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SUBDIRECTLY IRREDUCIBLE SECTIONALLY PSEUDOCOMPLEMENTED SEMILATTICES

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Abstract. Sectionally pseudocomplemented semilattices are an extension of relatively pseudocomplemented semilattices—they are meet-semilattices with a greatest element such that every section, i.e., every principal filter, is a pseudocomplemented semilattice. In the paper, we give a simple equational characterization of sectionally pseudocomplemented semilattices and then investigate mainly their congruence kernels which leads to a characterization of subdirectly irreducible sectionally pseudocomplemented semilattices.

Keywords: sectionally pseudocomplemented semilattice, weakly standard element

MSC 2000: 06A12, 06D15

A *pseudocomplemented semilattice* is an algebra $(S,\wedge,^*,0)$ of type $(2,1,0)$ such that $(S,\wedge,0)$ is a meet-semilattice with a least element and for all $x,y\in S$,

$$y\leqslant x^* \quad\text{iff}\quad y\wedge x=0. \tag{1}$$

A *relatively pseudocomplemented semilattice* is an algebra $(S,\wedge,*,1)$ of type $(2,2,0)$, where $(S,\wedge,1)$ is a meet-semilattice with a greatest element and for all $x,y,z\in S$,

$$z\leqslant x*y \quad\text{iff}\quad z\wedge x\leqslant y. \tag{2}$$

Relatively pseudocomplemented semilattices appear in the literature also under the name *Brouwerian semilattices* or *implicative semilattices*, respectively (see [7], [8]).

For every $a\in S$, we call the principal filter $[a)=\{x\in S\colon x\geqslant a\}$ a *section* of S. It is easy to see that if $(S,\wedge,*,1)$ is a relatively pseudocomplemented semilattice then for any $a\in S$ and $x\in[a)$, $x^a:=x*a$ is the pseudocomplement of x in

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the section $[a)$, i.e., $y \leqslant x^a$ iff $y \wedge x = a$ for every $y \in [a)$. Thus $([a), \wedge, ^a, a)$ is a pseudocomplemented semilattice, and consequently, for every interval $[a, b]$ of S, $([a, b], \wedge, ^{ab}, a)$ is a pseudocomplemented semilattice with $x^{ab} := x^a \wedge b = (x * a) \wedge b$.

We know that a lattice is relatively complemented if every interval is a complemented lattice. From this point of view, the concept of a relatively pseudocomplemented (semi)lattice may seem to be a bit misleading since a (semi)lattice whose intervals are pseudocomplemented (semi)lattices in general need not be relatively pseudocomplemented. For instance, the pentagon N_5 (see Figure 1) is such a (semi)lattice. This observation leads to the extension of relative pseudocomplementation we deal with in this paper.

1. Sectionally pseudocomplemented semilattices

Definition 1 [4]**.** A meet-semilattice $(S, \wedge, 1)$ with a greatest element is said to be *sectionally pseudocomplemented* if for every $a \in S$, the structure $([a), \wedge, ^a, a)$ is a pseudocomplemented semilattice, i.e., every $x \in [a)$ has the pseudocomplement x^a in the section $[a)$.

Remark. The concept of a sectionally pseudocomplemented lattice was invented by I. Chajda in [2]; similar ideas are contained also in J. C. Varlet's paper [9].

The difficulity arises concerning the number of partial unary operations $^a\colon [a) \to [a)$ which we overcome by defining a new total binary operation "$\circ$" as follows:

$$x \circ y := x^{x \wedge y}. \tag{3}$$

Thus $x \circ y$ is the pseudocomplement of x in the section $[x \wedge y)$.

It can be easily seen that if the relative pseudocomplement $x * y$ of x with respect to y exists then $x \circ y = x * y$. Indeed, we have $(x * y) \wedge x = x \wedge y$, so that $x * y \leqslant x \circ y$, and conversely, $(x \circ y) \wedge x = x \wedge y \leqslant y$ entails $x \circ y \leqslant x * y$.

On the other hand, $x \circ y$ need not be the relative pseudocomplement of x with respect to y. For instance, a non-distributive lattice (that is not relatively pseudocomplemented since relatively pseudocomplemented lattices are distributive) may be sectionally pseudocomplemented.

Example 2. Consider the (meet-semi)lattice that is shown in Figure 1. We obviously have $c \circ a = c^a = a$, while $c * a$ does not exist since the set of all x with $x \wedge c \leqslant a$ has no top element. The operation "$\circ$" is given by Table 1.

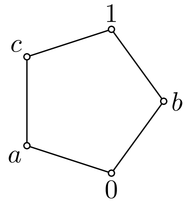

Figure 1

$\circ$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	c	c	1	c	1
c	b	a	b	1	1
1	0	a	b	c	1

Table 1

Remark. In the case of sectionally pseudocomplemented lattices, one can define another binary operation "$\bullet$" by

$$x \bullet y := (x \vee y)^y.$$

Obviously, $x \circ y = x \bullet (x \wedge y)$ and $x \bullet y = (x \vee y) \circ y$.

Theorem 3. *A meet-semilattice* $(S, \wedge, 1)$ *is sectionally pseudocomplemented if and only if there exists a binary operation* "$\circ$" *on* S *such that, for all* $x, y, z \in S$,

(4) $$x \circ x = 1,$$

(5) $$x \wedge (x \circ y) = x \wedge y,$$

(6) $$x \wedge ((x \wedge y) \circ z) = x \wedge (y \circ (x \wedge z)).$$

Proof. Let $(S, \wedge, 1)$ be a sectionally pseudocomplemented semilattice and let a binary operation "$\circ$" be defined by (3). Then clearly $x \circ x = x^x = 1$ and $x \wedge (x \circ y) = x \wedge x^{x \wedge y} = x \wedge y$ since $x^{x \wedge y}$ is the pseudocomplement of x in the section $[x \wedge y)$. Thus (4) and (5) hold.

It is known and straightforward to show that any pseudocomplemented semilattice satisfies the identity

$$x \wedge (x \wedge y)^* = x \wedge y^*.$$

Hence for the section $[x \wedge y \wedge z)$ we have

$$x \wedge ((x \wedge y) \circ z) = x \wedge (x \wedge y)^{x \wedge y \wedge z} = x \wedge y^{x \wedge y \wedge z} = x \wedge (y \circ (x \wedge z))$$

which is (6).

Conversely, let "$\circ$" be a binary operation on S that fulfils the identities (4), (5) and (6). Let $a \in S$. We have to show that for any $x \in [a)$, $x \circ a$ is the pseudocomplement of x in the section. By (5) we see that $x \wedge (x \circ a) = x \wedge a = a$, and so $y \wedge x = a$ for every $y \in [a)$ with $y \leqslant x \circ a$. On the other hand, if $y \wedge x = a$ then (6) and (4) yield $y \wedge (x \circ a) = y \wedge (x \circ (y \wedge a)) = y \wedge ((x \wedge y) \circ a) = y \wedge (a \circ a) = y \wedge 1 = y$, so $y \leqslant x \circ a$. Thus $x^a = x \circ a$. $\square$

Remark. In the light of the previous theorem, sectionally pseudocomplemented semilattices can be treated as algebras $(S, \wedge, \circ, 1)$ of type $(2, 2, 0)$. Of course, they form a variety that is axiomatized, relatively to the variety of meet-semilattices with 1, by the above identities (4), (5) and (6).

Let us recall that a meet-semilattice is said to be *distributive* if $a \geqslant b \wedge c$ implies the existence of $b_1 \geqslant b$ and $c_1 \geqslant c$ with $a = b_1 \wedge c_1$ (see [6]). It is worth noticing that a semilattice is distributive if and only if its filters form a distributive lattice.

Theorem 4. *Every distributive sectionally pseudocomplemented semilattice is relatively pseudocomplemented.*

Proof. Let $(S, \wedge, \circ, 1)$ be a distributive sectionally pseudocomplemented semilattice. We prove that $z \leqslant x \circ y$ is equivalent to $z \wedge x \leqslant y$. Obviously, if $z \leqslant x \circ y$ then $z \wedge x \leqslant (x \circ y) \wedge x = x \wedge y \leqslant y$. Conversely, if $z \wedge x \leqslant y$ then $y = x_1 \wedge z_1$, where $x_1 \geqslant x$ and $z_1 \geqslant z$, whence we obtain $x \wedge y = x \wedge x_1 \wedge z_1 = x \wedge z_1$ which yields $z \leqslant z_1 \leqslant x^{x \wedge y} = x \circ y$. Therefore $x \circ y$ is the relative pseudocomplement of x with respect to y. □

2. Congruence kernels

First, we recall several well-known concepts from universal algebra (see e.g. [1], [3]).

Let A be an algebra with a constant 1. By the *kernel* of a congruence $\Theta \in \mathrm{Con}(A)$ we mean the equivalence class $[1]_\Theta = \{a \in A\colon (a, 1) \in \Theta\}$. An algebra A is called *weakly regular* if $\Theta = \Phi$ whenever $[1]_\Theta = [1]_\Phi$ for any $\Theta, \Phi \in \mathrm{Con}(A)$. A variety $\mathscr{V}$ with a constant 1 is *weakly regular* if every $A \in \mathscr{V}$ is weakly regular. It is known that $\mathscr{V}$ is weakly regular if and only if there exist binary terms $t_1, \ldots, t_n$ $(n \in \mathbb{N})$ such that

$$t_1(x, y) = \ldots = t_n(x, y) = 1 \quad \text{iff} \quad x = y.$$

An algebra A with a constant 1 is *arithmetical at* 1 if for all $\Theta, \Phi, \Psi \in \mathrm{Con}(A)$,

$$[1]_{\Theta \circ \Phi} = [1]_{\Phi \circ \Theta} \quad \text{and} \quad [1]_{\Theta \cap (\Phi \vee \Psi)} = [1]_{(\Theta \cap \Phi) \vee (\Theta \cap \Psi)}.$$

A variety $\mathscr{V}$ is *arithmetical at* 1 if so is each $A \in \mathscr{V}$. Arithmeticity at 1 can be captured by a simple Maltsev type condition: a variety $\mathscr{V}$ is arithmetical at 1 if and only if there exists a binary term t with

$$t(x, x) = t(1, x) = 1 \quad \text{and} \quad t(x, 1) = x.$$

Finally, a variety $\mathscr{V}$ is called *congruence distributive* if the congruence lattice $\mathrm{Con}(A)$ of every $A \in \mathscr{V}$ is distributive.

Theorem 5. *The variety of all sectionally pseudocomplemented semilattices is weakly regular and arithmetical at* 1.

Proof. Consider the terms $t_1(x, y) = x \circ y$ and $t_2(x, y) = y \circ x$. Clearly, $t_1(x, x) = t_2(x, x) = 1$. Conversely, if $t_1(x, y) = t_2(x, y) = 1$ then $x = x \wedge (x \circ y) = x \wedge y = y \wedge (y \circ x) = y$ by the identity (5) of Theorem 3.

For arithmeticity at 1, consider the term $t(x, y) = y \circ x$. Then certainly $t(x, x) = 1$, $t(x, 1) = 1$ and $t(1, x) = x$. □

Corollary 6. *The variety of all sectionally pseudocomplemented semilattices is congruence distributive.*

As known (e.g. [8], [7]), filters of relatively pseudocomplemented semilattices are in a one-to-one correspondence with their congruence relations. More precisely, given a filter F of $(S, \wedge, *, 1)$, the relation Θ_F defined via

$$(x, y) \in \Theta_F \quad \text{iff} \quad (x * y) \wedge (y * x) \in F,$$

or equivalently,

$$(x, y) \in \Theta_F \quad \text{iff} \quad x \wedge a = y \wedge a \text{ for some } a \in F,$$

is a congruence on $(S, \wedge, *, 1)$ such that $[1]_{\Theta_F} = F$. This is in contrast to the situation for sectionally pseudocomplemented semilattices: there exist filters that are not congruence kernels (see Example 8). However, any congruence is determined by its kernel in the following manner:

Lemma 7. *Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice and let F be a filter of a semilattice $(S, \wedge)$. Define a binary relation Φ_F by*

$$(x, y) \in \Phi_F \quad \textit{iff} \quad x \wedge a = y \wedge a \textit{ for some } a \in F. \tag{7}$$

Then F is the kernel of a congruence $\Theta \in \mathrm{Con}(S)$ if and only if $\Theta = \Phi_F$.

In particular, a principal filter $[a)$ is the kernel of $\Theta \in \mathrm{Con}(S)$ if and only if $\Theta = \Phi_a$, where

$$(x, y) \in \Phi_a \quad \textit{iff} \quad x \wedge a = y \wedge a. \tag{8}$$

P r o o f. Let Θ be a congruence such that $F = [1]_\Theta$. If $(x, y) \in \Theta$ then $(x \circ y, 1), (y \circ x, 1) \in \Theta$, i.e., $x \circ y, y \circ x \in F$ whence it follows that $(x \circ y) \wedge (y \circ x) \in F$. It is obvious that $x \wedge (x \circ y) \wedge (y \circ x) = x \wedge y = y \wedge (x \circ y) \wedge (y \circ x)$, so we may take $a = (x \circ y) \wedge (y \circ x)$ which yields $(x, y) \in \Phi_F$. If $(x, y) \in \Phi_F$ then $x \wedge a = y \wedge a$ for some $a \in F = [1]_\Theta$. Since $(a, 1) \in \Theta$ implies $(x, x \wedge a) \in \Theta$ and $(y, y \wedge a) \in \Theta$, we have $(x, y) \in \Theta$. Thus $\Theta = \Phi_F$.

Conversely, one readily sees that $[1]_{\Phi_F} = F$, so if $\Theta = \Phi_F \in \mathrm{Con}(S)$ then F is the kernel of Θ. $\square$

Example 8. Let us return to Example 2. Then Φ_b is an equivalence with the partition $\{b, 1\}, \{0, a, c\}$, but it is not a congruence since $(a, c) \in \Phi_b$ while $(c \circ a, c \circ c) = (a, 1) \notin \Phi_b$.

Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice, $a \in S$ and $(a] = \{x \in S \colon\ x \leqslant a\}$. Then upon defining

$$x \circ_a y := (x \circ y) \wedge a,$$

the structure $((a], \wedge, \circ_a, a)$ is a sectionally pseudocomplemented semilattice, too. Hence

Corollary 9. *A principal filter $[a)$ is a congruence kernel if and only if the mapping $f \colon\ x \mapsto x \wedge a$ is a homomorphism of $(S, \wedge, \circ, 1)$ onto $((a], \wedge, \circ_a, a)$.*

P r o o f. Assume first that $[a) = [1]_\Theta$, where $\Theta = \Phi_a \in \mathrm{Con}(S)$. From $(a, 1) \in \Theta$ it follows that $(x, x \wedge a), (y, y \wedge a) \in \Theta$ and hence $(x \circ y, (x \wedge a) \circ (y \wedge a)) \in \Theta$. Seeing that $\Theta = \Phi_a$, we obtain $(x \circ y) \wedge a = ((x \wedge a) \circ (y \wedge a)) \wedge a$, i.e., $f(x \circ y) = f(x) \circ_a f(y)$. On the other hand, if $f \colon\ x \mapsto x \wedge a$ is a homomorphism then Φ_a is equal to Θ_f, the congruence induced by f, thus $[a) = [1]_{\Theta_f}$. $\square$

As an immediate consequence we obtain:

Corollary 10. *A principal filter $[a)$ is a congruence kernel if and only if*

$$(x \circ y) \wedge a = ((x \wedge a) \circ (y \wedge a)) \wedge a$$

for all $x, y \in S$.

It turns out that relatively pseudocomplemented semilattices are those sectionally pseudocomplemented semilattices in which every filter is a congruence kernel:

Theorem 11. *Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice. Then the mapping $\Theta \mapsto [1]_\Theta$ is a one-to-one correspondence between congruences and filters, the inverse of which is given by $F \mapsto \Phi_F$, if and only if $(S, \wedge, \circ, 1)$ is a relatively pseudocomplemented semilattice.*

P r o o f. If every filter F is a congruence kernel then the lattice of all filters is isomorphic to the congruence lattice $\mathrm{Con}(S)$. Thus the lattice of filters is distributive which implies that S is a distributive semilattice, and hence by Theorem 4, $(S, \wedge, \circ, 1)$ is a relatively pseudocomplemented semilattice. $\square$

Given a lattice $(L, \vee, \wedge)$, an element $a \in L$ is called *standard* (see [6]) if

$$x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$$

for all $x, y \in L$.

Let $(S, \wedge, 1)$ be a meet-semilattice with a greatest element. A filter F is called *standard* if it is a standard element of the lattice of all filters of $(S, \wedge, 1)$; this is equivalent to

$$[x) \cap (F \vee [y)) = ([x) \cap F) \vee ([x) \cap [y))$$

for all $x, y \in S$.

It was proved in [5] that the congruence kernels of sectionally pseudocomplemented lattices are precisely the standard filters, but this is not the case of sectionally pseudocomplemented semilattices:

Example 12. In the pentagon (cf. Example 2, Figure 1), $F = \{c, 1\}$ is the kernel of the congruence given by the partition $\{c, 1\}$, $\{a\}$, $\{0, b\}$, but F is not a standard filter since $[a) \cap (F \vee [b)) = [a)$ while $([a) \cap F) \vee ([a) \cap [b)) = [c)$.

In order to capture the congruence kernels of sectionally pseudocomplemented semilattice, we extend the concept of standardness as follows:

Let $(L, \vee, \wedge)$ be a lattice. We say that $a \in L$ is *weakly standard* if for all $x, y \in L$,

$$x \leqslant y \text{ implies } x \vee (a \wedge y) = (x \vee a) \wedge y.$$

Theorem 13. *Let $(L, \vee, \wedge)$ be a lattice. An element $a \in L$ is weakly standard if and only if there exist no $x_1, y_1 \in L$ such that $a \wedge x_1 = a \wedge y_1$, x_1, a, y_1 and $a \vee x_1 = a \vee y_1$ form a sublattice isomorphic to the pentagon N_5 (see Figure 2).*

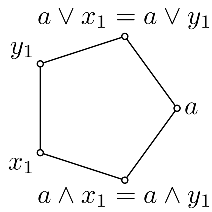

Figure 2

P r o o f. It is clear that if there exist such $x_1, y_1 \in L$ then a is not weakly standard since $x_1 \vee (a \wedge y_1) = x_1$ while $(x_1 \vee a) \wedge y_1 = y_1$.

Conversely, assume that $a \in L$ is not weakly standard, i.e., there are $x, y \in L$ with $x \leqslant y$, but $x \vee (a \wedge y) < (x \vee a) \wedge y$. We put $x_1 = x \vee (a \wedge y)$ and $y_1 = (x \vee a) \wedge y$. Then $a \vee x_1 = a \vee x \vee (a \wedge y) = a \vee x$ and $a \vee y_1 = a \vee ((x \vee a) \wedge y) \leqslant a \vee (x \vee a) = a \vee x \leqslant a \vee ((x \vee a) \wedge y) = a \vee y_1$ as $x = (x \vee a) \wedge x \leqslant (x \vee a) \wedge y$, so $a \vee x_1 = a \vee y_1$.

Similarly, $a \wedge y_1 = a \wedge (x \vee a) \wedge y = a \wedge y$ and $a \wedge x_1 = a \wedge (x \vee (a \wedge y)) \geqslant a \wedge y \geqslant a \wedge (x \vee (a \wedge y)) = a \wedge x_1$ since $y = y \vee (a \wedge y) \geqslant x \vee (a \wedge y)$, thus $a \wedge x_1 = a \wedge y_1$.

Therefore, one readily sees that $a \wedge x_1 = a \wedge y_1$, x_1, a, y_1, $a \vee x_1 = a \vee y_1$ form a sublattice of L that is isomorphic to N_5 (cf. Figure 2). □

In a semilattice $(S, \wedge, 1)$, a filter F is called *weakly standard* if F is a weakly standard element of the lattice of all filters of S. It is straightforward to prove that F is weakly standard if and only if for all $x, y \in S$,

$$x \leqslant y \;\text{ implies }\; [x) \cap (F \vee [y)) = ([x) \cap F) \vee [y).$$

Lemma 14. *Let $(S, \wedge, \circ, 1)$ be a sectionally pseudocomplemented semilattice, F a weakly standard filter and $x, y \in S$ with $x \leqslant y$. Then $(x, y) \in \Phi_F$ if and only if there exists $b \in F$ such that $x = y \wedge b$.*

P r o o f. Let $(x, y) \in \Phi_F$, i.e. $x \wedge a = y \wedge a$ for some $a \in F$. Clearly, $x \wedge a = y \wedge a \in F \vee [y)$ and so $x \in [x) \cap (F \vee [y)) = ([x) \cap F) \vee [y)$ since F is a weakly standard filter. Then $x \geqslant b \wedge y$ for some $b \in [x) \cap F$, i.e. $b \geqslant x$, whence $b \wedge y \geqslant x \geqslant b \wedge y$, so $x = b \wedge y$.

Conversely, if there is $b \in F$ with $x = y \wedge b$, then $x \wedge b = y \wedge b$, so that $(x, y) \in \Phi_F$. □

For sectionally pseudocomplemented semilattices we have the following analogue of [5]:

Theorem 15. *Let* $(S, \wedge, \circ, 1)$ *a sectionally pseudocomplemented semilattice and* F *a filter of a semilattice* $(S, \wedge, 1)$. *Then* F *is a congruence kernel if and only if* F *is a weakly standard filter.*

P r o o f. Let $F = [1]_\Theta$ for some $\Theta \in \mathrm{Con}(S)$. We have to show that $[x) \cap (F \vee [y)) \subseteq ([x) \cap F) \vee [y)$ provided $x \leqslant y$. For, let $z \in [x) \cap (F \vee [y))$. Then $z \geqslant x$ and $z \geqslant a \wedge y$ for some $a \in F$, so that $a \wedge y \wedge z = a \wedge y$ which means $(y, y \wedge z) \in \Phi_F = \Theta$. Hence $1 = y \circ y \;\Theta\; y \circ (y \wedge z) = y \circ z$, thus $y \circ z \in F = [1]_\Theta$. But we also have $y \circ z \in [x)$ since $x \leqslant y \wedge z \leqslant y \circ z$, so $y \circ z \in F \cap [x)$. Finally, $y \wedge (y \circ z) = y \wedge z \leqslant z$ yields $z \in (F \cap [x)) \vee [y)$.

Conversely, suppose that a filter F is weakly standard. First, we note that the relation Φ_F defined by (8) is compatible with "$\wedge$".

Now, we put $\Theta = \Phi_F$. It is obvious that $F = [1]_\Theta$, so we have to prove that Θ is compatible with the operation "$\circ$". For that purpose, it suffices to show that the quotient semilattice $(S/\Theta, \wedge, [1]_\Theta)$ is a sectionally pseudocomplemented semilattice in which $[x]_\Theta \circ [y]_\Theta = [x \circ y]_\Theta$.

Let $[a]_\Theta \in S/\Theta$ and $[x]_\Theta \geqslant [a]_\Theta$. Without loss of generality we may assume that $x \geqslant a$. We show that $[x \circ a]_\Theta$ is the pseudocomplement of $[x]_\Theta$ in the section $[[a]_\Theta)$ of the quotient semilattice. One readily sees that $[x]_\Theta \wedge [x \circ a]_\Theta = [x \wedge (x \circ a)]_\Theta = [x \wedge a]_\Theta = [a]_\Theta$. Let now $[y]_\Theta \wedge [x]_\Theta = [a]_\Theta$; again, we assume that $y \geqslant a$. Then by Lemma 14, $[x \wedge y]_\Theta = [a]_\Theta$ along with $a \leqslant x \wedge y$ yields the existence of $b \in F$ with $a = x \wedge y \wedge b$ whence $y \wedge b \leqslant x^a = x \circ a$. This implies that $[y]_\Theta = [y \wedge b]_\Theta \leqslant [x \circ a]_\Theta$.

Therefore, $(S/\Theta, \wedge, [1]_\Theta)$ is a sectionally pseudocomplemented semilattice with $[x]_\Theta \circ [y]_\Theta = [x]_\Theta^{[x]_\Theta \wedge [y]_\Theta} = [x]_\Theta^{[x \wedge y]_\Theta} = [x \circ (x \wedge y)]_\Theta = [x \circ y]_\Theta$. $\square$

Corollary 16. *A sectionally pseudocomplemented semilattice* $(S, \wedge, \circ, 1)$ *is subdirectly irreducible if and only if it has a smallest non-trivial weakly standard filter.*

Since each standard filter is weakly standard, we obtain

Corollary 17. *Let* $(S, \wedge, \circ, 1)$ *be a sectionally pseudocomplemented semilattice. Then every standard filter of* $(S, \wedge, 1)$ *is the kernel of some congruence* $\Theta \in \mathrm{Con}(S)$.

It is well-known (e.g. [7], [8]) that a relatively pseudocomplemented semilattice $(S, \wedge, *, 1)$ is subdirectly irreducible if and only if it has a smallest non-trivial filter; in other words, the set $S \setminus \{1\}$ has a greatest element. This easily follows from the fact that filters agree with congruence kernels. Sectionally pseudocomplemented semilattices generalize relative pseudocomplemented ones, however, the situation is rather different.

Lemma 18. *For any sectionally pseudocomplemented semilattice* $(S, \wedge, \circ, 1)$*, if* $S \setminus \{1\}$ *has a greatest element then* $(S, \wedge, \circ, 1)$ *is subdirectly irreducible.*

P r o o f. Let $\Theta \in \mathrm{Con}(S) \setminus \{\omega\}$ and let u be a greatest element of $S \setminus \{1\}$. Then $(a, b) \in \Theta$ for some $a \neq b$; of course, we may assume that $a < b$. If $b = 1$ then clearly $(u, 1) \in \Theta$. If $b \leqslant u$ then $(a, b) \in \Theta$ yields $(b \circ a, 1) \in \Theta$. But $b \circ a \leqslant u$ since $b \circ a$ is the pseudocomplement of b in the section $[a)$, and hence $(b \circ a, 1) \in \Theta$ implies $(u, 1) \in \Theta$. Thus $\Theta(u, 1) \subseteq \Theta$ proving that $\Theta(u, 1)$ is the monolith of the congruence lattice $\mathrm{Con}(S)$. □

Unfortunately, the converse statement fails to be true:

Example 19. Consider the sectionally pseudocomplemented (semi)lattice S as shown in Figure 3; the operation "$\circ$" is given by Table 2. By Theorem 13 it is easy to see that S and $[1)$ are the only weakly standard filters of S and so S is simple by Theorem 15.

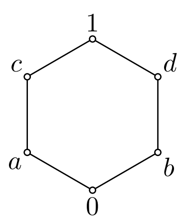

Figure 3

$\circ$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	c	1	1
c	d	a	d	1	d	1
d	c	c	b	c	1	1
1	0	a	b	c	d	1

Table 2

Example 20. Another example of a subdirectly irreducible sectionally pseudocomplemented semilattice such that the set of all $x \neq 1$ has no greatest element is that from Example 2. There are two proper weakly standard filters, namely, $[a)$ and $[c)$. Thus the congruence lattice is a four-element chain $\omega \subset \Phi_c \subset \Phi_a \subset \iota$, and consequently, N_5 (as a sectionally pseudocomplemented semilattice) is subdirectly irreducible.

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