Shariefuddin Pirzada; T. A. Naikoo; F. A. Dar

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SIGNED DEGREE SETS IN SIGNED GRAPHS

S. Pirzada, T. A. Naikoo and F. A. Dar, Srinagar

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Abstract. The set D of distinct signed degrees of the vertices in a signed graph G is called its signed degree set. In this paper, we prove that every non-empty set of positive (negative) integers is the signed degree set of some connected signed graph and determine the smallest possible order for such a signed graph. We also prove that every non-empty set of integers is the signed degree set of some connected signed graph.

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1. Introduction

All graphs in this paper are finite, undirected, without loops and multiple edges. A signed graph G is a graph in which each edge is assigned a positive or a negative sign. These were first introduced by Harary [3]. The signed degree of a vertex v_i in a signed graph G is denoted by $\operatorname{sdeg}(v_i)$ (or simply by d_i) and is defined as the number of positive edges incident with v_i less the number of negative edges incident with v_i. So, if v_i is incident with d_i^+ positive edges and d_i^- negative edges, then $\operatorname{sdeg}(v_i) = d_i^+ - d_i^-$. A signed degree sequence $\sigma = [d_1, d_2, \ldots, d_n]$ of a signed graph G is formed by listing the vertex signed degrees in non-increasing order. A sequence $\sigma = [d_1, d_2, \ldots, d_n]$ of integers is graphical if σ is a signed degree sequence of some signed graph. Also, a non-zero sequence $\sigma = [d_1, d_2, \ldots, d_n]$ is a standard sequence if σ is non-increasing, $\sum\limits_{i=1}^{n} d_i$ is even, $d_1 > 0$, each $|d_i| < n$, and $|d_1| \geqslant |d_n|$.

The following result, due to Chartrand et al. [1], gives a necessary and sufficient condition for a sequence of integers to be graphical, which is similar to Hakimi's result for degree sequences [2].

Theorem 1.1. *Let $\sigma = [d_1, d_2, \ldots, d_n]$ be a standard sequence. Then, σ is graphical if and only if there exist integers r and s with $d_1 = r - s$ and $0 \leqslant s \leqslant \frac{1}{2}(n-1-d_1)$ such that*

$$\sigma' = [d_2 - 1, d_3 - 1, \ldots, d_{r+1} - 1, d_{r+2}, d_{r+3}, \ldots, d_{n-s}, d_{n-s+1} + 1, \ldots, d_n + 1]$$

is graphical.

The next characterization for signed degrees in signed graphs is given by Yan et al. [5].

Theorem 1.2. *A standard integral sequence $\sigma = [d_1, d_2, \ldots, d_n]$ is graphical if and only if*

$$\sigma'_m = [d_2 - 1, \ldots, d_{d_1+m+1} - 1, d_{d_1+m+2}, \ldots, d_{n-m}, d_{n-m+1} + 1, \ldots, d_n + 1]$$

is graphical, where m is the maximum non-negative integer such that $d_{d_1+m+1} > d_{n-m+1}$.

In [4], Kapoor et al. proved that every non-empty set of distinct positive integers is the degree set of a connected graph and determined the smallest order for such a graph.

2. Main results

First we have the following definition.

Definition. The set D of distinct signed degrees of the vertices in a signed graph G is called its signed degree set.

Now, we obtain the following results.

Theorem 2.1. *Every non-empty set D of positive integers is the signed degree set of some connected signed graph and the minimum order of such a signed graph is $N+1$, where N is the maximum integer in the set D.*

Proof. Let D be a signed degree set and $n_0(D)$ denotes the minimum order of a signed graph G realizing D. Since N is the maximum integer in D, therefore there is a vertex in G which is adjacent to at least N other vertices with a positive sign. Then, $n_0(D) \geqslant N+1$. Now, if there exists a signed graph of order $N+1$ with D as signed degree set, then $n_0(D) = N+1$. The existence of such a signed graph is obtained by using induction on the number of elements of D.

Let $D = \{d_1, d_2, \ldots, d_n\}$, where $d_1 < d_2 < \ldots < d_n$, be a set of positive integers. For $n = 1$, let G be a complete graph on $d_1 + 1$ vertices, that is K_{d_1+1}, in which each edge is assigned a positive sign. Then,

$$\text{sdeg}(v) = (d_1 + 1 - 1) - 0 = d_1, \quad \text{for all } v \in V(G).$$

Therefore, G is a signed graph with signed degree set $D = \{d_1\}$.

For $n = 2$, let G_1 be a complete graph on d_1 vertices, that is K_{d_1}, in which each edge is assigned a positive sign and let G_2 be a null graph on $d_2 - d_1 + 1 > 0$ vertices, that is $\overline{K}_{d_2-d_1+1}$. Join every vertex of G_1 to each vertex of G_2 with a positive edge, so that we obtain a signed graph G on $d_1 + d_2 - d_1 + 1 = d_2 + 1$ vertices with

$$\text{sdeg}(u) = (d_1 - 1) + (d_2 - d_1 + 1) - 0 = d_2, \quad \text{for all } u \in V(G_1),$$

and

$$\text{sdeg}(v) = (0) + (d_1) - 0 = d_1, \quad \text{for all } v \in V(G_2).$$

Therefore, the signed degree set of G is $D = \{d_1, d_2\}$.

For $n = 3$, let G_1 be a complete graph on d_1 vertices, that is K_{d_1}, in which each edge is assigned a positive sign, G_2 be a complete graph on $d_2 - d_1 + 1 > 0$ vertices, that is $\overline{K}_{d_2-d_1+1}$, in which each edge is assigned a positive sign, and G_3 be a null graph on $d_3 - d_2 > 0$ vertices, that is $\overline{K}_{d_3-d_2}$. Join every vertex of G_1 to each vertex of G_2 with a positive edge and join every vertex of G_1 to each vertex of G_3 with a positive edge, so that we obtain a signed graph G on $d_1 + d_2 - d_1 + 1 + d_3 - d_2 = d_3 + 1$ vertices with

$$\text{sdeg}(u) = (d_1 - 1) + (d_2 - d_1 + 1) + (d_3 - d_2) - 0 = d_3, \quad \text{for all } u \in V(G_1),$$
$$\text{sdeg}(v) = (d_2 - d_1 + 1 - 1) + (d_1) - 0 = d_2, \quad \text{for all } v \in V(G_2),$$

and

$$\text{sdeg}(w) = (0) + (d_1) - 0 = d_1, \quad \text{for all } w \in V(G_3).$$

Therefore, the signed degree set of G is $D = \{d_1, d_2, d_3\}$.

Assume that the result holds for k. We show that the result is true for $k + 1$.

Let $D = \{d_1, d_2, \ldots, d_k, d_{k+1}\}$ be a set of $k + 1$ positive integers with $d_1 < d_2 < \ldots < d_k < d_{k+1}$. Clearly, $0 < d_2 - d_1 < d_3 - d_1 < \ldots < d_k - d_1$. Therefore, by induction hypothesis, there is a signed graph G_1 realizing the signed degree set $D_1 = \{d_2 - d_1, d_3 - d_1, \ldots, d_k - d_1\}$ on $d_k - d_1 + 1$ vertices as $|V(D_1)| < k$. Let G_2 be a complete graph on d_1 vertices, that is K_{d_1}, in which each edge is assigned a positive sign and G_3 be a null graph on $d_{k+1} - d_k > 0$ vertices, that is $\overline{K}_{d_{k+1}-d_k}$.

Join every vertex of G_2 to each vertex of G_1 with a positive edge and join every vertex of G_2 to each vertex of G_3 with a positive edge, so that we obtain a signed graph G on $d_k - d_1 + 1 + d_1 + d_{k+1} - d_k = d_{k+1} + 1$ vertices with

$$\operatorname{sdeg}(u) = (d_i - d_1) + (d_1) - 0 = d_i, \quad \text{for all } u \in V(G_1) \text{ where } 2 \leqslant i \leqslant k,$$
$$\operatorname{sdeg}(v) = (d_1 - 1) + (d_k - d_1 + 1) + (d_{k+1} - d_k) - 0 = d_{k+1}, \quad \text{for all } v \in V(G_2),$$

and

$$\operatorname{sdeg}(w) = (0) + (d_1) - 0 = d_1, \quad \text{for all } w \in V(G_3).$$

Therefore, the signed degree set of G is $D = \{d_1, d_2, \ldots, d_k, d_{k+1}\}$. Clearly, by construction, all the signed graphs are connected. Hence, the result follows.

Theorem 2.2. *Every non-empty set D of negative integers is the signed degree set of some connected signed graph and the minimum order of such a graph is $|M|+1$, where M is the minimum integer in the set D.*

P r o o f. Let D be a signed degree set and let $m_0(D)$ denote the minimum order of a signed graph G realizing D. Since $|M|$ is the maximum integer in D, therefore there is a vertex in G which is adjacent to at least $|M|$ other vertices with a negative sign. Then, $m_0(D) \geqslant |M| + 1$. Now, if there exists a signed graph of order $|M| + 1$ with D as signed degree set, then $m_0(D) = |M| + 1$.

Let $D = \{-d_1, -d_2, \ldots, -d_n\}$, $-d_1 > -d_2 > \ldots > -d_n$, be a set of negative integers where $d_1, d_2, \ldots, d_n$ are positive integers. Now, $D_1 = \{d_1, d_2, \ldots, d_n\}$ is a set of positive integers with $d_1 < d_2 < \ldots < d_n$. By Theorem 2.1, there exists a connected signed graph G_1 on $d_n + 1 = |-d_n| + 1$ vertices with signed degree set $D_1 = \{d_1, d_2, \ldots, d_n\}$. Now, construct a signed graph G from G_1 by interchanging positive edges with negative edges. Then, G is a connected signed graph on $|-d_n|+1$ vertices with signed degree set $D = \{-d_1, -d_2, \ldots, -d_n\}$. This proves the result.

Theorem 2.3. *Every non-empty set D of integers is the signed degree set of some connected signed graph.*

P r o o f. Let D be a set of n integers. We have the following cases.

Case I. D is a set of positive (negative) integers. Then, the result follows by Theorem 2.1 (Theorem 2.2).

Case II. $D = \{0\}$. Then, a null graph G on one vertex, that is K_1, has signed degree set $D = \{0\}$.

Case III. D is a set of non-negative (non-positive) integers. Let $D = D_1 \cup \{0\}$, where D_1 is a set of positive (negative) integers. Then, by Theorem 2.1 (Theorem 2.2), there is a signed graph G_1 with signed degree set D_1. Let G_2 be a null graph on two vertices, that is $\overline{K}_2$. Let $e = uv$ be an edge in G_1 with positive (negative) sign and let $x, y \in V(G_2)$. Add the positive (negative) edges ux and vy, and the negative (positive) edges uy and vx, so that we obtain a connected signed graph G with signed degree set D. We note that addition of such edges do not effect the signed degrees of the vertices of G_1, and the vertices x and y have signed degrees zero each.

Case IV. D is a set of non-zero integers. Let $D = D_1 \cup D_2$, where D_1 is a set of positive integers and D_2 is a set of negative integers. Then, by Theorem 2.1 and Theorem 2.2, there are connected signed graphs G_1 and G_2 with signed degree sets D_1 and D_2. Let $e_1 = uv$ be an edge in G_1 with positive sign and $e_2 = xy$ be an edge in G_2 with negative sign. Add the positive edges ux and vy, and the negative edges uy and vx, so that we obtain a connected signed graph G with signed degree set D. We note that addition of such edges do not effect the signed degrees of the vertices of G_1 and G_2.

Case V. D is a set of integers. Let $D = D_1 \cup D_2 \cup \{0\}$, where D_1 and D_2 are the sets of positive and negative integers respectively. Then, by Theorem 2.1 and Theorem 2.2, there are connected signed graphs G_1 and G_2 with signed degree sets D_1 and D_2. Let G_3 be a null graph on one vertex, that is K_1. Let $e_1 = uv$ be an edge in G_1 with positive sign, and let $x \in V(G_2)$ and $y \in V(G_3)$. Add the positive edges uy and vx, and the negative edges ux and vy, so that we obtain a connected signed graph G with signed degree set D. We note that addition of such edges do not effect the signed degrees of the vertices of G_1 and G_2, and the vertex y has signed degree zero. This completes the proof. □

Theorem 2.4. *If G is a signed graph with vertex set V, where $|V| = r$, and signed degree set $\{d_1, d_2, \ldots, d_n\}$. Then, for each $k \geqslant 1$, there is a signed graph with kr vertices and signed degree set $\{d_1, d_2, \ldots, d_n\}$.*

P r o o f. For each i, $1 \leqslant i \leqslant k$, let G_i be a copy of G with vertex set V_i. Define a signed graph H with vertex set $W = \bigcup_{i=1}^{k} V_i$ where $V_i \cap V_j = \emptyset$ $(i \neq j)$ and the edges of H are the edges of G_i for all i, where $1 \leqslant i \leqslant k$. Then, H is a signed graph on kr vertices with signed degree set $\{d_1, d_2, \ldots, d_n\}$. □

References

[1] *G. Chartrand, H. Gavlas, F. Harary and M. Schultz*: On signed degrees in signed graphs. Czech. Math. J. *44* (1994), 677–690. zbl

[2] *S. L. Hakimi*: On the realizability of a set of integers as degrees of the vertices of a graph. SIAM J. Appl. Math. *10* (1962), 496–506. zbl

[3] *F. Harary*: On the notion of balance in a signed graph. Michigan Math. J. *2* (1953), 143–146. zbl

[4] *S. F. Kapoor, A. O. Polimeni and C. E. Wall*: Degree sets for graphs. Fund. Math. *65* (1977), 189–194. zbl
[5] *J. H. Yan, K. W. Lih, D. Kuo and G. J. Chang*: Signed degree sequences of signed graphs. J. Graph Theory *26* (1997), 111–117. zbl

Authors' address: S. Pirzada, T. A. Naikoo, F. A. Dar, Department of Mathematics, University of Kashmir, Srinagar-190006, India, e-mail: sdpirzada@yahoo.co.in, tariqnaikoo@rediffmail.com, sfarooqdar@yahoo.co.in.