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Ramiro H. Lafuente-Rodríguez

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DIVISIBILITY IN CERTAIN AUTOMORPHISM GROUPS

Ramiro H. Lafuente-Rodríguez, La Paz

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Abstract. We study solvability of equations of the form $x^n = g$ in the groups of order automorphisms of archimedean-complete totally ordered groups of rank 2. We determine exactly which automorphisms of the unique abelian such group have square roots, and we describe all automorphisms of the general ones.

Keywords: totally ordered groups, ordered automorphisms, divisible groups, archimedean rank

MSC 2000: 06F15, 20B27

1. Introduction

The algebraic properties of the automorphism group $\mathrm{Aut}(\Omega)$ of a relational structure Ω have been a topic of interest since the 19th century. In the case when Ω is an ordered structure, the question of divisibility of $\mathrm{Aut}(\Omega)$ is of particular interest, that is: for which $n \in \mathbb{N}$ and $g \in \mathrm{Aut}(\Omega)$ is the equation $x^n = g$ solvable? For example, Holland [8] showed that if Ω is a totally ordered set and $\mathrm{Aut}(\Omega)$ is order-2-transitive then $\mathrm{Aut}(\Omega)$ is divisible, and from this concluded that every lattice-ordered group can be embedded in a divisible lattice-ordered group. More than 50 years ago, the question (recently answered in the negative by Bludov [2]) of whether every *totally* ordered group can be embedded in a divisible totally ordered group, led to the interesting investigation of divisibility of the automorphism group $\mathrm{Aut}(\Omega)$ when Ω is a totally ordered group. For example, Conrad [3] gave an example of an abelian totally ordered group which has infinite archimedean rank and is archimedean-complete (see definition in Section 2) whose automorphism group is not divisible. And Holland [7] then noted that if Ω is the abelian totally ordered group $\mathbb{R} \overleftarrow{\oplus} \mathbb{R}$ (also archimedean-complete and of rank 2), then $\mathrm{Aut}(\Omega)$ has an element with no square root, and so it is not divisible.

In this paper we first (Section 3) determine exactly which elements of $\mathrm{Aut}(\mathbb{R} \overleftarrow{\oplus} \mathbb{R})$ have square roots (Theorem 3.1) and illustrate the complexity of the general question. We then (Section 4) generalize to the same question in arbitrary archimedean-complete totally ordered groups of rank 2.

2. Definitions and background

The reader can find further background on ordered groups, for example, in the book by Darnel [5]. Here we will include some standard definitions.

A *totally ordered* group G is a group which is a totally ordered set under a relation $\leqslant$ and such that $x \leqslant y$ implies $xz \leqslant yz$ and $zx \leqslant zy$. Two elements $e < a, b \in G$ are *archimedean equivalent*, denoted $a \sim b$, if for some $n \in \mathbb{N}$, $a \leqslant b^n$ and $b \leqslant a^n$. The equivalence classes are convex, that is, $a \leqslant b \leqslant c$ and $a \sim c$ implies $a \sim b$, and so there is a naturally induced total order on the set of equivalence classes, whereby $(a \sim) < (b \sim)$ iff $a \sim a'$ and $b \sim b'$ imply $a' < b'$. The *archimedean rank* of a totally ordered group is just the order type of its ordered set of archimedean equivalence classes. If H is a totally ordered group and G is a subgroup of H, then H is an *archimedean extension* of G if for every $e < h \in H$, there exists $g \in G$ such that $g \sim h$. A totally ordered group is *archimedean-complete* if it has no proper archimedean extension. The totally ordered group $\mathbb{R}$ of real numbers has rank 1, and the anti-lexicographically ordered direct sum $\mathbb{R} \overleftarrow{\oplus} \mathbb{R}$ has rank 2. The former is the unique archimedean-complete ordered group of rank 1, and the latter is the unique *abelian* archimedean-complete ordered group of rank 2.

An ordered group G of finite rank n has just n nontrivial convex subgroups, and they form a tower under inclusion:

$$\{e\} \subset C_1 \subset C_2 \subset \ldots \subset C_n = G.$$

Each C_i is normal (in G) and $C_i/C_{i-1} = D_i$ is an ordered group of rank 1 called a *component* of G. Since each D_i has archimedean rank 1, by Hölder's theorem [6] each D_i is an ordered subgroup of the ordered additive group of real numbers $\mathbb{R}$. An *o-automorphism* (or just *automorphism*) of a totally ordered group G is a group automorphism of G which preserves the order, and the group of all such automorphisms is $\mathrm{Aut}(G)$. It is obvious that all inner automorphisms preserve order.

3. Abelian Archimedean rank 2

In what follows, we will sometimes denote the value of a function δ at an element x of its domain as $(x)\delta$. Note also that the homomorphisms from $\mathbb{R}$ to $\mathbb{R}$ form a ring containing, for all $r \in \mathbb{R}$, the function $\tau_r\colon\ x \mapsto xr$. Notationally, we will identify $\tau_r = r$. Also, it is important to note that if $r \in \mathbb{Q}$, then $r(= \tau_r)$ commutes with all homomorphisms δ. That is, $\tau_r\delta = \delta\tau_r$, or in our notation, $r\delta = \delta r$.

We will now investigate the divisibility properties of the automorphism groups of archimedean-complete totally ordered groups of small archimedean rank. Beginning with rank 1, the only such group is the totally ordered group $\mathbb{R}$ of all real numbers, and the only automorphisms are just multiplication by positive real numbers. Thus, $\mathrm{Aut}(\mathbb{R})$ is divisible. Next, we consider groups of rank 2. The only abelian archimedean-complete totally ordered group of rank 2 is $\mathbb{R} \overleftarrow{\oplus} \mathbb{R}$. And according to Conrad [4], $\mathrm{Aut}(\mathbb{R} \overleftarrow{\oplus} \mathbb{R})$ consists of the group of all matrices of the form

$$\alpha = \begin{pmatrix} r_1 & 0 \\ \gamma & r_2 \end{pmatrix}$$

with $0 < r_1, r_2 \in \mathbb{R}$ and $\gamma\colon\ \mathbb{R} \to \mathbb{R}$ an additive group homomorphism (i.e., a rational linear map), where for each $(x, y) \in \mathbb{R} \overleftarrow{\oplus} \mathbb{R}$,

$$(x, y)\alpha = (x, y) \begin{pmatrix} r_1 & 0 \\ \gamma & r_2 \end{pmatrix} = (xr_1 + (y)\gamma, yr_2).$$

Composition of two automorphisms in this representation is just matrix multiplication, where we must remember that multiplication of rational linear maps is not commutative. We wish to determine exactly which automorphisms of $\mathbb{R} \overleftarrow{\oplus} \mathbb{R}$ have square roots.

With this notation, we consider

$$\beta^2 = \begin{pmatrix} t & 0 \\ \pi & s \end{pmatrix} \begin{pmatrix} t & 0 \\ \pi & s \end{pmatrix} = \begin{pmatrix} t^2 & 0 \\ \pi t + s\pi & s^2 \end{pmatrix}.$$

Now let

$$\alpha = \begin{pmatrix} r_1 & 0 \\ \gamma & r_2 \end{pmatrix}$$

be an automorphism with a square root β as above. Then $t^2 = r_1$, $s^2 = r_2$, and $\pi t + s\pi = \gamma$. Thus, when trying to construct a square root of α, we must choose positive numbers $t = \sqrt{r_1}$, $s = \sqrt{r_2}$ and a homomorphism π such that

$$\pi t + s\pi = \gamma. \tag{1}$$

This functional equation means that for all $x \in \mathbb{R}$,

$$(x)\pi t + (xs)\pi = (x)\gamma. \tag{2}$$

Let

$$p(z) = \sum_{i=0}^{n} a_i z^i \in \mathbb{Q}[z]$$

be a polynomial with rational coefficients. For each $j = 1, \dots, n$, let

$$p_j(z) = \sum_{k=j}^{n} a_k z^{k-j}.$$

Theorem 3.1. *The element*

$$\alpha = \begin{pmatrix} t^2 & 0 \\ \gamma & s^2 \end{pmatrix} \in \operatorname{Aut}(\mathbb{R} \overset{\leftarrow}{\oplus} \mathbb{R})$$

fails to have a square root in $\operatorname{Aut}(\mathbb{R} \overset{\leftarrow}{\oplus} \mathbb{R})$ *if and only if* s *and* $-t$ *are both roots of the same irreducible polynomial*

$$p(z) \in \mathbb{Q}[z]$$

and for all $c \in \mathbb{R}$,

$$\sum_{j=1}^{n} (cs^{j-1})\gamma p_j(-t) \neq 0.$$

P r o o f. Suppose first that there exists

$$\beta = \begin{pmatrix} t & 0 \\ \pi & s \end{pmatrix} \in A$$

such that $\beta^2 = \alpha$. From equation (2), for any $c \in \mathbb{R}$ we must have

$$\begin{aligned} (c)\pi &= (c)\pi, \\ (cs)\pi &= (c)\gamma - (c)\pi t, \\ (cs^2)\pi &= (cs)\gamma - (cs)\pi t = (cs)\gamma - ((c)\gamma - (c)\pi t)t \\ &= (cs)\gamma - (c)\gamma t + (c)\pi t^2 \end{aligned}$$

and in general, for $i \geqslant 1$,

$$(cs^i)\pi = \sum_{k=0}^{i-1} (cs^{i-1-k})\gamma \cdot (-t)^k + (c)\pi \cdot (-t)^i. \tag{3}$$

Using the fact that π is a homomorphism, if $p(z) = \sum_{i=0}^{n} a_i z^i$ is a polynomial in $\mathbb{Q}[z]$, then

$$
\begin{aligned}
(4) \qquad (cp(s))\pi &= \sum_{i=0}^{n} a_i (cs^i)\pi \\
&= a_0(c)\pi + a_1(cs)\pi + a_2(cs^2)\pi + \ldots + a_n(cs^n)\pi \\
&= a_0(c)\pi + a_1((c)\gamma - (c)\pi t) + a_2((cs)\gamma - (cs)\pi t) + a_3((cs^2)\gamma - (cs^2)\pi t) + \ldots \\
&= a_0(c)\pi + a_1((c)\gamma - (c)\pi t) + a_2((cs)\gamma - ((c)\gamma - (c)\pi t)t) + \ldots \\
&= ((c\pi)(a_0 - a_1 t + a_2 t^2 - a_3 t^3 - \ldots + a_n(-t)^n) \\
&\quad + ((c)\gamma)(a_1 - a_2 t + a_3 t^2 - \ldots + a_n(-t)^{n-1}) \\
&\quad + ((cs)\gamma)(a_2 - a_3 t + a_4 t^2 \ldots + a_n(-t)^{n-2}) + \ldots + ((cs^{n-1})\gamma)a_n \\
&= ((c)\pi)p(-t) + \sum_{j=1}^{n} (cs^{j-1})\gamma p_j(-t).
\end{aligned}
$$

If s and $-t$ are both roots of $p(z)$, then we must have

$$\sum_{j=1}^{n} (cs^{j-1})\gamma p_j(-t) = 0.$$

We note for later use the following fact.

Lemma 3.2. *For $j = 0, \ldots, n-1$, $p_j(z) = p_{j+1}(z) \cdot (-z) + a_j$.*

Now we deal with the converse assertion of the theorem.

Suppose first that t is algebraic, say $-t$ is a root of $p(z)$ for some irreducible polynomial, but s is not a root of $p(z)$. In order to show that α has a square root in this case, we see from equation (4) that there must exist some π such that

$$(cp(s))\pi = \sum_{j=1}^{n} (cs^{j-1})\gamma p_j(-t)$$

for every c. In fact, we can define π by this equation as

$$(x)\pi = \sum_{j=1}^{n} \Big(\frac{x}{p(s)} s^{j-1}\Big)\gamma p_j(-t).$$

Then π is obviously a homomorphism. Moreover, to see that equation (2) is satisfied, we proceed as follows:

$$\begin{aligned}
(xs)\pi &= \sum_{j=1}^{n} \Big(\frac{xs}{p(s)}s^{j-1}\Big)\gamma p_j(-t) \\
&= \sum_{j=1}^{n} \Big(\frac{x}{p(s)}s^{j}\Big)\gamma p_j(-t) \\
&= \sum_{j=1}^{n} \Big(\frac{x}{p(s)}s^{j}\Big)\gamma((-t)\cdot p_{j+1}(-t) + a_j) \text{ (from Lemma 3.2)} \\
&= \sum_{j=2}^{n+1} \Big(\frac{x}{p(s)}s^{j-1}\Big)\gamma((-t)\cdot p_{j}(-t) + a_{j-1}) \\
&= \Bigg(\sum_{j=1}^{n} \Big(\frac{x}{p(s)}s^{j-1}\Big)\gamma p_j(-t)\Bigg)(-t) - \Big(\frac{x}{p(s)}\Big)\gamma p_1(-t)\cdot(-t) \\
&\quad + \sum_{j=2}^{n+1} \Big(\frac{x}{p(s)}s^{j-1}\Big)\gamma a_{j-1} \\
&= (x)\pi(-t) - \Big(\frac{x}{p(s)}\Big)\gamma(p(-t) - a_0) + \sum_{j=2}^{n+1} \Big(\frac{x}{p(s)}a_{j-1}s^{j-1}\Big)\gamma \\
&= (x)\pi(-t) + \Big(\frac{x}{p(s)}a_0\Big)\gamma + \sum_{j=1}^{n} \Big(\frac{x}{p(s)}a_j s^{j}\Big)\gamma \\
&= (x)\pi(-t) + \Big(\frac{x}{p(s)}p(s)\Big)\gamma \\
&= -(x)\pi t + (x)\gamma
\end{aligned}$$

as required.

Next, suppose s is a root of an irreducible $p(z)$ but $-t$ is not. Then we must have

$$0 = ((c)\pi)p(-t) + \sum_{j=1}^{n}(cs^{j-1})\gamma p_j(-t).$$

With this in mind, we can define

$$(c)\pi = -\Bigg(\sum_{j=1}^{n}(cs^{j-1})\gamma\Bigg)\frac{p_j(-t)}{p(-t)}.$$

Then π is clearly a homomorphism. A straightforward calculation as before shows that $s\pi + \pi t = \gamma$ as required.

There remain two cases: when either both s and $-t$ are transcendental, or both are roots of the same irreducible polynomial $p(z) \in \mathbb{Q}[z]$ but $\sum_{j=1}^{n}(cs^{j-1})\gamma p_j(-t) = 0$ for all c. For each of these cases, we proceed as follows. We choose a basis $\mathscr{D}$ for $\mathbb{R}$ as a vector space over the subfield $\mathbb{Q}(s)$. Then for each $d \in \mathscr{D}$ we define π on the one-dimensional subspace $d\mathbb{Q}(s)$ in such a way that equation (2) is satisfied on that subspace. Then the following lemma shows that the extension of π to all of $\mathbb{R}$ still satisfies equation (2). Thus, we have a square root.

Lemma 3.3. *If $(us)\pi = (u)\gamma - (u)\pi t$ and $(vs)\pi = (v)\gamma - (v)\pi t$ for some numbers u, v, then for all linear combinations with rational coefficients a, b, also $((au + bv)s)\pi = (au + bv)\gamma - (au + bv)\pi t$.*

The proof, which uses the fact that π and γ are homomorphisms, is trivial.

Let us suppose now that both s and $-t$ are transcendental. Choose any basis $\mathscr{D}$ for $\mathbb{R}$ over $\mathbb{Q}(s)$, and let $d \in \mathscr{D}$. Observing from equation (4) that to be successful we must have

$$(dq(s))\pi = ((d)\pi)q(-t) + \sum_{j=1}^{n}(ds^{j-1})\gamma q_j(-t),$$

we see that

$$(dp(s))\pi = \Big(\frac{dp(s)}{q(s)}q(s)\Big)\pi = \Big(\Big(\frac{dp(s)}{q(s)}\Big)\pi\Big)q(-t) + \sum_{j=1}^{n}\Big(\frac{dp(s)}{q(s)}s^{j-1}\Big)\gamma q_j(-t)$$

and this implies that

$$\begin{aligned}\Big(\frac{dp(s)}{q(s)}\Big)\pi &= ((dp(s))\pi - \sum_{j=1}^{n}\Big(\frac{dp(s)}{q(s)}s^{j-1}\Big)\gamma q_j(-t))\frac{1}{q(-t)}\\ &= \Big(((d)\pi)p(-t) + \sum_{j=1}^{n}(ds^{j-1})\gamma p_j(-t) - \sum_{j=1}^{n}\Big(\frac{dp(s)}{q(s)}s^{j-1}\Big)\gamma q_j(-t)\Big)\frac{1}{q(-t)}.\end{aligned}$$

We can use this last equation to define π on the one-dimensional subspace $d\mathbb{Q}(s)$ for each $d \in \mathscr{D}$, with $(d)\pi$ arbitrary, and then extend it in the natural way to all of $\mathbb{R}$; in fact, we may as well take $(d)\pi = 0$ for each $d \in \mathscr{D}$. That is, we define

$$\Big(\frac{dp(s)}{q(s)}\Big)\pi = \Big(\sum_{j=1}^{n}(ds^{j-1})\gamma p_j(-t) - \sum_{j=1}^{n}\Big(\frac{dp(s)}{q(s)}s^{j-1}\Big)\gamma q_j(-t)\Big)\frac{1}{q(-t)}.$$

It is routine to show that π is well defined, it is a homomorphism, and that equation (2) is satisfied.

Finally, we suppose that both s and $-t$ are roots of the irreducible $p(z) \in \mathbb{Q}[z]$ of degree n and that $\sum_{j=1}^{n} (cs^{j-1})\gamma p_j(-t) = 0$ for all c. Let $\mathscr{D}$ be any basis for $\mathbb{R}$ over $\mathbb{Q}(s)$. For each $d \in \mathscr{D}$ we arbitrarily define $(d)\pi$ and then we use equation (3) to define $(ds^i)\pi$ for $i = 1, \ldots, n-1$. From our assumption, it follows that equation (2) is satisfied on the rational span of $\{d, ds, \ldots, ds^{n-1}\}$ which is $\mathrm{d}\mathbb{Q}(s)$. Then as before, the extension of π to $\mathbb{R}$ satisfies equation (2) as well and we have a square root. This completes the proof of Theorem 3.1.

Corollary 3.4. $\mathrm{Aut}(\mathbb{R} \overleftarrow{\oplus} \mathbb{R})$ *is not divisible.*

Proof. We may take, for example, the automorphism

$$\alpha = \begin{pmatrix} 2 & 0 \\ \gamma & 2 \end{pmatrix},$$

where γ is any homomorphism such that $(1)\gamma = 1$ and $(\sqrt{2})\gamma = 0$.

An earlier (but different) example illustrating Corollary 3.4 is found in Holland [7].

The complexity of these conditions indicates that the answer to the following still open question may be complicated:

Question. Exactly what conditions on the parameters r_1, r_2, γ of α determine whether α has a cube root, or an nth root?

4. General Archimedean rank 2

In this section we describe the automorphism group of an archimedean-complete totally ordered group of archimedean rank 2, and make some observations about divisibility of the automorphism group.

Let G be an archimedean-complete totally ordered group of archimedean rank 2. Then by Conrad [3] G has a normal convex subgroup N isomorphic to $\mathbb{R}$, and the ordered group $H = G/N$ is also isomorphic to $\mathbb{R}$. Let $(\mathbb{R}^+, \cdot)$ denote the multiplicative group of positive real numbers. By [3] again, G is isomorphic to a totally ordered group $(\mathbb{R}, \mathbb{R}, \varphi, f)$ constructed in the following way. On the antilexicographically ordered *set* $\mathbb{R} \times \mathbb{R}$, we define multiplication by

$$(a, b) \cdot (x, y) = (a + x \cdot \varphi(-b) + f(b, y), b + y)$$

where $\varphi\colon (\mathbb{R}, +) \to (\mathbb{R}^+, \cdot)$ is a group homomorphism and $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function such that

1. $f(x, 0) = f(0, y) = 0$ for all $x, y \in \mathbb{R}$; and

2. $f(x+y,z)+f(x,y)=f(x,y+z)+f(y,z)\cdot\varphi(-x)$.

Henceforth, we identify G with $(\mathbb{R},\mathbb{R},\varphi,f)$.

Theorem 4.1. *The automorphisms of* $(\mathbb{R},\mathbb{R},\varphi,f)$ *are just the functions* σ *of the form* $(x,y)\sigma=(xs+(y)\gamma,yt)$ *where* $0<s,t\in\mathbb{R}$, *either* $t=1$ *or* $\mathbb{R}=\operatorname{Ker}(\varphi)$, *and*

$$(x+y)\gamma=(x)\gamma+(y)\gamma\cdot(\varphi(-xt))+f(xt,yt)-f(x,y)\cdot s. \tag{5}$$

Proof. Suppose that σ is an automorphism of G. Then it is straight-forward to check that σ must have the form $(x,y)\sigma=(xs+(y)\gamma,yt)$ where $0<s,t\in\mathbb{R}$, and $\gamma\colon\ \mathbb{R}\to\mathbb{R}$ is a function. We now determine the properties of γ:

$$[(0,x)(0,y)]\sigma=(f(x,y),x+y)\sigma=(f(x,y)\cdot s+(x+y)\gamma,(x+y)t),$$

and

$$[(0,x)\sigma][(0,y)\sigma]=((x)\gamma,xt)((y)\gamma,yt)=((x)\gamma+(y)\gamma\cdot\varphi(-xt)+f(xt,yt),xt+yt).$$

Equating the left members gives rise to equation (5). To get the remaining condition, we first take $x=0$ in equation (5), and conclude that $(0)\gamma=0$. Then using this fact and applying σ to the product $(0,x)(a,0)$ for any $x\in\mathbb{R}$ and an arbitrary $0\neq a\in\mathbb{R}$, we deduce that $\varphi(-x)=\varphi(-xt)$, and so $x(t-1)\in\operatorname{Ker}(\varphi)$. Thus, $t=1$ or $\mathbb{R}=\operatorname{Ker}(\varphi)$.

Conversely, it is straighforward to verify that any s, t, γ satisfying the conditions of the theorem give rise to an automorphism σ.

Because $\mathbb{R}=\operatorname{Ker}(\varphi)$ precisely when the extension is central, it is natural to consider separately the central and the non-central cases. Here we will consider mainly the non-central case and make some remarks on the central case which, perhaps surprisingly, is much more difficult.

Let $(\mathbb{R},\mathbb{R},\varphi,f)$ be a non-central extension, and consider an automorphism determined by s, t, γ as in Theorem 4.1. Then $t=1$ and from equation (5) we also have

$$(x+y)\gamma=(y+x)\gamma=(y)\gamma+(x)\gamma\varphi(-y)+f(y,x)-f(y,x)\cdot s.$$

Therefore,

$$(x)\gamma+(y)\gamma(\varphi(-x))+f(x,y)-f(x,y)s=(y)\gamma+(x)\gamma\varphi(-y)+f(y,x)-f(y,x)\cdot s$$

and so

$$\begin{aligned}(x)\gamma[1-\varphi(-y)]&\\ =(y)\gamma[1-\varphi(-x)]+f(y,x)-f(x,y)+s[f(x,y)-f(y,x)]&\\ =(y)\gamma[1-\varphi(-x)]+(s-1)[f(x,y)-f(y,x)].&\end{aligned} \tag{6}$$

Since the extension is not central, there exists $c \in H$ such that $\varphi(c) \neq 1$. We see that γ is completely determined by the value $(-c)\gamma$, because setting $y = -c$ in the last equation, we have

$$(x)\gamma = \frac{(-c)\gamma[1-\varphi(-x)] + (s-1)[f(x,-c) - f(-c,x)]}{1-\varphi(c)}. \tag{7}$$

Since it is straightforward to verify that any γ of this form satisfies equation (5), we have the non-central version of Theorem 4.1:

Theorem 4.2. *Let the extension* $(\mathbb{R}, \mathbb{R}, \varphi, f)$ *be non-central, and let* $\varphi(c) \neq 1$ *for some* $c \in \mathbb{R}$. *Then the automorphisms of* $(\mathbb{R}, \mathbb{R}, \varphi, f)$ *are just the functions* σ *of the form* $(x,y)\sigma = (xs + (y)\gamma, y)$ *where* $0 < s \in \mathbb{R}$, *and* γ *has the form given in equation* (7).

Theorem 4.3. *If the extension* $(\mathbb{R}, \mathbb{R}, \varphi, f)$ *is non-central, then its group of automorphisms is divisible.*

Proof. Let n be a positive integer and σ an automorphism. Then by Theorem 4.2, $(x,y)\sigma = (xs + (y)\gamma, y)$. Let $0 < r$ and $r^n = s$. Define $\beta\colon \mathbb{R} \to \mathbb{R}$ by

$$\beta = \gamma \cdot \frac{1}{r^{n-1} + \ldots + r + 1}.$$

Then equation (7) is satisfied with β in place of γ and r in place of s. Now let τ be the mapping defined by $(x,y)\tau = (xr + (y)\beta, y)$. Then by direct calculation,

$$(x,y)\tau^n = (xr^n + (y)\beta \cdot (r^{n-1} + \ldots + r + 1), y) = (xs + (y)\gamma, y) = (x,y)\sigma.$$

Therefore, $\tau^n = \sigma$.

The case when the extension $(\mathbb{R}, \mathbb{R}, \varphi, f)$ is central, that is, when $\mathrm{Ker}(\varphi) = \mathbb{R}$, is more difficult. We have already seen that the automorphism group is not divisible in case the group is abelian (Corollary 3.4). But the situation with the remaining cases of non-abelian central extensions is still not known. Thus, we have an open question:

Question. Among archimedean-complete ordered groups of rank 2, is $\mathbb{R} \overleftarrow{\oplus} \mathbb{R}$ the only one whose automorphism group is not divisible?

The other extreme is also possible:

Question. Among archimedean-complete ordered groups of rank 2, are the non-central extensions the only ones whose automorphism groups are divisible?

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Author's address: Ramiro H. Lafuente-Rodríguez, Universidad Mayor de San Andrés, P.O. Box 8669, La Paz, Bolivia, e-mail: ramiro_lafuente@yahoo.com.