}\psi\, \mathbf{n}\rangle \,\mathrm{d}\Gamma = \int_\Omega [\operatorname{div}(\varphi - \mathbf{g}) \operatorname{div}\psi + \langle \varphi - \mathbf{g}, \nabla \operatorname{div}\psi\rangle]\,\mathrm{d}x.$$

Finally we obtain a new Lagrangian

$$\begin{aligned} L(\Omega, \varphi, \psi) = {} & \int_\Omega \langle \mathscr{L}\varphi + \mathrm{D}\varphi\cdot\mathbf{w} - \mathbf{f}, \psi\rangle\,\mathrm{d}x \\ & - \alpha \int_\Omega [\langle \varphi - \mathbf{g}, \Delta\psi\rangle + \mathrm{D}(\varphi - \mathbf{g})\colon\ \mathrm{D}\psi]\,\mathrm{d}x \\ & - \beta \int_\Omega [\operatorname{div}(\varphi - \mathbf{g}) \operatorname{div}\psi + \langle \varphi - \mathbf{g}, \nabla \operatorname{div}\psi\rangle]\,\mathrm{d}x. \end{aligned} \tag{3.8}$$

This domain (volume) integral is advantageous for the computation of the shape gradient and the shape Hessian below.

4. Shape gradient by minimax differentiability

4.1. Function space embedding. Consider the cost functional

$$J(\Omega) \stackrel{\text{def}}{=} \frac{1}{2}\int_\Omega |\mathbf{y}(\Omega) - \mathbf{y}_d|^2\,\mathrm{d}x$$

associated with the solution $\mathbf{y} = \mathbf{y}(\Omega)$ of the Oseen problem (3.2) where $\mathbf{y}_d$ is fixed in $H^1(\mathbb{R}^N)^N$ and given by the designer for some purposes.

Now we can introduce a new Lagrangian

$$G(\Omega, \varphi, \psi) = \frac{1}{2}\int_\Omega |\varphi - \mathbf{y}_d|^2\,\mathrm{d}x + L(\Omega, \varphi, \psi)$$

on $H^2(\Omega)^N \times H^2(\Omega)^N$. It is easy to show that this Lagrangian has a unique saddle point $(\mathbf{y}, \mathbf{p})$ which is given by the systems

State equations.

$$\begin{cases} \mathscr{L}\mathbf{y} + \mathrm{D}\mathbf{y}\cdot\mathbf{w} = \mathbf{f} & \text{in } \Omega, \\ \mathbf{y} = \mathbf{g} & \text{on } \Gamma. \end{cases} \tag{4.1a}$$

Adjoint state equations.

$$\text{(4.1b)} \qquad \begin{cases} \mathscr{L}\mathbf{p} - \mathrm{D}\mathbf{p} \cdot \mathbf{w} = \mathbf{y}_d - \mathbf{y} & \text{in } \Omega, \\ \mathbf{p} = 0 & \text{on } \Gamma. \end{cases}$$

Naturally, the cost functional is given by

$$J(\Omega) = \inf_{\varphi \in H^2(\Omega)^N} \sup_{\psi \in H^2(\Omega)^N} G(\Omega, \varphi, \psi).$$

Now we shall use the above Lagrangian formulation coupling it with the velocity method (see [10], [22]) to compute the shape gradient of $J(\Omega)$. Recall that the initial domain Ω is perturbed by a velocity field $\mathbf{V}$ which generates the transformations

$$T_t \colon \mathbb{R}^N \to \mathbb{R}^N, \qquad T_t(X) = x(t, X)$$

and the transformed domain $\Omega_t(\mathbf{V}) = T_t(\mathbf{V})(\Omega)$.

It is readily seen that under the action of $\mathbf{V}$,

$$\text{(4.2)} \qquad J(\Omega_t) = \inf_{\varphi \in H^2(\Omega_t)^N} \sup_{\psi \in H^2(\Omega_t)^N} G(\Omega_t, \varphi, \psi),$$

where the saddle point $(\mathbf{y}_t, \mathbf{p}_t) \in H^2(\Omega_t)^N \times H^2(\Omega_t)^N$ is characterized by the previous saddle point equations (4.1a) and (4.1b) over Ω_t,

State equations.

$$\text{(4.3a)} \qquad \begin{cases} \mathscr{L}\mathbf{y}_t + \mathrm{D}\mathbf{y}_t \cdot \mathbf{w} = \mathbf{f} & \text{in } \Omega_t, \\ \mathbf{y}_t = \mathbf{g} & \text{on } \Gamma_t \overset{\text{def}}{=} \partial\Omega_t. \end{cases}$$

Adjoint state equations.

$$\text{(4.3b)} \qquad \begin{cases} \mathscr{L}\mathbf{p}_t - \mathrm{D}\mathbf{p}_t \cdot \mathbf{w} = \mathbf{y}_d - \mathbf{y}_t & \text{in } \Omega_t, \\ \mathbf{p}_t = 0 & \text{on } \Gamma_t. \end{cases}$$

We are looking for a theorem that would give an expression for the derivative of an inf-sup with respect to the parameter t. However, the space in (4.2) depends on the parameter t. There are two ways to get rid of this time dependence (see Delfour & Zolésio [10], [6], [7], [8]):

- ◇ *Function Space Parametrization* technique;
- ◇ *Function Space Embedding* technique.

In the first case, we can parametrize the functions in $H^m(\Omega_t)^N$ by elements of $H^m(\Omega)^N$ through the transformation

$$\varphi \mapsto \varphi \circ T_t^{-1}\colon \quad H^m(\Omega)^N \to H^m(\Omega_t)^N.$$

Thus we have a new Lagrangian $G(\Omega_t, \varphi \circ T_t^{-1}, \psi \circ T_t^{-1})$ on $H^2(\Omega)^N \times H^2(\Omega)^N$.

We shall use the latter technique. Since $\mathbb{R}^N$ contains the set of transformations $\{\Omega_t\colon\ t \in [0,\tau]\}$ of Ω for some small $t > 0$, we have

$$J(\Omega_t) = \inf_{\mathbf{\Phi} \in H^2(\mathbb{R}^N)^N} \sup_{\mathbf{\Psi} \in H^2(\mathbb{R}^N)^N} G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi}), \tag{4.4}$$

where

$$G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi}) = \frac{1}{2}\int_{\Omega_t} |\mathbf{\Phi} - \mathbf{y}_d|^2 \,\mathrm{d}x + L(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi})$$

and the restrictions of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ on Ω_t are $\mathbf{y}_t$ and $\mathbf{p}_t$, respectively.

Our next objective is to find an expression for the limit

$$\lim_{t \searrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}$$

where $J(\Omega_t)$ is given by (4.4).

4.2. A theorem on differentiability of a minimax. In this section, we first introduce a theorem concerning the differentiability of a saddle point (or a minimax) with respect to a parameter, then we apply it to our case and obtain the shape gradient of the given cost functional J.

Define a functional

$$\mathscr{G}\colon\ [0,\tau] \times X \times Y \to \mathbb{R}$$

where $\tau > 0$ and X, Y are two topological spaces.

For any $t \in [0,\tau]$, define

$$g(t) = \inf_{x \in X} \sup_{y \in Y} \mathscr{G}(t,x,y)$$

and the sets

$$X(t) = \{x^t \in X\colon\ g(t) = \sup_{y \in Y} \mathscr{G}(t,x^t,y)\},$$
$$Y(t,x) = \{y^t \in Y\colon\ \mathscr{G}(t,x,y^t) = \sup_{y \in Y} \mathscr{G}(t,x,y)\}.$$

Similarly, we can define dual functionals

$$h(t) = \sup_{y \in Y} \inf_{x \in X} \mathscr{G}(t, x, y)$$

and the corresponding sets

$$Y(t) = \{y^t \in Y\colon\ h(t) = \inf_{x \in X} \mathscr{G}(t, x, y^t)\},$$
$$X(t, y) = \{x^t \in X\colon\ \mathscr{G}(t, x^t, y) = \inf_{x \in X} \mathscr{G}(t, x, y)\}.$$

Furthermore, we introduce the set of saddle points

$$S(t) = \{(x, y) \in X \times Y\colon\ g(t) = \mathscr{G}(t, x, y) = h(t)\}.$$

Now we can introduce the following theorem (see [4] or page 427 of [10]):

Theorem 4.1. *Assume that the following hypotheses hold:*

(H1) $S(t) \neq \emptyset$, $t \in [0, \tau]$;

(H2) *the partial derivative* $\partial_t \mathscr{G}(t, x, y)$ *exists in* $[0, \tau]$ *for all*

$$(x, y) \in \Big[\bigcup_{t \in [0,\tau]} X(t) \times Y(0)\Big] \bigcup \Big[X(0) \times \bigcup_{t \in [0,\tau]} Y(t)\Big];$$

(H3) *there exists a topology* $\mathscr{T}_X$ *on* X *such that for any sequence* $\{t_n\colon\ t_n \in [0, \tau]\}$ *with* $\lim_{n \nearrow \infty} t_n = 0$ *there exists* $x^0 \in X(0)$ *and a subsequence* $\{t_{n_k}\}$, *and for each* $k \geqslant 1$ *there exists* $x_{n_k} \in X(t_{n_k})$ *such that*

(i) $\lim_{n \nearrow \infty} x_{n_k} = x^0$ *in the* $\mathscr{T}_X$ *topology,*

(ii)

$$\liminf_{\substack{t \searrow 0 \\ k \nearrow \infty}} \partial_t \mathscr{G}(t, x_{n_k}, y) \geqslant \partial_t \mathscr{G}(0, x^0, y), \quad \forall y \in Y(0);$$

(H4) *there exists a topology* $\mathscr{T}_Y$ *on* Y *such that for any sequence* $\{t_n\colon\ t_n \in [0, \tau]\}$ *with* $\lim_{n \nearrow \infty} t_n = 0$ *there exists* $y^0 \in Y(0)$ *and a subsequence* $\{t_{n_k}\}$, *and for each* $k \geqslant 1$ *there exists* $y_{n_k} \in Y(t_{n_k})$ *such that*

(i) $\lim_{n \nearrow \infty} y_{n_k} = y^0$ *in the* $\mathscr{T}_Y$ *topology,*

(ii)

$$\limsup_{\substack{t \searrow 0 \\ k \nearrow \infty}} \partial_t \mathscr{G}(t, x, y_{n_k}) \leqslant \partial_t \mathscr{G}(0, x, y^0), \quad \forall x \in X(0).$$

Then there exists $(x^0, y^0) \in X(0) \times Y(0)$ *such that*

$$\begin{aligned} \mathrm{d}g(0) &= \lim_{t \searrow 0} \frac{g(t) - g(0)}{t} \\ &= \inf_{x \in X(0)} \sup_{y \in Y(0)} \partial_t \mathscr{G}(0, x, y) = \partial_t \mathscr{G}(0, x^0, y^0) = \sup_{y \in Y(0)} \inf_{x \in X(0)} \partial_t \mathscr{G}(0, x, y). \end{aligned}$$

This means that $(x^0, y^0) \in X(0) \times Y(0)$ *is a saddle point of* $\partial_t \mathscr{G}(0, x, y)$*.*

In our situation, the set of saddle points

$$S(t) = X(t) \times Y(t) \subset H^2(\mathbb{R}^N)^N \times H^2(\mathbb{R}^N)^N$$

is not a singleton since

$$\begin{aligned} X(t) &= \{\mathbf{\Phi} \in H^2(\mathbb{R}^N)^N : \ \mathbf{\Phi}|_{\Omega_t} = \mathbf{y}_t\}, \\ Y(t) &= \{\mathbf{\Psi} \in H^2(\mathbb{R}^N)^N : \ \mathbf{\Psi}|_{\Omega_t} = \mathbf{p}_t\}, \end{aligned}$$

where $(\mathbf{y}_t, \mathbf{p}_t) \in H^2(\Omega_t)^N \times H^2(\Omega_t)^N$ is the unique solution of (4.3a) and (4.3b).

We are able to apply Theorem 4.1 under appropriate assumptions (to be verified in Section 4.3) to obtain

$$\mathrm{d}J(\Omega; \mathbf{V}) = \inf_{\mathbf{\Phi} \in X(0)} \sup_{\mathbf{\Psi} \in Y(0)} \partial_t G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi})\big|_{t=0}. \tag{4.5}$$

Since the sets $X(0)$ and $Y(0)$ have been given, we only have to compute the partial derivative of the Lagrangian

$$G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi}) = \int_{\Omega_t} F(\mathbf{\Phi}, \mathbf{\Psi}) \, \mathrm{d}x \tag{4.6}$$

where

$$\begin{aligned} F(\mathbf{\Phi}, \mathbf{\Psi}) \stackrel{\text{def}}{=} &\langle \mathscr{L}\mathbf{\Phi} + \mathrm{D}\mathbf{\Phi} \cdot \mathbf{w} - \mathbf{f}, \mathbf{\Psi} \rangle - \alpha \left[\langle \mathbf{\Phi} - \mathbf{g}, \Delta \mathbf{\Psi} \rangle + \mathrm{D}(\mathbf{\Phi} - \mathbf{g}) : \ \mathrm{D}\psi \right] \\ &- \beta \left[\operatorname{div} (\mathbf{\Phi} - \mathbf{g}) \operatorname{div} \mathbf{\Psi} + \langle \mathbf{\Phi} - \mathbf{g}, \nabla \operatorname{div} \mathbf{\Psi} \rangle \right] + \frac{1}{2} |\mathbf{\Phi} - \mathbf{y}_d|^2. \end{aligned}$$

If we assume that Ω_t is of class C^3 (at least), then the solutions $\mathbf{y}_t$ and $\mathbf{p}_t$ belong to $H^3(\Omega_t)^N$ since $\mathbf{f}, \mathbf{y}_d \in H^1(\mathbb{R}^N)^N, \mathbf{w} \in H^1_{\operatorname{div}}(\mathbb{R}^N)$, and $\mathbf{g} \in H^{5/2}(\mathbb{R}^N)^N$. Now we consider saddle points $S(t)$ in $H^3(\mathbb{R}^N)^N \times H^3(\mathbb{R}^N)^N$ rather than $H^2(\mathbb{R}^N)^N \times H^2(\mathbb{R}^N)^N$, hence we can use the Hadamard formula [10], [22]

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_t} F(t, x) \, \mathrm{d}x = \int_{\Omega_t} \frac{\partial F}{\partial t}(t, x) \, \mathrm{d}x + \int_{\partial \Omega_t} F(t, x) \langle \mathbf{V}, \mathbf{n}_t \rangle \, \mathrm{d}\Gamma_t \tag{4.7}$$

for a sufficiently smooth functional $F\colon\ [0,\tau]\times\mathbb{R}^N\to\mathbb{R}$. So the partial derivative of

$$\partial_t G(\Omega_t,\mathbf{\Phi},\mathbf{\Psi}) = \int_{\Gamma_t} F(\mathbf{\Phi},\mathbf{\Psi})\langle\mathbf{V},\mathbf{n}_t\rangle\,\mathrm{d}\Gamma_t \tag{4.8}$$

can be derived since $\mathbf{\Phi}$ and $\mathbf{\Psi}$ belong to the space $H^3(\mathbb{R}^N)^N$ rather than $H^2(\mathbb{R}^N)^N$.

We also note that the expression (4.8) is a boundary integral on Γ_t which will not depend on $\mathbf{\Phi}$ and $\mathbf{\Psi}$ outside of $\overline{\Omega}_t$, and the restriction of the elements of $S(0)$ is unique so the inf and sup in (4.5) can be dropped,

$$\begin{aligned}
\mathrm{d}J(\Omega;\mathbf{V}) &= \int_\Gamma F(\mathbf{y},\mathbf{p})\langle\mathbf{V},\mathbf{n}\rangle\,\mathrm{d}\Gamma\\
&= \int_\Gamma \Big\{\frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2 + \langle\mathscr{L}\mathbf{y}+\mathrm{D}\mathbf{y}\cdot\mathbf{w}-\mathbf{f},\mathbf{p}\rangle\\
&\quad -\alpha\left[\langle\mathbf{y}-\mathbf{g},\Delta\mathbf{p}\rangle + \mathrm{D}(\mathbf{y}-\mathbf{g})\colon\ \mathrm{D}\mathbf{p}\right]\\
&\quad -\beta\left[\operatorname{div}(\mathbf{y}-\mathbf{g})\operatorname{div}\mathbf{p} + \langle\mathbf{y}-\mathbf{g},\nabla\operatorname{div}\mathbf{p}\rangle\right]\Big\}\langle\mathbf{V},\mathbf{n}\rangle\,\mathrm{d}\Gamma.
\end{aligned}$$

However, $\mathbf{y}=\mathbf{g}$ and $\mathbf{p}=0$ imply that

$$\mathrm{D}(\mathbf{y}-\mathbf{g})\colon\ \mathrm{D}\mathbf{p} = \langle\mathrm{D}(\mathbf{y}-\mathbf{g})\,\mathbf{n},\mathrm{D}\mathbf{p}\,\mathbf{n}\rangle \quad\text{and}\quad \operatorname{div}(\mathbf{y}-\mathbf{g}) = \langle\mathrm{D}(\mathbf{y}-\mathbf{g})\,\mathbf{n},\mathbf{n}\rangle \text{ on } \Gamma.$$

Finally, we obtain

$$\mathrm{d}J(\Omega;\mathbf{V}) = \int_\Gamma \Big[\frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2 - \langle\mathrm{D}(\mathbf{y}-\mathbf{g})\mathbf{n},\ \alpha\mathrm{D}\mathbf{p}\,\mathbf{n}+\beta\operatorname{div}\mathbf{p}\,\mathbf{n}\rangle\Big]\langle\mathbf{V},\mathbf{n}\rangle\,\mathrm{d}\Gamma. \tag{4.9}$$

This expression for the shape gradient accords with the Hadamard-Zolesio theorem (see Theorem 2.1).

4.3. Verification of the assumptions of theorem 4.1. As we have seen that the computation of the shape gradient is both compact and efficient, we must verify the four assumptions of Theorem 4.1.

First, we can always construct linear and continuous extensions (see Adams [1]):

$$\Pi\colon\ H^3(\Omega)^N\to H^3(\mathbb{R}^N)^N$$

and

$$\Pi_t\colon\ H^3(\Omega_t)^N\to H^3(\mathbb{R}^N)^N.$$

Therefore we can define extensions

$$\mathbf{Y}_t = \Pi_t\mathbf{y}_t \quad\text{and}\quad \mathbf{P}_t = \Pi_t\mathbf{p}_t$$

of $\mathbf{y}_t$ and $\mathbf{p}_t$. So $\mathbf{Y}_t \in X(t)$ and $\mathbf{P}_t \in Y(t)$, and this shows the existence of a saddle point, i.e., $S(t) \neq \emptyset$. Hence (H1) is satisfied.

The next step is to verify (H2). Using the transformation T_t, formula (4.6) can be rewritten as

$$G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi}) = \int_\Omega F(\mathbf{\Phi}, \mathbf{\Psi}) \circ T_t \, J_t \, \mathrm{d}x. \tag{4.10}$$

Furthermore, we can compute its partial derivative for $\mathbf{\Phi}$ and $\mathbf{\Psi}$ in $H^3(\mathbb{R}^N)^N$,

$$\partial_t G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi}) = \int_\Omega [\nabla F(\mathbf{\Phi}, \mathbf{\Psi}) \cdot \mathbf{V}_t + F(\mathbf{\Phi}, \mathbf{\Psi}) \operatorname{div} \mathbf{V}_t] \circ T_t \, J_t \, \mathrm{d}x \tag{4.11}$$

where $\mathbf{V}_t \stackrel{\text{def}}{=} \mathbf{V}(t)$. By the choice of the velocity field $\mathbf{V} \in \mathscr{D}^1(\mathbb{R}^N, \mathbb{R}^N)$, the maps $t \mapsto \mathbf{V}_t$ and $t \mapsto \operatorname{div} \mathbf{V}_t$ are continuous on $[0, \tau]$. Moreover, $\mathbf{f}, \mathbf{y}_d \in H^1(\mathbb{R}^N)^N$, $\mathbf{w} \in H^1_{\mathrm{div}}(\mathbb{R}^N)$ and $\mathbf{g} \in H^{5/2}(\mathbb{R}^N)^N$, hence (4.11) is well-defined and $\partial_t G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi})$ exists everywhere in $[0, \tau]$ for all $\mathbf{\Phi}$ and $\mathbf{\Psi}$ in $H^3(\mathbb{R}^N)^N$. Assumption (H2) is verified.

To check (H3) and (H4), we introduce two basic theorems.

Theorem 4.2. *For the velocity field $\mathbf{V} \in \mathscr{D}^1(\mathbb{R}^N, \mathbb{R}^N)$ and $\mathbf{\Phi} \in L^2(\mathbb{R}^N)^N$,*

$$\lim_{t \searrow 0} \mathbf{\Phi} \circ T_t = \mathbf{\Phi} \quad \textit{and} \quad \lim_{t \searrow 0} \mathbf{\Phi} \circ T_t^{-1} = \mathbf{\Phi} \quad \textit{in } L^2(\mathbb{R}^N)^N. \tag{4.12}$$

Proof. See Delfour & Zolesio [10], [7] for a similar proof. □

Corollary 4.1. *For $k \geqslant 1$, $\mathbf{V}$ in $\mathscr{D}^k(\mathbb{R}^N, \mathbb{R}^N)$ and $\mathbf{\Phi} \in H^k(\mathbb{R}^N)^N$ we have*

$$\lim_{t \searrow 0} \mathbf{\Phi} \circ T_t = \mathbf{\Phi} \quad \textit{and} \quad \lim_{t \searrow 0} \mathbf{\Phi} \circ T_t^{-1} = \mathbf{\Phi} \quad \textit{in } H^k(\mathbb{R}^N)^N \tag{4.13}$$

Proof. See Delfour & Zolesio [10], [7] for a similar proof. □

Theorem 4.3. *Under the hypotheses of Corollary 4.1,*

$$\mathbf{y}^t \to \mathbf{y}^0 \quad \textit{in } H^k(\Omega)^N\textit{-strong (respectively, weak)}$$

implies that

$$\mathbf{Y}_t \to \mathbf{Y}_0 \quad \textit{in } H^k(\mathbb{R}^N)^N\textit{-strong (respectively, weak).}$$

Proof. See Delfour & Zolesio [10], [7] for a similar proof. □

To check (H3) (i) and (H4) (i), we transform $(\mathbf{y}_t, \mathbf{p}_t)$ on Ω_t to $(\mathbf{y}^t, \mathbf{p}^t) = (\mathbf{y}_t \circ T_t, \mathbf{p}_t \circ T_t)$ on Ω. Since $(\mathbf{y}_t, \mathbf{p}_t)$ satisfies (4.3a) and (4.3b), $(\mathbf{y}^t, \mathbf{p}^t)$ is completely characterized by the following variational systems:

State system. $\mathbf{y}^t \in H^1(\Omega)^N, \qquad \forall\, \psi \in H_0^1(\Omega)^N$

$$\sigma \int_\Omega \langle \mathbf{y}^t, \psi \rangle J_t \,\mathrm{d}x + \int_\Omega \mathscr{A}(t)\mathrm{D}\mathbf{y}^t \colon \mathrm{D}\psi J_t \,\mathrm{d}x + \int_\Omega \langle \mathrm{D}\mathbf{y}^t \mathbf{w}, \psi \rangle J_t \,\mathrm{d}x = \int_\Omega \langle \mathbf{f} \circ T_t, \psi \rangle J_t \,\mathrm{d}x \tag{4.16a}$$

Adjoint state system. $\mathbf{p}^t \in H^1(\Omega)^N, \qquad \forall\, \varphi \in H_0^1(\Omega)^N$

$$\sigma \int_\Omega \langle \mathbf{p}^t, \varphi \rangle J_t \,\mathrm{d}x + \int_\Omega \mathscr{A}(t)\mathrm{D}\mathbf{p}^t \colon \mathrm{D}\varphi J_t \,\mathrm{d}x - \int_\Omega \langle \mathrm{D}\mathbf{p}^t \mathbf{w}, \varphi \rangle J_t \,\mathrm{d}x = \int_\Omega \langle \mathbf{y}_d \circ T_t - \mathbf{y}^t, \varphi \rangle J_t \,\mathrm{d}x \tag{4.16b}$$

with the notation

$$\mathscr{A}(t)\tau \colon \sigma \overset{\text{def}}{=} \alpha[\tau(\mathrm{D}T_t)^{-1}] \colon [\sigma(\mathrm{D}T_t)^{-1}] + \beta[\tau \colon {}^*(\mathrm{D}T_t)^{-1}][\sigma \colon {}^*(\mathrm{D}T_t)^{-1}].$$

We now show that $\mathbf{y}^t$ is bounded in $H^1(\Omega)^N$. Assume that the velocity field $\mathbf{V} \in \mathscr{D}^1(\mathbb{R}^N, \mathbb{R}^N)$. Choose $\tau > 0$ small enough such that there exist two constants α_0, β_0 $(0 < \alpha_0 < \beta_0)$,

$$\alpha_0 \leqslant J_t (= |J_t|) \leqslant \beta_0, \ \forall\, t \in [0, \tau].$$

Then taking $\psi = \mathbf{y}^t$, we obtain from (4.16a)

$$\exists\, c > 0, \qquad c\alpha_0 \|\mathbf{y}^t\|_{1,\Omega} \leqslant \beta_0 \|\mathbf{f} \circ T_t\|_{0,\Omega}.$$

Finally, we get

$$\|\mathbf{y}^t\|_{1,\Omega} \leqslant c\|\mathbf{f}\|_{0,\Omega}$$

by Theorem 4.2.

Similarly, taking $\varphi = \mathbf{p}^t$, from (4.16b) we can obtain the boundness of $\mathbf{p}^t$:

$$\|\mathbf{p}^t\|_{1,\Omega} \leqslant c\|\mathbf{y}^t - \mathbf{y}_d\|_{0,\Omega}$$

Now, we can subtract weakly convergent subsequences from $(\mathbf{y}^t, \mathbf{p}^t)$ to some $(\mathbf{z}, \mathbf{q})$ in $H^1(\Omega)^N \times H^1(\Omega)^N$. However, by linearity of the equations with respect to $(\mathbf{y}^t, \mathbf{p}^t)$ and continuity of the coefficients with respect to t, $(\mathbf{z}, \mathbf{q})$ will coincide with $(\mathbf{y}, \mathbf{p})$

since the solution of system (4.1a) and (4.1b) is unique. After that, we go back to the equations for $\mathbf{y}^t$ and $\mathbf{y}$, i.e., (4.1a) and (4.3a). It is readily seen that the convergence is strong in $H^1(\Omega)^N$. Finally by using the regularity of the data and the classical regularity theorem (see [13]), we can show that $(\mathbf{y}^t, \mathbf{p}^t)$ is strongly convergent to $(\mathbf{y}, \mathbf{p})$ in $H^3(\Omega)^N \times H^3(\Omega)^N$. Hence assumptions (H3)(i) and (H4)(i) are satisfied by virtue of Theorem 4.3.

To verify (H3)(ii) and (H4)(ii), we rewrite 4.8 as a domain integral by the Stokes formula,

$$\partial_t G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi}) = \int_{\Omega_t} \operatorname{div}\, [F(\mathbf{\Phi}, \mathbf{\Psi})\mathbf{V}]\, \mathrm{d}x. \tag{4.17}$$

Now the map

$$(\mathbf{\Phi}, \mathbf{\Psi}) \mapsto F(\mathbf{\Phi}, \mathbf{\Psi})\mathbf{V}\colon\ H^3(\mathbb{R}^N)^N \times H^3(\mathbb{R}^N)^N \to H^1(\mathbb{R}^N)^N$$

is bilinear and continuous. Furthermore, by the transformations T_t, the map

$$(t, F(\mathbf{\Phi}, \mathbf{\Psi})\mathbf{V}) \mapsto \int_{\Gamma_t} F(\mathbf{\Phi}, \mathbf{\Psi})\langle \mathbf{V}, \mathbf{n}_t\rangle\, \mathrm{d}\Gamma_t = \int_{\Omega} [\operatorname{div}\,(F(\mathbf{\Phi}, \mathbf{\Psi})\mathbf{V})] \circ T_t\ J_t\, \mathrm{d}x$$

is also continuous. Finally,

$$(t, \mathbf{\Phi}, \mathbf{\Psi}) \mapsto \partial_t G(\Omega_t, \mathbf{\Phi}, \mathbf{\Psi}) = \int_{\Gamma_t} F(\mathbf{\Phi}, \mathbf{\Psi})\langle \mathbf{V}, \mathbf{n}_t\rangle\, \mathrm{d}\Gamma_t$$

is continuous, and hence (H3)(ii) and (H4)(ii) are verified. This completes the verification of the four assumptions of Theorem 4.1.

5. Shape Hessian by minimax differentiability

We proceed as in Section 3 and 4.

5.1. Statement of the problem. For the study of the shape Hessian, we need two time invariant fields $\mathbf{V}$ and $\mathbf{W}$ on $\mathbb{R}^N$. From Section 4, the expression of the first order Eulerian derivative at $s \geqslant 0$ is given by

$$\begin{aligned} &\mathrm{d}J(\Omega_s(\mathbf{W}); \mathbf{V}) \\ &\quad = \int_{\Gamma_s} \Big[\frac{1}{2}|\mathbf{y}_s - \mathbf{y}_d|^2 - \alpha \mathrm{D}(\mathbf{y}_s - \mathbf{g})\colon \mathrm{D}\mathbf{p}_s - \beta \operatorname{div}\,(\mathbf{y}_s - \mathbf{g}) \operatorname{div} \mathbf{p}_s\Big] \langle \mathbf{V}, \mathbf{n}_s\rangle\, \mathrm{d}\Gamma_s \end{aligned}$$

or equivalently,

$$\begin{aligned}(5.1)\ & \mathrm{d}J(\Omega_s(\mathbf{W});\mathbf{V}) \\ &= \int_{\Omega_s} \operatorname{div}\left\{\left[\frac{1}{2}|\mathbf{y}_s-\mathbf{y}_d|^2 - \alpha \mathrm{D}(\mathbf{y}_s-\mathbf{g})\colon\ \mathrm{D}\mathbf{p}_s - \beta \operatorname{div}(\mathbf{y}_s-\mathbf{g}) \operatorname{div}\mathbf{p}_s\right]\mathbf{V}\right\} \mathrm{d}x\end{aligned}$$

where $\Omega_s(\mathbf{W})$ is the perturbation of the domain Ω by the velocity field $\mathbf{W}$ and $(\mathbf{y}_s, \mathbf{p}_s) \in H^3(\Omega_s(\mathbf{W}))^N \times H^3(\Omega_s(\mathbf{W}))^N$ is the unique solution of the system

$$\begin{cases} \mathscr{L}\mathbf{y}_s + \mathrm{D}\mathbf{y}_s \cdot \mathbf{w} = \mathbf{f} & \text{in } \Omega_s(\mathbf{W}), \\ \mathbf{y}_s = \mathbf{g} & \text{on } \Gamma_s(\mathbf{W}) \end{cases} \tag{5.2a}$$

and

$$\begin{cases} \mathscr{L}\mathbf{p}_s - \mathrm{D}\mathbf{p}_s \cdot \mathbf{w} = \mathbf{y}_d - \mathbf{y}_s & \text{in } \Omega_s(\mathbf{W}), \\ \mathbf{p}_s = 0 & \text{on } \Gamma_s(\mathbf{W}). \end{cases} \tag{5.2b}$$

Our objective is to study the differential quotient

$$\lim_{s \searrow 0} \frac{\mathrm{d}J(\Omega_s(\mathbf{W});\mathbf{V}) - \mathrm{d}J(\Omega;\mathbf{V})}{s}. \tag{5.3}$$

5.2. Formal application of the theorem on differentiability of a minimax. We can proceed as in Section 4. (5.1) can be expressed as a minimax over a new Lagrange functional,

$$\mathrm{d}J(\Omega_s(\mathbf{W});\mathbf{V}) = \inf_{\mathbf{\Phi},\mathbf{\Psi}\in H^3(\mathbb{R}^N)^N} \ \sup_{\mathbf{\Theta},\mathbf{\Xi}\in H^2(\mathbb{R}^N)^N} G(\Omega_s, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}, \mathbf{\Xi}), \tag{5.4}$$

where the Lagrange functional is defined by

$$G(\Omega_s, \mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}, \mathbf{\Xi}) \stackrel{\text{def}}{=} \Lambda(\Omega_s, \mathbf{\Phi}, \mathbf{\Psi}) + L(\Omega_s, \mathbf{\Phi}, \mathbf{\Theta}) + M(\Omega_s, \mathbf{\Psi}, \mathbf{\Xi}) \tag{5.5}$$

with the notation

$$\begin{aligned}&\Lambda(\Omega_s, \mathbf{\Phi}, \mathbf{\Psi}) \\ &\quad= \int_{\Omega_s} \operatorname{div}\left\{\left[\frac{1}{2}|\mathbf{\Phi}-\mathbf{y}_d|^2 - \alpha \mathrm{D}(\mathbf{\Phi}-\mathbf{g})\colon\ \mathrm{D}\mathbf{\Psi} - \beta \operatorname{div}(\mathbf{\Phi}-\mathbf{g}) \operatorname{div}\mathbf{\Psi}\right]\mathbf{V}\right\} \mathrm{d}x \\ &\quad M(\Omega_s, \mathbf{\Psi}, \mathbf{\Xi}) = \int_{\Omega_s} \langle \mathscr{L}\mathbf{\Psi} - \mathrm{D}\mathbf{\Psi}\cdot\mathbf{w} + \mathbf{\Phi} - \mathbf{y}_d,\ \mathbf{\Xi}\rangle\, \mathrm{d}x \\ &\quad - \int_{\Omega_s} \{\alpha\,[\langle \mathbf{\Psi}, \Delta\mathbf{\Xi}\rangle + \mathrm{D}\mathbf{\Psi}\colon\ \mathrm{D}\mathbf{\Xi}] + \beta\,[\operatorname{div}\mathbf{\Psi} \operatorname{div}\mathbf{\Xi} + \langle \mathbf{\Psi}, \nabla \operatorname{div}\mathbf{\Xi}\rangle]\}\ \mathrm{d}x\end{aligned}$$

and $L(\Omega_s, \mathbf{\Phi}, \mathbf{\Theta})$ was defined by (3.8).

We can easily find that the functional $[L(\Omega_s,\mathbf{\Phi},\mathbf{\Theta})+M(\Omega_s,\mathbf{\Psi},\mathbf{\Xi})]$ is convex in $(\mathbf{\Phi},\mathbf{\Psi})$ but the cost functional $\Lambda(\Omega_s,\mathbf{\Phi},\mathbf{\Psi})$ is not convex in $(\mathbf{\Phi},\mathbf{\Psi})$ and we shall see that Theorem 4.1 can still be applied to our case of study (to be proved in Section 5.3) provided the sets

$$X(s)\subset H^3(\mathbb{R}^N)^N\times H^3(\mathbb{R}^N)^N,$$
$$Y(s)\subset H^2(\mathbb{R}^N)^N\times H^2(\mathbb{R}^N)^N$$

are be characterized by the following systems:

State system.

$$\text{(5.6)}\quad \int_{\Omega_s}\langle\mathscr{L}\widehat{\mathbf{\Phi}}+\mathrm{D}\widehat{\mathbf{\Phi}}\cdot\mathbf{w}-\mathbf{f},\mathbf{\Theta}\rangle\,\mathrm{d}x-\int_{\Gamma_s}\langle\widehat{\mathbf{\Phi}}-\mathbf{g},\mathscr{B}_s\mathbf{\Theta}\rangle\,\mathrm{d}\Gamma_s=0,\ \forall\mathbf{\Theta}\in H^2(\mathbb{R}^N)^N.$$

$$\text{(5.7)}\quad \int_{\Omega_s}\langle\mathscr{L}\widehat{\mathbf{\Psi}}-\mathrm{D}\widehat{\mathbf{\Psi}}\cdot\mathbf{w}+\widehat{\mathbf{\Phi}}-\mathbf{y}_d,\mathbf{\Xi}\rangle\,\mathrm{d}x-\int_{\Gamma_s}\langle\widehat{\mathbf{\Psi}},\mathscr{B}_s\mathbf{\Xi}\rangle\,\mathrm{d}\Gamma_s=0,\ \forall\mathbf{\Xi}\in H^2(\mathbb{R}^N)^N.$$

Adjoint state system.

$$\begin{aligned}\text{(5.8)}\quad &-\int_{\Omega_s}\operatorname{div}\{[\alpha\mathrm{D}(\widehat{\mathbf{\Phi}}-\mathbf{g})\colon\,\mathrm{D}\mathbf{\Psi}+\beta\operatorname{div}(\widehat{\mathbf{\Phi}}-\mathbf{g})\operatorname{div}\mathbf{\Psi}]\mathbf{V}\}\,\mathrm{d}x\\ &+\int_{\Omega_s}\langle\mathscr{L}\mathbf{\Psi}-\mathrm{D}\mathbf{\Psi}\cdot\mathbf{w},\widehat{\mathbf{\Xi}}\rangle\,\mathrm{d}x-\int_{\Gamma_s}\langle\mathbf{\Psi},\mathscr{B}_s\widehat{\mathbf{\Xi}}\rangle\,\mathrm{d}\Gamma_s=0,\quad\forall\mathbf{\Psi}\in H^2(\mathbb{R}^N)^N,\\ \text{(5.9)}\quad &\int_{\Omega_s}\operatorname{div}\{[(\widehat{\mathbf{\Phi}}-\mathbf{y}_d)\mathbf{\Phi}-\alpha\mathrm{D}\mathbf{\Phi}\colon\,\mathrm{D}\widehat{\mathbf{\Psi}}-\beta\operatorname{div}\mathbf{\Phi}\operatorname{div}\widehat{\mathbf{\Psi}}]\mathbf{V}\}\,\mathrm{d}x+\int_{\Omega_s}\langle\mathbf{\Phi},\widehat{\mathbf{\Xi}}\rangle\,\mathrm{d}x\\ &+\int_{\Omega_s}\langle\mathscr{L}\mathbf{\Phi}+\mathrm{D}\mathbf{\Phi}\cdot\mathbf{w},\widehat{\mathbf{\Theta}}\rangle\,\mathrm{d}x-\int_{\Gamma_s}\langle\mathbf{\Phi},\mathscr{B}_s\widehat{\mathbf{\Theta}}\rangle\,\mathrm{d}\Gamma_s=0,\quad\forall\mathbf{\Phi}\in H^2(\mathbb{R}^N)^N,\end{aligned}$$

where

$$\mathscr{B}_s\varphi\overset{\text{def}}{=}\alpha\mathrm{D}\varphi\,\mathbf{n}_s+\beta\operatorname{div}\varphi\,\mathbf{n}_s;\qquad\mathscr{B}_0\varphi=\mathscr{B}\varphi=\alpha\mathrm{D}\varphi\,\mathbf{n}+\beta\operatorname{div}\varphi\,\mathbf{n}.$$

It is easy to find that (5.6) and (5.7) yield

$$\text{(5.10)}\qquad \widehat{\mathbf{\Phi}}|_{\Omega_s}=\mathbf{y}_s,\qquad\widehat{\mathbf{\Psi}}|_{\Omega_s}=\mathbf{p}_s,$$

where $\mathbf{y}_s$ and $\mathbf{p}_s$ solve (5.2a) and (5.2b), respectively.

Since $(\mathbf{y}_s-\mathbf{g})|_{\Gamma_s}=0$, we have

$$\mathrm{D}(\mathbf{y}_s-\mathbf{g})\colon\,\mathrm{D}\mathbf{\Psi}=\langle\mathrm{D}(\mathbf{y}_s-\mathbf{g})\,\mathbf{n}_s,\mathrm{D}\mathbf{\Psi}\,\mathbf{n}_s\rangle,\ \operatorname{div}(\mathbf{y}_s-\mathbf{g})=\langle\mathrm{D}(\mathbf{y}_s-\mathbf{g})\,\mathbf{n}_s,\mathbf{n}_s\rangle,\ \text{on }\Gamma_s.$$

Hence by Green formula and (5.10), (5.8) reduces to

$$\int_{\Omega_s} \langle \mathscr{L}\widehat{\boldsymbol{\Xi}} + \mathrm{D}\widehat{\boldsymbol{\Xi}} \cdot \mathbf{w}, \boldsymbol{\Psi} \rangle \, \mathrm{d}x - \int_{\Gamma_s} \langle \widehat{\boldsymbol{\Xi}} + \mathbf{V} \cdot \mathbf{n}_s \, \mathrm{D}(\mathbf{y}_s - \mathbf{g})\mathbf{n}_s, \mathscr{B}_s \boldsymbol{\Psi} \rangle \, \mathrm{d}\Gamma_s = 0. \tag{5.11}$$

Similarly, (5.9) reduces to

$$\int_{\Omega_s} \langle \mathscr{L}\widehat{\boldsymbol{\Theta}} - \mathrm{D}\widehat{\boldsymbol{\Theta}} \cdot \mathbf{w} + \widehat{\boldsymbol{\Xi}}, \boldsymbol{\Phi} \rangle \, \mathrm{d}x - \int_{\Gamma_s} \langle \widehat{\boldsymbol{\Theta}} + \mathbf{V} \cdot \mathbf{n}_s \, \mathrm{D}\mathbf{p}_s \, \mathbf{n}_s, \mathscr{B}_s \boldsymbol{\Phi} \rangle \, \mathrm{d}\Gamma_s = 0. \tag{5.12}$$

Therefore, systems (5.8) and (5.9) have solutions $(\widehat{\boldsymbol{\Xi}}, \widehat{\boldsymbol{\Theta}})$ in $H^2(\mathbb{R}^N)^N \times H^2(\mathbb{R}^N)^N$ such that

$$\widehat{\boldsymbol{\Xi}}|_{\Omega_s} = \mathbf{y}'_s, \qquad \text{and} \quad \widehat{\boldsymbol{\Theta}}|_{\Omega_s} = \mathbf{p}'_s$$

are the unique solution of the following adjoint state systems in $H^2(\Omega_s)^N \times H^2(\Omega_s)^N$:

$$\begin{cases} \mathscr{L}\mathbf{y}'_s + \mathrm{D}\mathbf{y}'_s \cdot \mathbf{w} = 0 & \text{in } \Omega_s(\mathbf{W}), \\ \mathbf{y}'_s = -\mathbf{V} \cdot \mathbf{n}_s \, \mathrm{D}(\mathbf{y}_s - \mathbf{g}) \, \mathbf{n}_s & \text{on } \Gamma_s(\mathbf{W}), \end{cases} \tag{5.13a}$$

$$\begin{cases} \mathscr{L}\mathbf{p}'_s - \mathrm{D}\mathbf{p}'_s \cdot \mathbf{w} = -\mathbf{y}'_s & \text{in } \Omega_s(\mathbf{W}), \\ \mathbf{p}'_s = -\mathbf{V} \cdot \mathbf{n}_s \, \mathrm{D}\mathbf{p}_s \, \mathbf{n}_s & \text{on } \Gamma_s(\mathbf{W}). \end{cases} \tag{5.13b}$$

The reason for the notation $\mathbf{y}'_s$ and $\mathbf{p}'_s$ comes from the fact that (5.13a) and (5.13b) are respectively the equations for the "partial derivative" of the state $\mathbf{y}_s$ and $\mathbf{p}_s$ with respect to the parameter s. In general, if $\mathbf{V}$ and Ω are sufficiently smooth, $\mathbf{y}'_s$ and $\mathbf{p}'_s$ belong to $H^k(\Omega_s)^N$ whenever $\mathbf{y}_s$ and $\mathbf{p}_s$ belong to $H^{k+1}(\Omega_s)^N$.

Assuming that $\mathbf{f}, \mathbf{y}_d \in H^2(\mathbb{R}^N)^N$ and $\mathbf{g} \in H^{\frac{7}{2}}(\mathbb{R}^N)^N$, we can deduce $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, $\boldsymbol{\Theta}$, $\boldsymbol{\Xi} \in H^3(\mathbb{R}^N)^N$, and further we have $\mathbf{y}_s, \mathbf{p}_s, \mathbf{p}'_s, \mathbf{y}'_s \in H^3(\Omega)^N$. However, $\mathbf{p}'_s, \mathbf{y}'_s \in H^3(\Omega)^N$ require that $\mathbf{y}_s, \mathbf{p}_s \in H^4(\Omega)^N$. Thus we can consider our saddle points $S(s) = X(s) \times Y(s)$ in $(H^4(\mathbb{R}^N)^N \times H^4(\mathbb{R}^N)^N) \times (H^3(\mathbb{R}^N)^N \times H^3(\mathbb{R}^N)^N)$,

$$X(s) = \{(\boldsymbol{\Phi}, \boldsymbol{\Psi}) \in H^4(\mathbb{R}^N)^N \times H^4(\mathbb{R}^N)^N \colon \boldsymbol{\Phi}|_{\Omega_s} = \mathbf{y}_s, \ \boldsymbol{\Psi}|_{\Omega_s} = \mathbf{p}_s\}, \tag{5.14}$$

$$Y(s) = \{(\boldsymbol{\Theta}, \boldsymbol{\Xi}) \in H^3(\mathbb{R}^N)^N \times H^3(\mathbb{R}^N)^N \colon \boldsymbol{\Theta}|_{\Omega_s} = \mathbf{p}'_s, \ \boldsymbol{\Xi}|_{\Omega_s} = \mathbf{y}'_s\} \tag{5.15}$$

and we can use Hadamard's formula (4.7) to derive the expression for $\partial_s G$:

$$\partial_s G(\Omega_s, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}, \boldsymbol{\Xi}) \stackrel{\text{def}}{=} \partial_s \Lambda(\Omega_s, \boldsymbol{\Phi}, \boldsymbol{\Psi}) + \partial_s L(\Omega_s, \boldsymbol{\Phi}, \boldsymbol{\Theta}) + \partial_s M(\Omega_s, \boldsymbol{\Psi}, \boldsymbol{\Xi}) \tag{5.16}$$

where

$$\partial_s \Lambda(\Omega_s, \mathbf{\Phi}, \mathbf{\Psi}) = \int_{\Gamma_s} \operatorname{div}\Big\{\Big[\frac{1}{2}|\mathbf{\Phi} - \mathbf{y}_d|^2 - \alpha \mathrm{D}(\mathbf{\Phi} - \mathbf{g})\colon \mathrm{D}\mathbf{\Psi} - \beta \operatorname{div}(\mathbf{\Phi} - \mathbf{g}) \operatorname{div} \mathbf{\Psi}\Big] \mathbf{V}\Big\}\langle \mathbf{W}, \mathbf{n}_s\rangle \,\mathrm{d}\Gamma_s, \tag{5.17}$$

$$\begin{aligned} \partial_s L(\Omega_s, \mathbf{\Phi}, \mathbf{\Theta}) = &\int_{\Gamma_s} \langle \mathscr{L}\mathbf{\Phi} + \mathrm{D}\mathbf{\Phi}\cdot \mathbf{w} - \mathbf{f}, \mathbf{\Theta}\rangle\langle \mathbf{W}, \mathbf{n}_s\rangle \,\mathrm{d}\Gamma_s \\ &- \alpha \int_{\Gamma_s} [\langle \mathbf{\Phi} - \mathbf{g}, \Delta\mathbf{\Theta}\rangle + \mathrm{D}(\mathbf{\Phi} - \mathbf{g})\colon \mathrm{D}\mathbf{\Theta}]\langle \mathbf{W}, \mathbf{n}_s\rangle \,\mathrm{d}\Gamma_s \\ &- \beta \int_{\Gamma_s} [\operatorname{div}(\mathbf{\Phi} - \mathbf{g}) \operatorname{div} \mathbf{\Theta} + \langle \mathbf{\Phi} - \mathbf{g}, \nabla \operatorname{div} \mathbf{\Theta}\rangle]\langle \mathbf{W}, \mathbf{n}_s\rangle \,\mathrm{d}\Gamma_s \end{aligned} \tag{5.18}$$

$$\begin{aligned} \partial_s M(\Omega_s, \mathbf{\Psi}, \mathbf{\Xi}) = &\int_{\Gamma_s} \langle \mathscr{L}\mathbf{\Psi} - \mathrm{D}\mathbf{\Psi}\cdot \mathbf{w} + \mathbf{\Phi} - \mathbf{y}_d,\ \mathbf{\Xi}\rangle\langle \mathbf{W}, \mathbf{n}_s\rangle \,\mathrm{d}\Gamma_s \\ &- \int_{\Gamma_s} \{\alpha[\langle \mathbf{\Psi}, \Delta\mathbf{\Xi}\rangle + \mathrm{D}\mathbf{\Psi}\colon \mathrm{D}\mathbf{\Xi}] + \beta[\operatorname{div} \mathbf{\Psi} \operatorname{div} \mathbf{\Xi} \\ &\qquad + \langle \mathbf{\Psi}, \nabla \operatorname{div} \mathbf{\Xi}\rangle]\}\langle \mathbf{W}, \mathbf{n}_s\rangle \,\mathrm{d}\Gamma_s. \end{aligned} \tag{5.19}$$

Finally, the expression for $\partial_s G$ is a functional on Γ_s which does not depend on $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{\Theta}$ and $\mathbf{\Xi}$ outside of $\bar{\Omega}_s$. As a result, the inf and sup can be dropped and

$$\begin{aligned} \mathrm{d}^2 J(\Omega; \mathbf{V}; \mathbf{W}) = \partial_s \Lambda(\Omega_s, \mathbf{y}_s, \mathbf{p}_s)|_{s=0} &+ \partial_s L(\Omega_s, \mathbf{y}_s, \mathbf{p}'_s)|_{s=0} \\ &+ \partial_s M(\Omega_s, \mathbf{p}_s, \mathbf{y}'_s)|_{s=0} \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} &\partial_s \Lambda(\Omega_s, \mathbf{y}_s, \mathbf{p}_s)|_{s=0} \\ &\quad = \int_{\Gamma} \operatorname{div}\Big\{\Big[\frac{1}{2}|\mathbf{y} - \mathbf{y}_d|^2 - \alpha \mathrm{D}(\mathbf{y} - \mathbf{g}) : \mathrm{D}\mathbf{p} - \beta \operatorname{div}(\mathbf{y} - \mathbf{g}) \operatorname{div} \mathbf{p}\Big] \mathbf{V}\Big\}\langle \mathbf{W}, \mathbf{n}\rangle \,\mathrm{d}\Gamma; \end{aligned}$$

$$\begin{aligned} \partial_s L(\Omega_s, \mathbf{y}_s, \mathbf{p}'_s)|_{s=0} = &\int_{\Gamma} \langle \mathscr{L}\mathbf{y} + \mathrm{D}\mathbf{y}\cdot \mathbf{w} - \mathbf{f}, \mathbf{p}'\rangle\langle \mathbf{W}, \mathbf{n}\rangle \,\mathrm{d}\Gamma \\ &- \alpha \int_{\Gamma} [\langle \mathbf{y} - \mathbf{g}, \Delta\mathbf{p}'\rangle + \mathrm{D}(\mathbf{y} - \mathbf{g}) : \mathrm{D}\mathbf{p}']\langle \mathbf{W}, \mathbf{n}\rangle \,\mathrm{d}\Gamma \\ &- \beta \int_{\Gamma} [\operatorname{div}(\mathbf{y} - \mathbf{g}) \operatorname{div} \mathbf{p}' + \langle \mathbf{y} - \mathbf{g}, \nabla \operatorname{div} \mathbf{p}'\rangle]\langle \mathbf{W}, \mathbf{n}\rangle \,\mathrm{d}\Gamma; \\ \partial_s M(\Omega_s, \mathbf{p}_s, \mathbf{y}'_s)|_{s=0} = &\int_{\Gamma} \langle \mathscr{L}\mathbf{p} - \mathrm{D}\mathbf{p}\cdot \mathbf{w} + \mathbf{y} - \mathbf{y}_d, \mathbf{y}'\rangle\langle \mathbf{W}, \mathbf{n}\rangle \,\mathrm{d}\Gamma \\ &- \int_{\Gamma} \{\alpha[\langle \mathbf{p}, \Delta\mathbf{y}'\rangle + \mathrm{D}\mathbf{p} : \mathrm{D}\mathbf{y}'] \\ &+ \beta[\operatorname{div} \mathbf{p} \operatorname{div} \mathbf{y}' + \langle \mathbf{p}, \nabla \operatorname{div} \mathbf{y}'\rangle]\}\langle \mathbf{W}, \mathbf{n}\rangle \,\mathrm{d}\Gamma. \end{aligned}$$

Since $\mathbf{y}$ and $\mathbf{p}$ satisfy equations (4.1a) and (4.1b) respectively, (5.20) can be reduced to

$$\text{(5.21)}\quad \mathrm{d}^2J(\Omega;\mathbf{V};\mathbf{W}) = \int_\Gamma \operatorname{div}\Big\{\Big[\frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2 - \langle \mathrm{D}(\mathbf{y}-\mathbf{g})\mathbf{n}, \mathscr{B}\mathbf{p}\rangle\Big]\mathbf{V}\Big\}\langle \mathbf{W},\mathbf{n}\rangle\,\mathrm{d}\Gamma$$
$$- \int_\Gamma [\langle \mathrm{D}(\mathbf{y}-\mathbf{g})\,\mathbf{n}, \mathscr{B}\mathbf{p}'_V\rangle + \langle \mathrm{D}\mathbf{p}\,\mathbf{n}, \mathscr{B}\mathbf{y}'_V\rangle]\langle \mathbf{W},\mathbf{n}\rangle\,\mathrm{d}\Gamma$$

where $(\mathbf{y}'_V, \mathbf{p}'_V) \in H^3(\Omega)^N \times H^3(\Omega)^N$ uniquely solve

$$\text{(5.22a)}\qquad \begin{cases} \mathscr{L}\mathbf{y}'_V + \mathrm{D}\mathbf{y}'_V \cdot \mathbf{w} = 0 & \text{in } \Omega, \\ \mathbf{y}'_V = -\mathbf{V}\cdot\mathbf{n}\,\mathrm{D}(\mathbf{y}-\mathbf{g})\,\mathbf{n} & \text{on } \Gamma, \end{cases}$$
$$\text{(5.22b)}\qquad \begin{cases} \mathscr{L}\mathbf{p}'_V - \mathrm{D}\mathbf{p}'_V \cdot \mathbf{w} = -\mathbf{y}'_V & \text{in } \Omega, \\ \mathbf{p}'_V = -\mathbf{V}\cdot\mathbf{n}\,\mathrm{D}\mathbf{p}\,\mathbf{n} & \text{on } \Gamma \end{cases}$$

where we have added the subscript V to $\mathbf{y}'$ and $\mathbf{p}'$ to emphasize that they depend on the velocity field $\mathbf{V}$.

Before closing this section, we give two equivalent expressions for $\mathrm{d}^2J(\Omega;\mathbf{V};\mathbf{W})$. We denote by $(\mathbf{y}'_W, \mathbf{p}'_W)$ the solution of (5.22) with W instead of V; then it can be immediately shown that

$$\text{(5.23)}\quad \mathrm{d}^2J(\Omega;\mathbf{V};\mathbf{W}) = \int_\Gamma \operatorname{div}\Big\{\Big[\frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2 - \langle \mathrm{D}(\mathbf{y}-\mathbf{g})\mathbf{n}, \mathscr{B}\mathbf{p}\rangle\Big]\mathbf{V}\Big\}\langle \mathbf{W},\mathbf{n}\rangle\,\mathrm{d}\Gamma$$
$$- \int_\Gamma [\langle \mathbf{y}'_W, \mathscr{B}\mathbf{p}'_V\rangle + \langle \mathbf{p}'_W, \mathscr{B}\mathbf{y}'_V\rangle]\,\mathrm{d}\Gamma.$$

Another expression for $\mathrm{d}^2J(\Omega;\mathbf{V};\mathbf{W})$ can be obtained by eliminating $\mathbf{p}'_V$ from (5.21). Setting test function $\mathbf{\Psi}|_{\Omega_s} = \mathbf{p}'_V$ and the adjoint state $\widehat{\mathbf{\Xi}}|_{\Omega_s} = \mathbf{y}'_W$ with $\mathbf{V} = \mathbf{W}$ and $s = 0$ in (5.11), we obtain

$$\int_\Omega \langle \mathscr{L}\mathbf{y}'_W + \mathrm{D}\mathbf{y}'_W \cdot \mathbf{w}, \mathbf{p}'_V\rangle\,\mathrm{d}x - \int_\Gamma \langle \mathbf{y}'_W + \mathbf{W}\cdot\mathbf{n}\,\mathrm{D}(\mathbf{y}-\mathbf{g})\mathbf{n}, \mathscr{B}\mathbf{p}'_V\rangle\,\mathrm{d}\Gamma = 0$$
$$\Leftrightarrow a(\Omega;\mathbf{p}'_V,\mathbf{y}'_W) - (\mathrm{D}\mathbf{p}'_V\cdot\mathbf{w},\mathbf{y}'_W)_\Omega + \int_\Gamma \langle \mathbf{V}\cdot\mathbf{n}\mathrm{D}\mathbf{p}\,\mathbf{n}, \mathscr{B}\mathbf{y}'_W\rangle\,\mathrm{d}\Gamma = 0.$$

Similarly, if $\mathbf{\Phi}|_{\Omega_s} = \mathbf{y}'_W$, $\widehat{\mathbf{\Theta}}|_{\Omega_s} = \mathbf{p}'_V$ and $s = 0$ in (5.12), then

$$\int_\Omega \langle \mathscr{L}\mathbf{p}'_V - \mathrm{D}\mathbf{p}'_V\cdot\mathbf{w} + \mathbf{y}'_V, \mathbf{y}'_W\rangle\,\mathrm{d}x - \int_\Gamma \langle \mathbf{p}'_V + \mathbf{V}\cdot\mathbf{n}\,\mathrm{D}\mathbf{p}\,\mathbf{n}, \mathscr{B}\mathbf{y}'_W\rangle\,\mathrm{d}\Gamma = 0$$
$$\Leftrightarrow a(\Omega;\mathbf{p}'_V,\mathbf{y}'_W) - (\mathrm{D}\mathbf{p}'_V\cdot\mathbf{w},\mathbf{y}'_W)_\Omega + (\mathbf{y}'_V,\mathbf{y}'_W)_\Omega + \int_\Gamma \langle \mathbf{W}\cdot\mathbf{n}\mathrm{D}(\mathbf{y}-\mathbf{g}), \mathscr{B}\mathbf{p}'_V\rangle\,\mathrm{d}\Gamma = 0.$$

Comparing the above two expressions, we have

$$\int_\Gamma \langle \mathrm{D}(\mathbf{y}-\mathbf{g})\mathbf{n}, \mathscr{B}\mathbf{p}'_V\rangle \mathbf{W}\cdot\mathbf{n}\,\mathrm{d}\Gamma = \int_\Gamma \langle \mathrm{D}\mathbf{p}\,\mathbf{n}, \mathscr{B}\mathbf{y}'_W\rangle \mathbf{V}\cdot\mathbf{n}\,\mathrm{d}\Gamma - (\mathbf{y}'_V, \mathbf{y}'_W)_\Omega.$$

Substituting it into (5.21), we conclude

$$\begin{aligned}(5.24)\quad \mathrm{d}^2 J(\Omega;\mathbf{V};\mathbf{W}) = &\int_\Gamma \operatorname{div}\left\{\left[\frac{1}{2}|\mathbf{y}-\mathbf{y}_d|^2 - \langle \mathrm{D}(\mathbf{y}-\mathbf{g})\mathbf{n}, \mathscr{B}\mathbf{p}\rangle\right]\mathbf{V}\right\}\mathbf{W}\cdot\mathbf{n}\,\mathrm{d}\Gamma \\ &- \int_\Gamma \langle \mathrm{D}\mathbf{p}\,\mathbf{n}, \mathscr{B}\mathbf{y}'_V \mathbf{W}\cdot\mathbf{n} + \mathscr{B}\mathbf{y}'_W \mathbf{V}\cdot\mathbf{n}\rangle\,\mathrm{d}\Gamma + (\mathbf{y}'_V, \mathbf{y}'_W)_\Omega.\end{aligned}$$

Remark 5.1. The final expression (5.21) for the shape Hessian is consistent with the structure theorem 2.2 provided that both $\mathbf{y}'_V$ and $\mathbf{p}'_V$ are linear and continuous functions of $\mathbf{V}\cdot\mathbf{n}$ on the boundary Γ. Moreover, the non-symmetry of the shape Hessian is satisfied since the first boundary integral in (5.21) (or (5.23), (5.24)) is not symmetric in $\mathbf{V}\cdot\mathbf{n}$ and $\mathbf{W}\cdot\mathbf{n}$.

5.3. Verification of the assumptions of theorem 4.1. In Section 5.2 we have applied the conclusion of Theorem 4.1 to the Lagrangian $G(\Omega_s,\cdot,\cdot,\cdot,\cdot)$ which contains a non-convex cost functional $\Lambda(\Omega_s,\cdot,\cdot)$ in (5.5). This means that the Lagrangian G doesn't necessarily have saddle points.

The cost functional $\mathrm{d}J(\Omega_s(\mathbf{W});\mathbf{V})$ is a non-convex differentiable functional, hence we can choose a suitable constant $c>0$ such that

$$(5.25)\qquad \mathrm{d}J(\Omega_s(\mathbf{W});\mathbf{V}) + c(\|\mathbf{y}_s\|^2_{4,\Omega_s} + \|\mathbf{p}_s\|^2_{4,\Omega_s})$$

is convex and continuous on $H^4(\Omega_s)^N \times H^4(\Omega_s)^N$.

Then using (5.5) we can define a convex functional

$$(5.26)\qquad G_c(\Omega_s,\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Theta},\mathbf{\Xi}) \stackrel{\text{def}}{=} G(\Omega_s,\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Theta},\mathbf{\Xi}) + C(\Omega_s,\mathbf{\Phi},\mathbf{\Psi})$$

where the convex functional $C(\Omega_s,\mathbf{\Phi},\mathbf{\Psi}) \stackrel{\text{def}}{=} c(\|\mathbf{\Phi}\|^2_{4,\Omega_s} + \|\mathbf{\Psi}\|^2_{4,\Omega_s})$. Thus we have

$$(5.27)\qquad \mathrm{d}J_c(\Omega_s(\mathbf{W});\mathbf{V}) = \inf_{\mathbf{\Phi},\mathbf{\Psi}\in H^4(\mathbb{R}^N)^N}\ \sup_{\mathbf{\Theta},\mathbf{\Xi}\in H^3(\mathbb{R}^N)^N} G_c(\Omega_s,\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Theta},\mathbf{\Xi})$$

and

$$J_0(\Omega_s) = \inf_{\mathbf{\Phi},\mathbf{\Psi}\in H^4(\mathbb{R}^N)^N}\ \sup_{\mathbf{\Theta},\mathbf{\Xi}\in H^3(\mathbb{R}^N)^N} C(\Omega_s,\mathbf{\Phi},\mathbf{\Psi}).$$

It is readily seen that this new Lagrangian G_c is affine in $(\mathbf{\Theta},\mathbf{\Xi})$ and convex in $(\mathbf{\Phi},\mathbf{\Psi})$, so saddle points of G_c exist. In addition, saddle points of C exist obviously. Since the verification of the assumptions is essentially the same as for the shape gradient in Section 4.3 under the previous appropriate hypotheses on $\mathbf{f}$, $\mathbf{y}_d \in H^2(\mathbb{R}^N)^N$, $\mathbf{g} \in H^{\frac{7}{2}}(\mathbb{R}^N)^N$ and $\mathbf{w} \in H^1_{\mathrm{div}}(\mathbb{R}^N)$, we can use the conclusion of Theorem 4.1 to obtain

$$\mathrm{d}^2 J(\Omega; \mathbf{V}; \mathbf{W}) = \mathrm{d}^2 J_c(\Omega; \mathbf{V}; \mathbf{W}) - \mathrm{d}J_0(\Omega),$$

and eventually,

$$\mathrm{d}^2 J(\Omega; \mathbf{V}; \mathbf{W}) = \inf_{\boldsymbol{\Phi}, \boldsymbol{\Psi} \in X(0)} \sup_{\boldsymbol{\Theta}, \boldsymbol{\Xi} \in Y(0)} \partial_s G(\Omega_s, \boldsymbol{\Phi}, \boldsymbol{\Psi}, \boldsymbol{\Theta}, \boldsymbol{\Xi})\big|_{s=0}.$$

References

[1] *R. A. Adams*: Sobolev Spaces. Academic Press, London, 1975. zbl

[2] *J. Céa*: Lectures on Optimization: Theory and Algorithms. Springer-Verlag, 1978. zbl

[3] *J. Céa*: Problems of shape optimal design. "Optimization of Distributed Parameter Structures", Vol. II (E. J. Haug and J. Céa, eds.). Sijthoff and Noordhoff, Alphen aan denRijn, Netherlands, 1981, pp. 1005–1048. zbl

[4] *R. Correa and A. Seeger*: Directional derivative of a minmax function. Nonlinear Analysis, Theory Methods and Applications *9* (1985), 13–22. zbl

[5] *M. C. Delfour, G. Payre and J. -P. Zolésio*: An optimal triangulation for second order elliptic problems. Computer Methods in Applied Mechanics and Engineering *50* (1985), 231–261. zbl

[6] *M. C. Delfour and J.-P. Zolésio*: Shape sensitivity analysis via min max differentiability. SIAM J. Control and Optimization *26* (1988), 834–862. zbl

[7] *M. C. Delfour and J.-P. Zolésio*: Computation of the shape Hessian by a Lagrangian method. IFAC, Control of Distributed Parameter Systems. Perpignan, France, 1989, pp. 215–220.

[8] *M. C. Delfour and J.-P. Zolésio*: Shape Hessian by the velocity method: a Lagrangian approach. Lect. Notes. Cont. Inf. Sci, 147, Springer-Verlag, 1990, pp. 255–279. zbl

[9] *M. C. Delfour and J.-P. Zolésio*: Tangential calculus and shape derivative in Shape Optimization and Optimal Design (Cambridge, 1999), Dekker, New York. 2001, pp. 37–60. zbl

[10] *M. C. Delfour and J.-P. Zolésio*: Shapes and Geometries: Analysis, Differential Calculus, and Optimization. Advance in Design and Control, SIAM (2002). zbl

[11] *I. Ekeland and R. Temam*: Convex Analysis and Variational Problems. Series Classics in Applied Mathematics, SIAM, Philadelphia, 1999. zbl

[12] *N. Fujii*: Domain optimization problems with a boundary value problem as a constraint. "Control of Distributed Parameter Systems" 1986, Pergamon Press, Oxford, New York, 1986, pp. 5–9. zbl

[13] *D. Gilbarg and N. S. Trudinger*: Elliptic Partial Differential Equations of Second Order. Springer, Berlin, 1983. zbl

[14] *J. Hadamard*: Mémoire sur le problème d'analyse relatif à l'équilibre des plaques élastiques encastrées. Mém. Sav. étrang. (2) 33, Nr. 4, 128 S (1908). zbl

[15] *J. Haslinger and R. A. E. Mäkinen*: Introduction to Shape Optimization: Theory, Approximation, and Computation. SIAM (2003). zbl

[16] *J.-C. Nédélec*: Acoustic and Electromagnetic Equations. Integral representations for harmonic problems. Springer-Verlag, Berlin Heidelberg New York, 2001. zbl

[17] *O. Pironneau*: Optimal Shape Design for Elliptic systems. Springer, Berlin, 1984. zbl
[18] *J. Simon*: Differentiation with respect to the domain in boundary value problems. Numer. Funct. Anal. Optim. *2* (1980), 649–687. zbl
[19] *J. Simon*: Second variations for domain optimization problems. "Control of Distributed Parameter Systems", Birkhauser Verlag, 1988. zbl
[20] *R. Temam*: Navier-Stokes Equations: Theory and Numerical Analysis. Providence, RI: American Mathematical Society (AMS), 2001. zbl
[21] *J.-P. Zolésio*: Identification de domaines par déformation, Thèse de doctorat d'état, Université de Nice, France. 1979.
[22] *J. Sokolowski and J.-P. Zolésio*: Introduction to Shape Optimization: Shape Sensitivity Analysis. Springer-Verlag, Berlin, 1992. zbl

Authors' addresses: Zhiming Gao, School of Science, Xi'an Jiaotong University, P.O. Box 1844, Xi'an, Shaanxi, P.R. China, 710049, e-mail: dtgaozm@gmail.com; Yichen Ma, School of Science, Xi'an Jiaotong University, Shaanxi, P.R. China, 710049, e-mail: ycma@mail.xjtu.edu.cn; Hongwei Zhuang, Engineering College of Armed Police Force, Shaanxi, P.R. China, 710086.