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# TEST ELEMENTS AND THE RETRACT THEOREM FOR MONOUNARY ALGEBRAS

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*Abstract.* The term "Retract Theorem" has been applied in literature in connection with group theory. In the present paper we prove that the Retract Theorem is valid (i) for each finite structure, and (ii) for each monounary algebra. On the other hand, we show that this theorem fails to be valid, in general, for algebras of the form $\mathscr{A} = (A, F)$, where each $f \in F$ is unary and $\operatorname{card} F > 1$.

*Keywords*: monounary algebra, retract, test element

*MSC 2000*: 08A60

## 1. Introduction

An element $t$ of a structure $\mathscr{A}$ is said to be a *test element* if for any endomorphism $\varphi$ of $\mathscr{A}$, $\varphi(t) = t$ implies that $\varphi$ is an automorphism.

The notion of test elements was first considered in the context of free groups, in which they are called *test words*. The first example was provided in 1918 by Nielsen [5], who showed that the basic commutator $[a, b] = aba^{-1}b^{-1}$ is a test word in the free group $F(a, b)$.

Test elements can be applied for distinguishing automorphisms from nonautomorphisms: $t$ is a test element if $\varphi(t) = \alpha(t)$ for some automorphism $\alpha$ and any endomorphism $\varphi$ implies that $\varphi$ is an automorphism.

We recall that a substructure $\mathscr{B}$ of a structure $\mathscr{A}$ is a retract of $\mathscr{A}$ if there exists an endomorphism (called a retraction) $\varphi$ of $\mathscr{A}$ onto $\mathscr{B}$ such that $\varphi(b) = b$ for each element $b$ of $\mathscr{B}$.

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It is easy to show (cf. 2.1) that if $t$ is a test element of $\mathscr{A}$, then $t$ does not belong to any proper retract of $\mathscr{A}$.

The following theorem (called the *Retract Theorem for free groups*) was proved in [8]:

**Theorem.** *A word $w$ in a free group $F$ is a test word if and only if $w$ is not in any proper retract of $F$.*

In [4] validity of the Retract Theorem for torsion free stably hyperbolic groups and for finitely generated Fuchsian groups is verified. Further, it is remarked there that by [9] there exists a group for which the Retract Theorem does not hold (e.g. the fundamental group of the Klein bottle).

Let us consider the following conditions for a structure $\mathscr{A}$ and an element $t$ of $\mathscr{A}$:

$(*)$ $t$ is a test element of $\mathscr{A}$,

$(**)$ $t$ does not belong to any proper retract of $\mathscr{A}$.

In this paper we show that if $\mathscr{A}$ is a finite structure then $(*)$ and $(**)$ for any $t$ of $\mathscr{A}$ are equivalent, i.e., that the Retract Theorem is valid for each finite structure.

The main goal of the paper concerns monounary algebras. Let $\mathscr{A}$ be a monounary algebra. Necessary and sufficient conditions under which there is $t$ of $\mathscr{A}$ satisfying $(**)$ are found. Next it is proved that for each monounary algebra the Retract Theorem holds.

Finally, we give an example of a unary algebra $\mathscr{A}$ with two operations such that $\mathscr{A}$ contains an element $t$ which is not a test element and which does not belong to any proper retract of $\mathscr{A}$, hence, for unary algebras with two operations the Retract Theorem does not hold in general. Moreover, using this example, it is shown that this result remains valid for unary algebras with more than one operation.

## 2. Finite structures

By a structure we will understand any algebraic or topological structure (where endomorphisms and automorphisms are defined).

If a structure is denoted by a symbol $\mathscr{A}$, then $A$ will always denote the support of $\mathscr{A}$.

**Lemma 2.1.** *Let $\mathscr{A}$ be a structure with a finite support $A$ and let $t \in A$ satisfy $(*)$. Then $t$ satisfies $(**)$.*

Proof. Assume that $t$ does not satisfy $(**)$, i.e., there is a proper retract $\mathscr{B}$ of $\mathscr{A}$ containing $t$. Let $\varphi$ be the corresponding retraction. Then $\varphi$ is not surjective, therefore $\varphi$ is not an automorphism and $\varphi(t) = t$. Hence $t$ does not fulfil $(*)$. $\square$

**Lemma 2.2.** *Let $\mathscr{A}$ be a structure with a finite support $A$. If $t$ is not a test element of $\mathscr{A}$ then there is a proper retract $\mathscr{B}$ of $\mathscr{A}$ containing $t$.*

Proof. Suppose that $t$ is not a test element of $\mathscr{A}$, i.e., there is an endomorphism $\psi$ of $\mathscr{A}$ such that $\psi(t) = t$ and $\psi$ is not an automorphism. Since $\mathscr{A}$ is finite, $\psi$ is not surjective. Consider the seqence of mappings $\{\psi^n\}_{n\in\mathbb{N}}$. The members of the sequence cannot be distinct, thus there exist $j, k \in \mathbb{N}$, $j < k$ with $\psi^j = \psi^k$. We have $\psi^j \circ \psi^{k-j} = \psi^k = \psi^j$, hence $\psi^{k-j}$ is the identity on $\psi^j(A)$,

$$\psi^{k-j}(x) = x \text{ for each } x \in \psi^j(A).$$

Therefore $(\psi^{k-j})^j$ is again the identity on $\psi^j(A)$.

Denote $\varphi = \psi^{j(k-j)} = (\psi^j)^{k-j}$, $B = \varphi(A)$.

This yields

$$\varphi^2 = (\psi^{j(k-j)})^2 = \psi^{2j(k-j)} = \psi^{j(k-j)+j(k-j)} = (\psi^j)^{k-j} \circ (\psi^{k-j})^j = (\psi^j)^{k-j} = \varphi$$

since $\psi^j(A) \supseteq (\psi^j)^{(k-j)}(A)$.

The mapping $\varphi$ is an endomorphism of $\mathscr{A}$, $\mathscr{B}$ is a retract of $\mathscr{A}$. The fact that $\psi$ is not surjective implies that $\varphi$ is not surjective either, and $\psi(t) = t$ implies $\varphi(t) = t$. Thus $\mathscr{B}$ is a proper retract of $\mathscr{A}$ containing $t$. □

**Theorem 2.3.** *An element $t$ of a finite structure $\mathscr{A}$ is a test element of $\mathscr{A}$ if and only if $t$ does not belong to any proper retract of $\mathscr{A}$.*

Proof. It is a consequence of 2.1 and 2.2. □

## 3. Preliminaries on monounary algebras

By a *monounary algebra* we understand an algebra with a single unary operation, cf. e.g. [2], [3]. By $\mathbb{Z}$ and $\mathbb{N}$ we denote the set of all integers and all positive integers, respectively. A monounary algebra $\mathscr{A} = (A, f)$ is said to be *connected* if for each $x, y \in A$ there are $m, n \in \mathbb{N} \cup \{0\}$ with $f^n(x) = f^m(y)$. A maximal connected subalgebra of $\mathscr{A}$ is called a *connected component* of $\mathscr{A}$. If $x \in A$, $f^n(x) = x$ for some $n \in \mathbb{N}$, then $x$ is called *cyclic*. Let $C$ be a set of all cyclic elements of a connected component $\mathscr{B}$ of $\mathscr{A}$; denote $r(B) = \operatorname{card} C$. If $r(B) \neq 0$, then $C$ is said to be a *cycle* of $\mathscr{A}$ (and, as usual, also $(C, f)$ is called a cycle of $\mathscr{A}$).

We denote by $(\mathbb{Z}, \mathrm{succ})$ and $(\mathbb{N}, \mathrm{succ})$ the monounary algebras with the operation of the successor.

The notion of the *degree* $s(x)$ of an element $x$ in a unary algebra $\mathscr{A} = (A, f)$ was introduced in [6] (cf. e.g. also [1]) as follows. Let us denote by $A^{(\infty)}$ the set of all elements $x \in A$ such that there exists a sequence $\{x_n\}_{n\in\mathbb{N}\cup\{0\}}$ of elements belonging to $A$ with the property $x_0 = x$ and $f(x_n) = x_{n-1}$ for each $n \in \mathbb{N}$. Further, we put $A^{(0)} = \{x \in A\colon\ f^{-1}(x) = \emptyset\}$. We now define a set $A^{(\lambda)} \subseteq A$ for each ordinal $\lambda > 0$ by induction. Assume we have defined $A^{(\alpha)}$ for each ordinal $\alpha < \lambda$. Then we put

$$A^{(\lambda)} = \Big\{x \in A - \bigcup_{\alpha<\lambda} A^{(\alpha)}\colon\ f^{-1}(x) \subseteq \bigcup_{\alpha<\lambda} A^{(\alpha)}\Big\}.$$

The sets $A^{(\lambda)}$ are pairwise disjoint. For each $x \in A$, either $x \in A^{(\infty)}$ or there is an ordinal $\lambda$ with $x \in A^{(\lambda)}$. In the former case we put $s(x) = \infty$, in the latter we set $s(x) = \lambda$. We put $\lambda < \infty$ for each ordinal $\lambda$. If it is neccessary to express that the mapping $s$ is constructed for the algebra $(A, f)$, we write $s_{(A,f)}$ for $s$.

In [7] a connected monounary algebra $\mathscr{B} = (B, g)$ was defined to be *admissible* to a connected monounary algebra $\mathscr{A} = (A, f)$ if one of the following conditions is satisfied:

(i) $r(\mathscr{B}) \neq 0$ and $r(\mathscr{B})$ divides $r(\mathscr{A})$,

(ii) $r(\mathscr{B}) = 0 = r(\mathscr{A})$ and there exist $a \in A$, $b \in B$ such that

$$s_{(A,f)}(f^n(a)) \leqslant s_{(B,g)}(g^n(b)) \ \text{ for each } n \in \mathbb{N} \cup \{0\}.$$

From Thm. of Section 2 [7] we obtain

**Lemma 3.1.** *Let $\varphi$ be a homomorphism of a monounary algebra $\mathscr{A}$ into a monounary algebra $\mathscr{B}$. If $x$ is an element of $\mathscr{A}$, then $s_{(A,f)}(x) \leqslant s_{(B,g)}(\varphi(x))$.*

**Lemma 3.2.** *Let $(A, f)$ be a connected monounary algebra, $a, b \in A$ such that $a$, $b$ are not cyclic and $b \neq a \neq f(a) = f(b)$. If $x \in A$ and there exists $k \in \mathbb{N} \cup \{0\}$ such that $f^k(x) = a$, then $f^n(x) \neq b$ for all $n \in \mathbb{N} \cup \{0\}$.*

Proof. Let $x \in A$, $k \in \mathbb{N} \cup \{0\}$, $f^k(x) = a$. Assume that there is $n \in \mathbb{N} \cup \{0\}$ with $f^n(x) = b$.

Let $k \leqslant n$. If $k = n$, then $a = f^k(x) = f^n(x) = b$, which is a contradiction. Thus $k < n$ and $f^{n-k}(a) = f^{n-k}(f^k(x)) = f^n(x) = b$ and $n - k > 0$. Further, $f^{n-k}(b) = f^{n-k-1}(f(b)) = f^{n-k-1}(f(a)) = f^{n-k}(a) = b$ and we have that $b$ is cyclic, which is a contradiction.

Let $n < k$. Then $f^{k-n}(b) = f^{k-n}(f^n(x)) = f^k(x) = a$ and $k - n > 0$. Further, $f^{k-n}(a) = f^{k-n-1}(f(a)) = f^{k-n-1}(f(b)) = f^{k-n}(b) = a$, thus $a$ is cyclic, which is a contradiction.

Thus $f^n(x) \neq b$ holds for all $n \in \mathbb{N} \cup \{0\}$. $\square$

Now let us introduce the following notion:

**Definition 3.3.** Let $(A, f)$, $(B, g)$ be connected monounary algebras, $a \in A$, $b \in B$. We say that the ordered pair $(a, b)$ is *generating*, if it satisfies the following conditions:

(i) the assignment $f^n(a) \longmapsto g^n(b)$ for all $n \in \mathbb{N} \cup \{0\}$ is a mapping of $\{f^n(a)\colon n \in \mathbb{N} \cup \{0\}\}$ onto $\{g^n(b)\colon n \in \mathbb{N} \cup \{0\}\}$,

(ii) $s_{(A,f)}(f^n(a)) \leqslant s_{(B,g)}(g^n(b))$ for all $n \in \mathbb{N} \cup \{0\}$.

**Definition 3.4.** Let $(A, f)$ be a connected monounary algebra $a, b \in A$. We say that the ordered pair $(a, b)$ is *strongly generating* if it satisfies the following conditions:

(i) the element $a$ is not cyclic,

(ii) $b \neq a \neq f(a) = f(b)$,

(iii) $s(a) \leqslant s(b)$.

**Remark 3.5.** Let us notice that if the assumption of 3.3 is valid then by [7] there is a homomorphism $\varphi$ of $(A, f)$ into $(B, f)$ with $\varphi(a) = b$.

It is easy to see that each strongly generating pair is generating for the pair $(A, f)$, $(A, f)$ where $(A, f)$ is a connected monounary algebra. By a strongly generating pair of an algebra that is not connected we mean a strongly generating pair of one of its components.

**Lemma 3.6.** *Let $(A, f)$ be a monounary algebra, $x, y \in A$ elements such that $f(x) = f(y) \neq x \neq y$. Then there exists a strongly generating pair $(a, b)$ such that $\{a, b\} = \{x, y\}$. If neither $x$ nor $y$ are cyclic and $s(x) = s(y)$, then the pairs $(x, y)$ and $(y, x)$ are strongly generating.*

P r o o f. If both elements $x, y$ were cyclic, the condition $f(y) = f(x)$ would imply $y = x$, which is a contradiction.

Hence, if one of the elements is cyclic, we denote it by $b$ and the other by $a$. Then the conditions (i), (ii) and (iii) of 3.4 are satisfied. If none of the elements $x, y$ is cyclic, we compare $s(x)$ and $s(y)$. If $s(x) \leqslant s(y)$ we put $a = x$, $b = y$; if $s(x) > s(y)$, we set $a = y$, $b = x$. The conditions (i), (ii) and (iii) are satisfied in this case, too. The remaining fact of the assertion follows from the proof above. □

**Lemma 3.7.** *Let $(A, f)$ be a connected monounary algebra, $h$ an endomorphism which is not an automorphism and $x_0 \in A$ such that $h(x_0) = x_0$. Then there exists a strongly generating pair $(a, b)$ such that $f^n(x_0) \neq a$ for each $n \in \mathbb{N} \cup \{0\}$.*

P r o o f. Let $M = \{x \in A\colon h(x) = x\}$. Then $M$ is a closed subset of the algebra $(A, f)$. By assumption, $M \neq A$. Let $c$ be an arbitrary element of $A - M$ and $l \in \mathbb{N} \cup \{0\}$ such that $f^l(c) \in A - M$ and $f^{l+1}(c) \in M$.

Denote $a = f^l(c)$ and $b = h(a)$. Clearly $a \in A - M$, $f(a) \in M$, thus $a \neq f(a)$. Further we have $b = h(a) \neq a$ and $h(f(a)) = f(a)$. If $a$ is cyclic, then there exists $m \in \mathbb{N}$ such that $f^m(x_0) = a$ and we obtain $h(a) = h(f^m(x_0)) = f^m(h(x_0)) = f^m(x_0) = a$, a contradiction. Thus (i) of 3.4 holds. Further, $f(b) = f(h(a)) = h(f(a)) = f(a)$ and so (ii) holds. The last condition is satisfied because $h$ is an endomorphism and by 3.1, $s(a) \leqslant s(h(a))$. $\square$

**Lemma 3.8.** *Let $(A, f)$ be a connected monounary algebra, $(a, b)$ a strongly generating pair. There exists a retraction $h$ of the algebra $(A, f)$, such that the condition $h(x) \neq x$ is equivalent to $x \in \bigcup\limits_{k \in \mathbb{N} \cup \{0\}} f^{-k}(a)$.*

P r o o f. (1) If the element $b$ is cyclic then we construct $h\colon A \longmapsto A$ as follows: if $f^k(x) = a$ holds for some $k \in \mathbb{N} \cup \{0\}$ we find the unique cyclic element $y$ with $f^k(y) = b$ and put $h(x) = y$; for all other elements of $A$ we put $h(x) = x$.

We verify that $h$ is an endomorphism of the algebra $(A, f)$. Let $f^k(x) = a$ for some $k \in \mathbb{N} \cup \{0\}$. If $k = 0$, we have $x = a$, $h(x) = b$ and the elements $b$, $f(b)$ are cyclic, thus $h(b) = b$, $h(f(a)) = h(f(b)) = f(b) = f(h(a))$. By induction assume that $h(f(x')) = f(h(x'))$ holds for all $x' \in A$ with $f^n(x') = a$, where $n \in \mathbb{N} \cup \{0\}$. Let $x \in A$ be an element with $f^{n+1}(x) = a$. Then for $x' = f(x)$, by the assumption and the definition of $h$, we have that $f(h(x)) = h(x') = h(f(x))$.

Finally, for any $x \in A$ with the property $f^k(x) \neq a$ for all $k \in \mathbb{N} \cup \{0\}$ we have $h(x) = x$, $h(f(x)) = f(x)$, thus $h(f(x)) = f(x) = f(h(x))$.

Hence $h$ is an endomorphism and $h(x) \neq x$ holds if and only if $f^k(x) = a$ for some $k \in \mathbb{N} \cup \{0\}$, thus $h$ is a retraction.

(2) Assume that $b$ is not cyclic. Since the pair $(a, b)$ is strongly generating, according to [7] there exists an endomorphism $g$ of the algebra $(A, f)$ such that $g(a) = b$. We define $h$ as follows:

$$h(x) = \begin{cases} g(x) & \text{if there exists } k \in \mathbb{N} \cup \{0\} \text{ with } f^k(x) = a, \\ x & \text{otherwise.} \end{cases}$$

Let $x \in A$ be an element such that $f^n(x) \neq a$ for all $n \in \mathbb{N} \cup \{0\}$. Then $f(x)$ satisfies the same condition (i.e., $f^n(f(x)) \neq a$ for each $n \in \mathbb{N} \cup \{0\}$), therefore $h(f(x)) = f(x) = f(h(x))$.

If $f^n(x) = a$ for some $n \in \mathbb{N} \cup \{0\}$, then $h(x) = g(x)$. If $n > 0$, then $h(f(x)) = g(f(x)) = f(g(x)) = f(h(x))$. For $n = 0$ we have $x = a$, $h(a) = g(a) = b$ and therefore $f(h(a)) = f(b) = f(a) = h(f(a))$. Hence $h$ is an endomorphism.

If $f^n(x) = a$ holds for some $n \in \mathbb{N} \cup \{0\}$, then $f^n(h(x)) = h(f^n(x)) = h(a) = b$, thus $f^k(h(x)) \neq a$ for all $k \in \mathbb{N} \cup \{0\}$ according to 3.2. Hence $h$ is a retraction, whereas $h(x) \neq x$ if and only if $f^n(x) = a$ for some $n \in \mathbb{N} \cup \{0\}$. $\square$

In 4.1–4.3 assume that $(A, f)$ is a monounary algebra with the system $\{(A_i, f)\}_{i\in I}$ of connected components where $A_i \neq A_j$ for $i \neq j$; furthermore, let $t \in A_{i_0}$, $i_0 \in I$.

Let us start the section with the definition of a property of monounary algebras.

**Definition 4.1.** We say that $(A, f)$ has the property $V(t)$, if the following conditions are satisfied:

(i) if $i_1 \neq i_0$, then there does not exist $i_2 \neq i_1$ such that $(A_{i_2}, f)$ is admissible to $(A_{i_1}, f)$,

(ii) if $(a, b)$ is a strongly generating pair, then there exists $n \in \mathbb{N} \cup \{0\}$ such that $a = f^n(t)$.

**Theorem 4.2.** *Let $(A, f)$ have the property $V(t)$. Then every endomorphism $h$ of $(A, f)$ with $h(t) = t$ is an automorphism.*

P r o o f. Assume that there exists an endomorphism $h$ of $(A, f)$ with $h(t) = t$ which is not an automorphism. According to (i) of 4.1 $h$ maps every component of $(A, f)$ into itself and thus $h \restriction A_i$ is an endomorphism of $(A_i, f)$ for all $i \in I$. At least one of these endomorphisms is not an automorphism. Let $(A_j, f)$ be such component that $h \restriction A_j$ is no automorphism and $j \neq i_0$. Then this component is not a cycle and also it is not isomorphic to $(\mathbb{Z}, \mathrm{succ})$, because for these algebras, the only endomorphisms are automorphisms. Further, $(A_j, f)$ is not isomorphic to $(\mathbb{N}, \mathrm{succ})$, since $(\mathbb{N}, \mathrm{succ})$ is admissible to $(A_{i_0}, f)$. Consequently, there exist $x, y \in A_j$, with $x \neq y \neq f(x) = f(y)$. According to 3.6 there exists a strongly generating pair $(a, b)$ with $\{a, b\} = \{x, y\}$, which contradicts (ii) of 4.1.

Therefore $h \restriction A_{i_0}$ is not an automorphism. According to 3.7, there exists a strongly generating pair $(a, b)$, $a, b \in A_{i_0}$, with $f^n(t) \neq a$ for all $n \in \mathbb{N} \cup \{0\}$, which contradicts (ii).

Hence every endomorphism $h$ of $(A, f)$ with $h(t) = t$ is an automorphism of $(A, f)$. $\square$

**Theorem 4.3.** *Suppose that $(A, f)$ has not the property $V(t)$. Then there exists a retraction $h$ of $(A, f)$ with $h(t) = t$ which is different from the identity.*

P r o o f. Suppose that (i) of 4.1 fails to be valid. Then there exist $i_1 \neq i_0$ and $i_2 \neq i_1$ such that $(A_{i_2}, f)$ is admissible to $(A_{i_1}, f)$. There exists a homomorphism $g\colon A_{i_2} \longmapsto A_{i_1}$ which after being extended by the identity to all components different from $(A_{i_2}, f)$ gives the desired retraction $h$ of $(A, f)$, which is not the identity and $h(t) = t$.

Further, assume that (i) holds and (ii) fails. Then there exists a strongly generating pair $(a,b)$ such that $a \neq f^n(t)$ for all $n \in \mathbb{N} \cup \{0\}$.

If $(a,b)$ belongs to a component $(A_i, f)$, where $i \neq i_0$, then, according to 3.8, there exists a retraction $h'$ of algebra $(A_i, f)$ with $h'(a) \neq a$ , thus this retraction is not the identity on $A_i$. Extension of the mapping by the identity to the other components gives a retraction $h$ different from the identity with the property $h(t) = t$.

Let $(a,b)$ belong to the component $(A_{i_0}, f)$. Then, according to 3.8, there exists a retraction $h'$ of $(A_{i_0}, f)$ such that the condition $h'(x) \neq x$ is equivalent to the condition that there exists $k \in \mathbb{N} \cup \{0\}$ such that $f^k(x) = a$. The element $t$ does not fulfil this condition and thus $h'(t) = t$. Moreover, $h(a) \neq a$, so this retraction is not the identity on $A_{i_0}$. Again, the extension of this mapping by the identity to the other components gives a retraction $h$ different from the identity, with $h(t) = t$. $\square$

The following result is the Retract Theorem for monounary algebras:

**Theorem 4.4.** *Let $(A,f)$ be a monounary algebra, $t \in A$ an element. Then the following conditions are equivalent:*

(a) *$(A,f)$ has the property $V(t)$,*

(b) *$t$ is the test element of $(A,f)$,*

(c) *$t$ does not belong to any proper retract of $(A,f)$.*

P r o o f. (a) implies (b) due to 4.2. According to 2.1 (b) implies (c) and (c) implies (a) by virtue of 4.3. $\square$

Now we will give a more concrete description of a monounary algebra $(A,f)$ having the property $V(t)$, $t \in A$. Again suppose that $(A,f)$ is a monounary algebra with the system $\{(A_i,f)\}_{i \in I}$ of connected components where $A_i \neq A_j$ for $i \neq j$; let $t \in A_{i_0}$, $i_0 \in I$.

We introduce the following conditions.

(A) If $i \in I$, $i \neq i_0$, then either $(A_i, f)$ is a cycle or it is isomorphic to $(\mathbb{Z}, \mathrm{succ})$.

(B) If $|I| \geqslant 3$, $i \in I$, $i \neq i_0$, then $(A_i, f)$ is a cycle.

(C) If $i_1, i_2 \in I$, $i_0 \neq i_1 \neq i_2$ and $(A_{i_1}, f)$, $(A_{i_2}, f)$ contain cycles with $n_1, n_2$ elements respectively, then $n_2$ does not divide $n_1$.

(D) If there is $x \in A_{i_0}$ with $s(x) = \infty$, then $(A_i, f)$ is a cycle for each $i \in I$, $i \neq i_0$.

(E) There exist $l \in \mathbb{N} \cup \{0, \omega_0\}$ and $j_k \in \mathbb{N}$ and $v_k \in A_{i_0}$ for each $k \in \mathbb{N}$, $k < l$ such that

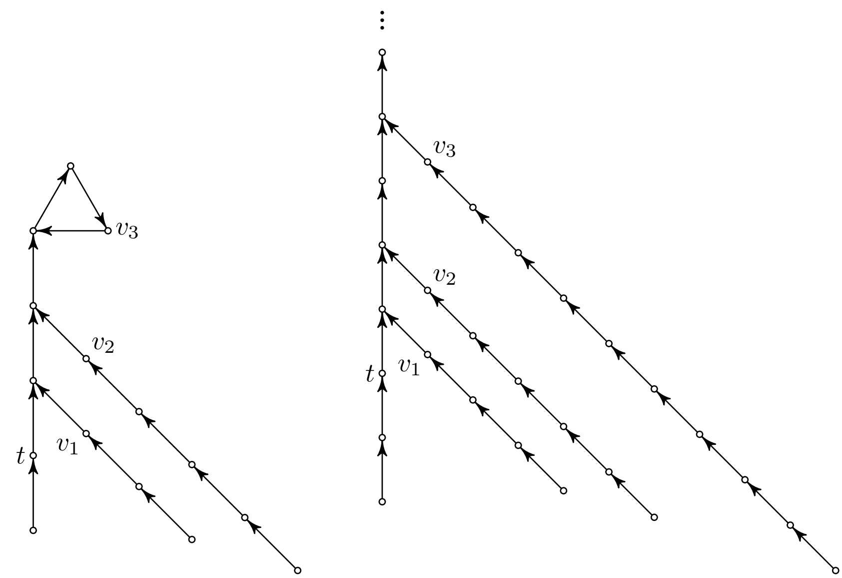
  (1) if $k \in \mathbb{N}$, $k < l$ then $f^{j_k - 1}(t)$ is not cyclic, $\operatorname{card} f^{-1}(f^{j_k}(t)) = 2$, $f^{-1}(f^{j_k}(t)) = \{f^{j_k-1}(t), v_k\}$ and either $s(f^{j_k-1}(t)) < s(v_k)$ or $\infty = s(f^{j_k-1}(t)) = s(v_k)$ where $v_k$ is cyclic,

  (2) if $x \in A_{i_0} - \{f^{j_k}(t)\colon\ k \in \mathbb{N}, k < l\}$ then $\operatorname{card} f^{-1}(x) \leqslant 1$.

(Let us notice that if $l = 0$, then no $j_k$ is determined and $(A_{i_0}, f) \cong (\mathbb{Z}, \mathrm{succ})$ or $(A_{i_0}, f) \cong (\mathbb{N}, \mathrm{succ})$ or $(A_{i_0}, f)$ is a cycle.)

**Example.** a) $l = 4$, $j_1 = 1$, $j_2 = 2$, $j_3 = 3$, $s(f^0(t)) = 1 < 2 = s(v_1)$, $s(f^1(t)) = 3 < 4 = s(v_2)$, $s(f^2(t)) = 5 < \infty = s(v_3)$,

b) $l = 4$, $j_1 = 1$, $j_2 = 2$, $j_3 = 4$, $s(f^0(t)) = 2 < 3 = s(v_1)$, $s(f^1(t)) = 4 < 5 = s(v_2)$, $s(f^3(t)) = 7 < 9 = s(v_3)$.




**Theorem 4.5.** *A monounary algebra $(A, f)$ has the property $V(t)$ if and only if it satisfies* (A)–(E).

P r o o f. Let the algebra $(A, f)$ have the property $V(t)$.

If $x, y \in A$, $f(x) = f(y) \neq x \neq y$, then there exists a strongly generating pair $(a, b)$ such that $\{a, b\} = \{x, y\}$ by 3.6. By 4.1(ii), we obtain $\{x, y\} = \{a, b\} \subseteq A_{i_0}$. Hence, if $i \neq i_0$, $x \in A_i$, $y \in A_i$, $x \neq y$, then $f(x) \neq f(y)$, which implies that $(A_i, f)$ is either a cycle or is isomorphic to $(\mathbb{Z}, \mathrm{succ})$ or to $(\mathbb{N}, \mathrm{succ})$. Since $(A_{i_0}, f)$ is admisible to $(\mathbb{N}, \mathrm{succ})$, the third possibility is excluded by 4.1(i). Thus (A) holds. The condition (B) follows from the fact that any cycle is admissible to $(\mathbb{Z}, \mathrm{succ})$ and from 4.1(i). The condition (C) is a consequence of 4.1(i).

If there exists $x \in A_{i_0}$ with $s(x) = \infty$ and $i \in I$, $i \neq i_0$ such that $(A_i, f)$ is not a cycle, then $(A_i, f)$ is isomorphic to $(\mathbb{Z}, \mathrm{succ})$ by (A). Clearly, if $y \in A_i$ is the element corresponding to $0 \in \mathbb{Z}$ in this isomorphism, then $(y, x)$ is a generating pair and, therefore, $(A_{i_0}, f)$ is admissible to $(A_i, f)$, which contradicts 4.1(i). Thus, (D) holds.

For any strongly generating pair $(u,v)$ there exists $j \in \mathbb{N}\cup\{0\}$ such that $u = f^j(t)$. If $(u,v')$ is a strongly generating pair with $v \neq v'$, then there exists a strongly generating pair $(a,b)$ with $\{a,b\} = \{v,v'\}$. Clearly, $a \neq f^k(t)$ for any $k$ with $k \in \mathbb{N}\cup\{0\}$, which contradicts (ii) of 4.1. It follows that the set of all strongly generating pairs is either finite or countable. Thus, there exists $l \in \mathbb{N} \cup \{0,\omega_0\}$, $j_k \in \mathbb{N}$ and $v_k \in A_{i_0}$ such that $\{(f^{j_k-1}(t), v_k); k \in \mathbb{N}, k < l\}$ is the set of all strongly generating pairs. Hence (1) and (2) of (E) hold.

Let the conditions (A)–(E) be satisfied.

If $i_1 \neq i_0$ and there exists $i_2 \neq i_1$ such that $(A_{i_2}, f)$ is admissible to $(A_{i_1}, f)$, then either $i_2 \neq i_0$ or $i_2 = i_0$.

In the former case $(A_{i_1}, f)$, $(A_{i_2}, f)$ are cycles by (B) and we obtain a contradiction by (C).

In the latter case, we obtain that $(A_{i_1}, f)$ is either a cycle or it is isomorphic to $(\mathbb{Z}, \mathrm{succ})$ by (A). If $(A_{i_0}, f) = (A_{i_2}, f)$ is admissible to $(A_{i_1}, f)$, then it contains an element $x \in A_{i_0}$ with $s_{(A_{i_0},f)}(x) = \infty$ by 3.1. By (D), the algebra $(A_{i_1}, f)$ is a cycle with $n_1$ elements. Since $(A_{i_0}, f)$ is admissible to $(A_{i_1}, f)$, it contains a cycle with $n_2$ elements where $n_2$ divides $n_1$. This contradicts (C).

Hence (i) of 4.1 holds. Furthermore, $(f^{j_k-1}(t), v_k)$ is a strongly generating pair for any $k \in \mathbb{N}$, $k < l$ by (1) of (E) and no other strongly generating pair exists by (2) of (E).

Thus (ii) of 4.1 holds. $\square$

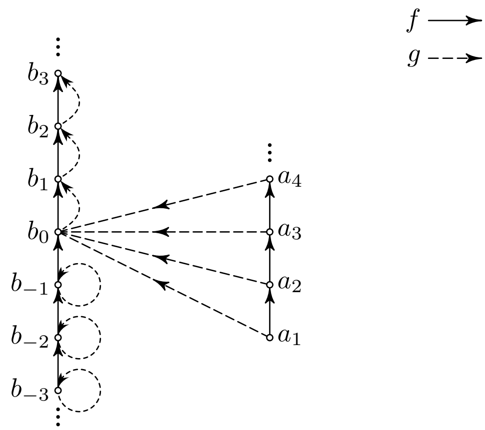
## 5. Unary algebras with several operations

**Example.** Consider a unary algebra $\mathscr{A} = (A, f, g)$ with two unary operations, with the support $A = \{a_n\colon n \in \mathbb{N}\} \cup \{b_i\colon i \in \mathbb{Z}\}$ (the elements written here are supposed to be distinct) and such that

$$f(x) = \begin{cases} a_{n+1} & \text{if } x = a_n,\ n \in \mathbb{N}, \\ b_{i+1} & \text{if } x = b_i,\ i \in \mathbb{Z}, \end{cases}$$

$$g(x) = \begin{cases} b_0 & \text{if } x = a_n,\ n \in \mathbb{N}, \\ b_{i+1} & \text{if } x = b_i,\ i \in \mathbb{N}\cup\{0\}, \\ x & \text{if } x = b_i,\ -i \in \mathbb{N}. \end{cases}$$

Put $t = b_0$.




Let $\alpha$ be a mapping of $\mathscr{A}$ into $\mathscr{A}$ defined as follows:

$$\alpha(x) = \begin{cases} x & \text{if } x = b_i,\ i \in \mathbb{Z}, \\ a_{n+1} & \text{if } x = a_n,\ n \in \mathbb{N}. \end{cases}$$

Then $\alpha$ is an endomorphism of $\mathscr{A}$, $\alpha(t) = t$, but $\alpha$ is not an automorphism of $\mathscr{A}$ since $a_1 \notin \alpha(A)$. Therefore $t$ is not a test element of $\mathscr{A}$.

We want to show that $t$ satisfies $(**)$. Suppose that there is a proper retract $\mathscr{B} = (B, f, g)$ of $\mathscr{A}$ such that $t \in B$; let $\varphi$ be the corresponding retraction. We have $\varphi(t) = t$, thus $\varphi \restriction \{b_i\colon\ i \in \mathbb{Z}\} = \mathrm{id}_{\{b_i\,:\, i \in \mathbb{Z}\}}$. If $\varphi(a_1) = a_{n+1}$ for some $n \in \mathbb{N}$, then $a_{n+1} \in B$, thus $a_{n+1} = \varphi(a_{n+1}) = \varphi(f^n(a_1)) = f^n(\varphi(a_1)) = f^n(a_{n+1}) = a_{2n+1}$, which is a contradiction. If $\varphi(a_1) = a_1$, then for each $k \in \mathbb{N}$ the condition $\varphi(a_{k+1}) = \varphi(f^k(a_1)) = f^k(\varphi(a_1)) = f^k(a_1) = a_{k+1}$ is satisfied, thus $\varphi = \mathrm{id}_A$, which is a contradiction: the retract $\varphi(\mathscr{A})$ is not proper.

Hence $\varphi(a_1) = b_i$ for some $i \in \mathbb{Z}$. This implies $b_0 = \varphi(b_0) = \varphi(g(a_1)) = g(\varphi(a_1)) = g(b_i)$, which is a contradiction. Therefore $t = b_0$ satisfies $(**)$.

**Proposition 5.1.** *There exist a unary algebra $\mathscr{A} = (A, f, g)$ and $t \in A$ such that $t$ does not belong to any proper retract of $\mathscr{A}$ and $t$ is not a test element of $\mathscr{A}$.*

Proof. The assertion follows from the previous example. $\square$

**Corollary 5.2.** *The Retract Theorem for unary algebras with two operations is not valid in general.*

**Proposition 5.3.** *Let $F$ be a type containing more than one operation symbol and assume that each $f \in F$ is unary. There exist a unary algebra $\mathscr{A} = (A, F)$ and $t \in A$ such that $t$ does not belong to any proper retract of $\mathscr{A}$ and $t$ is not a test element of $\mathscr{A}$.*

Proof. Let $(A, f, g)$ be the unary algebra as in the example above. We choose any $f_1, f_2 \in F$, $f_1 \neq f_2$ and for $x \in A$, $h \in F - \{f_1, f_2\}$ we put $f_1(x) = f(x), f_2(x) = g(x), h(x) = x$. The definition of $\mathscr{A} = (A, F)$ yields that if $\mathscr{B} = (B, F)$ is a proper retract of $(A, F)$ then $(B, f_1, f_2)$ is a proper retract of $(A, f_1, f_2)$, therefore $t$ is not contained in any proper retract of $\mathscr{A} = (A, F)$. Next, $\varphi$ is an endomorphism of $(A, F)$ if and only if $\varphi$ is an endomorphism of $(A, f_1, f_2)$, hence $t$ is not a test element of $(A, F)$. $\square$

**Corollary 5.4.** *The Retract Theorem for unary algebras of a given type $F$ with* $\operatorname{card} F > 1$ *is not valid in general.*

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