Jiří Močkoř

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EXTENSIONAL SUBOBJECTS IN CATEGORIES OF Ω-FUZZY SETS

Jiří Močkoř, Ostrava

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Abstract. Two categories $\mathbf{Set}(\Omega)$ and $\mathbf{SetF}(\Omega)$ of fuzzy sets over an MV-algebra Ω are investigated. Full subcategories of these categories are introduced consisting of objects $(\mathrm{sub}(A,\delta),\sigma)$, where $\mathrm{sub}(A,\delta)$ is a subset of all extensional subobjects of an object (A,δ). It is proved that all these subcategories are quasi-reflective subcategories in the corresponding categories.

Keywords: MV-algebras, similarity relation, quasi-reflective subcategory

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Introduction

In a fuzzy set theory there are several categories which play an important role in fuzzy logic interpretation. Two of these categories of fuzzy sets over an MV-algebra $\Omega = (L, \otimes, \rightarrow)$ will be investigated in the paper. The first one is the category $\mathbf{Set}(\Omega)$ with objects (A, δ) where A is a set and $\delta\colon\ A \times A \rightarrow \Omega$ is a similarity relation such that

(i) $(\forall x \in A) \quad \delta(x, x) = 1$,
(ii) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,
(iii) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes \delta(y, z) \leqslant \delta(x, z)$.

A morphism $f\colon\ (A, \delta) \rightarrow (B, \gamma)$ in $\mathbf{Set}(\Omega)$ is a map $f\colon\ A \times B \rightarrow \Omega$ satisfying the following conditions.

(1) $(\forall x, z \in A)(\forall y \in B) \quad \delta(x, z) \otimes f(x, y) \leqslant f(z, y)$,
(2) $(\forall x \in A)(\forall y, z \in B) \quad \gamma(y, z) \otimes f(x, y) \leqslant f(x, z)$,
(3) $(\forall x \in A)(\forall y, z \in B) \quad f(x, y) \otimes f(x, z) \leqslant \gamma(y, z)$,

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(4) $(\forall x \in A) \quad 1 = \bigvee\{f(x,y)\colon\ y \in B\}$.

If $f\colon\ \mathbf{A} \to \mathbf{B}$ and $g\colon\ \mathbf{B} \to \mathbf{C}$ are two morphisms then their composition is the function $g \circ f\colon\ A \times C \to \Omega$ such that

$$g \circ f(x,z) = \bigvee_{y \in B} (f(x,y) \otimes g(y,z)).$$

The other category $\mathbf{SetF}(\Omega)$ will have the same objects as the category $\mathbf{Set}(\Omega)$. A morphism $f\colon\ (A,\delta) \to (B,\gamma)$ in $\mathbf{SetF}(\Omega)$ is a map $f\colon\ A \to B$ such that $(\forall x, y \in A) \quad \gamma(f(x), f(y)) \geqslant \delta(x,y)$.

In [9] we investigated some principal properties of the category $\mathbf{SetF}(\Omega)$ and we proved that the *extensional subobjects* of objects (A,δ) in this category $\mathbf{SetF}(\Omega)$ can be identified with some *characteristic morphism* $(A,\delta) \to (\Omega^*, \mu)$. Namely we proved that if $S\colon\ \mathbf{SetF}(\Omega) \to \mathbf{Set}$ is a functor such that $S(A,\delta) = \{s\colon$ s is an extensional subobject of $(A,\delta)\}$ then there exists a natural isomorphism

$$\zeta\colon\ S(-) \to \mathrm{Hom}_{\mathbf{SetF}(\Omega)}(-, \Omega^*).$$

This classification property, which is one of the most important properties of a topos category, is frequently used for interpretation of formulas of fuzzy logic in the category $\mathbf{SetF}(\Omega)$ in such a way that interpretation of a fuzzy logic formula is defined as a special extensional subobject of some object (A,δ). Hence, it seems natural that extensional subobjects of objects in the category $\mathbf{SetF}(\Omega)$ play an important role for further investigation of that category.

In this paper we are interested in the following problem related to extensional subobjects. For any object (A,δ) of the category $\mathbf{SetF}(\Omega)$ we can define a set $\Omega^{(A,\delta)}$ of all (or some special, respectively) extensional subobjects of (A,δ). This set can be transformed (in several different ways) into an object of the category $\mathbf{SetF}(\Omega)$. In that way we obtain a full subcategory $\Omega^{\mathbf{SetF}(\Omega)}$ of the category $\mathbf{SetF}(\Omega)$ consisting of these special objects. We will be interested in conditions under which that subcategory $\Omega^{\mathbf{SetF}(\Omega)}$ is a quasi-reflective subcategory in $\mathbf{SetF}(\Omega)$. Recall that a subcategory $\mathbf{L}$ of a category $\mathbf{K}$ is a *quasi-reflective* subcategory in $\mathbf{K}$ if there exists a functor $G\colon\ \mathbf{K} \to \mathbf{L}$ such that for any object $a \in \mathbf{K}$ there is a morphism $a \xrightarrow{u_a} G(a)$ such that for any object $b \in \mathbf{L}$ and any morphism $f\colon\ a \to b$ (in $\mathbf{K}$) there exists a morphism (in general non unique) $\hat{f}\colon\ G(a) \to b$ such that the following diagram commutes:

$$\begin{tikzcd} a \arrow[r, "u_a"] \arrow[d, "f"'] & G(a) \arrow[d, "\hat{f}"] \\ b \arrow[r, equal] & b \end{tikzcd}$$

A functor G is then called a *quasi-reflector*. We will be interested in several subcategories of the category $\mathbf{SetF}(\Omega)$ consisting of various objects $(\mathrm{sub}(A,\delta),\sigma)$, where $\mathrm{sub}(A,\delta)$ will be a subset of the set of all extensional subobjects of (A,δ) and σ will be a similarity relation defined on that subset. We prove that all these subcategories are quasi-reflective subcategories in the category $\mathbf{SetF}(\Omega)$. We also introduce the notion of a weak singleton extensional subobject of (A,δ) in the category $\mathbf{Set}(\Omega)$ and we prove that a subcategory consisting of these subobjects is also a quasi-reflective subcategory of the category $\mathbf{Set}(\Omega)$.

Subcategories of extensional subobjects

We show firstly a simple result which states the existence of a functor between the categories $\mathbf{SetF}(\Omega)$ and $\mathbf{Set}(\Omega)$.

Lemma 1. *There exists a functor* $F\colon \mathbf{SetF}(\Omega)\to\mathbf{Set}(\Omega)$.

P r o o f. For $(A,\delta)\in\mathbf{SetF}(\Omega)$ we set $F(A,\delta)=(A,\delta)$ and for a morphism $f\colon (A,\delta)\to(B,\gamma)$ in $\mathbf{SetF}(\Omega)$ we define a map $F(f)\colon A\times B\to\Omega$ such that $F(f)(a,b)=\gamma(f(a),b)$ for any $a\in A, b\in B$. Then $F(f)$ is a morphism in $\mathbf{Set}(\Omega)$. In fact, we have for example

$$\begin{aligned}F(f)(a,b)\otimes\delta(a,a')&=\gamma(f(a),b)\otimes\delta(a,a')\\&\leqslant\gamma(f(a),b)\otimes\gamma(f(a),f(a'))\leqslant\gamma(f(a'),b)=F(f)(a',b).\end{aligned}$$

□

Recall that an *extensional subobject* of (A,δ) in the category $\mathbf{SetF}(\Omega)$ is a map $s\colon A\to\Omega$ such that

$$s(x)\otimes\delta(x,y)\leqslant s(y).$$

An extensional subobject can be defined in the category $\mathbf{Set}(\Omega)$ as well. In fact, it is clear that $(\Omega,\leftrightarrow)$ is an object in $\mathbf{SetF}(\Omega)$, where $\alpha\leftrightarrow\beta=(\alpha\to\beta)\wedge(\beta\to\alpha)$. Then s is an extensional subobject of (A,δ) in the category $\mathbf{SetF}(\Omega)$ if $s\colon (A,\delta)\to(\Omega,\leftrightarrow)$ is a morphism in $\mathbf{SetF}(\Omega)$. Analogously s will be called an *extensional subobject* of (A,δ) in $\mathbf{Set}(\Omega)$ if $s\colon (A,\delta)\to(\Omega,\leftrightarrow)$ is a morphism in $\mathbf{Set}(\Omega)$, i.e.

(1) $(\forall a,a'\in A)(\alpha\in\Omega)\quad s(a,\alpha)\otimes\delta(a,a')\leqslant s(a',\alpha)$,
(2) $(\forall a\in A)(\forall\beta,\alpha\in\Omega)\quad s(a,\alpha)\otimes(\alpha\leftrightarrow\beta)\leqslant s(a,\beta)$,
(3) $(\forall a\in A)(\alpha,\beta\in\Omega)\quad s(a,\alpha)\otimes s(a,\beta)\leqslant\alpha\leftrightarrow\beta$,
(4) $(\forall x\in A)\quad 1=\bigvee_{\alpha\in\Omega}s(x,\alpha)$.

An extensional subobject s of (A,δ) in $\mathbf{SetF}(\Omega)$ is called *normal* if $\bigvee_{a\in A} s(a)=1$. Then an extensional subobject t of (A,δ) in $\mathbf{Set}(\Omega)$ is called *normal* if $\bigvee_{a\in A} t(a,1) = 1$. Moreover, we say that t is a *weak singleton* of (A,δ) in $\mathbf{Set}(\Omega)$ if t is a normal extensional subobject of (A,δ) in $\mathbf{Set}(\Omega)$ and $t(a,1)\otimes t(b,1)\leqslant \delta(a,b)$ for all $a,b\in A$. A special normal extensional subobject in $\mathbf{SetF}(\Omega)$ is called a singleton. Recall that an extensional subobject $s\colon\ A\to\Omega$ of (A,δ) (in the category $\mathbf{SetF}(\Omega)$) is a *singleton* if it satisfies the condition

$$(\forall x,y\in A)\quad s(x)\otimes s(y)\leqslant \delta(x,y).$$

It is clear that the map $\{a\}=\delta(a,-)\colon\ A\to\Omega$ is an example of a singleton for any $a\in A$. On the other hand an object (A,δ) is called *complete* if for any singleton s of (A,δ) there exists $a\in A$ such that $s=\delta(a,-)$.

Let (A,δ) be an object of $\mathbf{Set}(\Omega)$ (or $\mathbf{SetF}(\Omega)$). We introduce the following notation.

$$\begin{aligned}
\Omega^{(A,\delta)} &= \{s\colon\ s \text{ is an extensional subobject of } (A,\delta) \text{ in } \mathbf{SetF}(\Omega)\},\\
\Omega_1^{(A,\delta)} &= \{s\colon\ s \text{ is an extensional and normal subobject of } (A,\delta) \text{ in } \mathbf{SetF}(\Omega)\},\\
\text{w-singl}(A,\delta) &= \{s\colon\ s \text{ is a weak singleton of } (A,\delta) \text{ in } \mathbf{Set}(\Omega)\},\\
\text{singl}(A,\delta) &= \{s\in\Omega_1^{(A,\delta)}\colon\ s \text{ is singleton of } (A,\delta) \text{ in } \mathbf{SetF}(\Omega)\}.
\end{aligned}$$

All the previous sets can be transformed into objects of categories $\mathbf{Set}(\Omega)$ and $\mathbf{SetF}(\Omega)$, respectively. In fact, for any object (A,δ) and for any $s,t\in\Omega^{(A,\delta)}$, $p,q\in$ w-singl(A,δ) we set

$$\begin{aligned}
\sigma(s,t) &= \sigma_{(A,\delta)}(s,t) = \bigwedge_{x\in A} s(x)\leftrightarrow t(x),\\
\tau(s,t) &= \tau_{(A,\delta)}(s,t) = \begin{cases} \bigvee_{x\in A} s(x)\otimes t(x), & \text{if } s\neq t,\\ 1, & \text{if } s=t,\end{cases}\\
\varrho(p,q) &= \varrho_{(A,\delta)}(p,q) = \bigwedge_{x\in A} p(x,1)\leftrightarrow q(x,1).
\end{aligned}$$

Lemma 2. *For any object (A,δ) there exists a morphism* $\hat{}\colon\ \text{singl}(A,\delta),\sigma_{(A,\delta)}) \to (\text{w-singl}(A,\delta),\varrho_{(A,\delta)})$.

Proof. For $s\in\text{singl}(A,\delta)$ we set $\hat{s}(a,\alpha)=s(a)\leftrightarrow\alpha$. It is then clear that $\hat{s}=F(s)$ (see Lemma 1), since $s\colon\ (A,\delta)\to(\Omega,\leftrightarrow)$ is a morphism in $\mathbf{SetF}(\Omega)$. It follows that $\hat{s}$ is an extensional (and clearly normal) subobject in $\mathbf{Set}(\Omega)$. Since s is a singleton, $\hat{s}$ is a weak singleton. Moreover according to Lemma 5 for any $s,t \in \Omega_1^{(A,\delta)}$ we have

$$\begin{aligned}\varrho_{A,\delta)}(\hat{s},\hat{t}) &= \bigwedge_{a\in A} (s(a) \leftrightarrow 1) \leftrightarrow (t(a) \leftrightarrow 1) \\ &= \bigwedge_{a\in A} s(a) \leftrightarrow t(a) = \sigma_{(A,\delta)}(s,t).\end{aligned}$$

□

Lemma 3. *For any object (A,δ) the pairs $(\Omega^{(A,\delta)}, \sigma_{(A,\delta)})$, $(\Omega^{(A,\delta)}, \tau_{(A,\delta)})$, $(\text{w-singl}(A,\delta), \varrho_{(A,\delta)})$ and $(\text{singl}(A,\delta), \tau_{(A,\delta)})$, respectively are objects of the category $\mathbf{Set}(\Omega)$ (and $\mathbf{SetF}(\Omega)$, simultaneously).*

The proof of this lemma can be done by a simple computation.

Lemma 4. *Let Ω be an MV-algebra. Let $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ be two sets of elements of Ω and let $p\colon I \to I$ be a bijection map.*

(1) $\bigvee_{i\in I} a_i \leftrightarrow \bigvee_{i\in I} b_{i\in I} \geqslant \bigwedge_{j\in I} (a_j \leftrightarrow b_{p(j)})$,

(2) $\bigwedge_{i\in I} a_i \leftrightarrow \bigwedge_{i\in I} b_i \geqslant \bigwedge_{j\in I} (a_j \leftrightarrow b_{p(j)})$.

The proof can be done by a simple computation and will be omitted.

Lemma 5. *Let Ω be an MV-algebra.*

(1) $(\forall a,b,x \in \Omega) \quad (x \leftrightarrow a) \leftrightarrow (x \leftrightarrow b) \geqslant a \leftrightarrow b$.

(2) $(\forall a,b,c,d \in \Omega) \quad (a \leftrightarrow c) \leftrightarrow (b \leftrightarrow d) \geqslant (a \leftrightarrow b) \otimes (c \leftrightarrow d)$.

P r o o f. (1) We have

$$\begin{aligned}(x \to a) \leftrightarrow (x \to b) \geqslant (x \leftrightarrow x) \otimes (a \leftrightarrow b) = a \leftrightarrow b,\\ (a \to x) \leftrightarrow (b \to x) \geqslant (a \leftrightarrow b) \otimes (x \leftrightarrow x) = a \leftrightarrow b,\end{aligned}$$

and it follows that

$$\begin{aligned}(x \leftrightarrow a) \leftrightarrow (x \leftrightarrow b) &= ((x \to a) \wedge (a \to x)) \leftrightarrow ((x \to b) \wedge (b \to x)) \\ &\geqslant ((x \to a) \leftrightarrow (x \to b)) \wedge ((a \to x) \leftrightarrow (b \to x)) \geqslant a \leftrightarrow b.\end{aligned}$$

(2) According to Lemma 4 we have

$$\begin{aligned}(a \leftrightarrow c) \leftrightarrow (b \leftrightarrow d) &= ((a \to c) \wedge (c \to a)) \leftrightarrow ((b \to d) \wedge (d \to b)) \\ &\geqslant ((a \to c) \leftrightarrow (b \to d)) \wedge ((c \to a) \leftrightarrow (d \to b)) \\ &\geqslant ((a \leftrightarrow b) \otimes (c \leftrightarrow d)) \wedge ((c \leftrightarrow d) \otimes (a \leftrightarrow b) = (a \leftrightarrow b) \otimes (c \leftrightarrow d).\end{aligned}$$

□

We consider the following subcategories of the categories $\mathbf{Set}(\Omega)$ and $\mathbf{SetF}(\Omega)$, respectively.

(1) A full subcategory $\mathbf{SetF}(\Omega)_{\mathrm{comp}} \hookrightarrow \mathbf{SetF}(\Omega)$ consisting of complete objects of the category $\mathbf{SetF}(\Omega)$,

(2) A full subcategory $\Omega^{\mathbf{SetF}(\Omega)}_{\leftrightarrow} \hookrightarrow \mathbf{SetF}(\Omega)$ with objects $(\Omega^{(A,\delta)}, \sigma_{(A,\delta)})$ for any object (A,δ),

(3) A full subcategory $\Omega^{\mathbf{SetF}(\Omega)}_{\otimes} \hookrightarrow \mathbf{SetF}(\Omega)$ with objects $(\Omega^{(A,\delta)}, \tau_{(A,\delta)})$ for any object (A,δ),

(4) A full subcategory $\Omega^{\mathbf{Set}(\Omega)}_{1,\otimes} \hookrightarrow \mathbf{Set}(\Omega)$ with objects $(\Omega_1^{(A,\delta)}, \tau_{(A,\delta)})$ for any object (A,δ),

(5) A full subcategory $\Omega^{\mathbf{Set}(\Omega)} \hookrightarrow \mathbf{Set}(\Omega)$ with objects $(\text{w-singl}(A,\delta), \varrho_{(A,\delta)})$ for any object (A,δ).

Theorem 1. *There is a functor $C\colon \mathbf{SetF}(\Omega) \to \mathbf{SetF}(\Omega)_{\mathrm{comp}}$ which is a quasi-reflector.*

P r o o f. Let (A,δ) be an object in $\mathbf{SetF}(\Omega)$. We show first that $(\mathrm{singl}(A,\delta),\tau)$ is a complete object. Let S be a singleton in $(\mathrm{singl}(A,\delta), \tau_{(A,\delta)})$. Then we define a map $e_S\colon A \to \Omega$ such that

$$e_S(x) = \bigvee_{t\in\mathrm{singl}(A,\delta)} t(x)\otimes S(t).$$

We show that $e_S \in \mathrm{singl}(A,\delta)$. It is easy to see that e_S is a normal extensional subobject. Moreover, we have

$$\begin{aligned}
&e_S(x)\otimes e_S(y) = \bigvee_{t,p\in\mathrm{singl}(A,\delta)} t(x)\otimes p(y)\otimes S(t)\otimes S(p)\\
&\leqslant \bigvee_{t,p\in\mathrm{singl}(A,\delta)} t(x)\otimes p(y)\otimes \tau(t,p) = \bigvee_{t,p\in\mathrm{singl}(A,\delta)} t(x)\otimes \Big(\bigvee_{a\in A} t(a)\otimes p(a)\otimes p(y)\Big)\\
&\leqslant \bigvee_{t,p\in\mathrm{singl}(A,\delta)} t(x)\otimes \Big(\bigvee_{a\in A} t(a)\otimes \delta(a,y)\Big) \leqslant \bigvee_{t\in\mathrm{singl}(A,\delta)} t(x)\otimes t(y)\leqslant \delta(x,y).
\end{aligned}$$

Then $S=\{e_S\}$. In fact, let $s\in\mathrm{singl}(A,\delta)$, then we have

$$\begin{aligned}
\{e_S\}(s) = \tau_{(A,\delta)}(e_S,s) &= \bigvee_{x\in A}\ \bigvee_{t\in\mathrm{singl}(A,\delta)} t(x)\otimes S(t)\otimes s(x)\\
&= \bigvee_{t\in\mathrm{singl}(A,\delta)} \Big(\bigvee_{x\in A} t(x)\otimes s(x)\Big)\otimes S(t)\\
&= \bigvee_{t\in\mathrm{singl}(A,\delta)} \tau_{(A,\delta)}(s,t)\otimes S(t) \geqslant \tau_{(A,\delta)}(s,s)\otimes S(s) = S(s),
\end{aligned}$$

and on the other hand since $\tau(s,t) \otimes S(t) \leqslant S(s)$, we obtain that $\{e_S\}(s) = S(s)$. We define a functor

$$C\colon \mathbf{SetF}(\Omega) \to \mathbf{SetF}(\Omega)_{\mathrm{comp}}$$

such that $C(A,\delta) = (\mathrm{singl}(A,\delta), \tau_{(A,\delta)})$ and for a morphism $f\colon (A,\delta) \to (B,\gamma)$ in $\mathbf{SetF}(\Omega)$ we set $C(f) = \overline{f}$, where $\overline{f}(s)(b) = \bigvee_{a \in A} s(a) \otimes \gamma(f(a), b)$. Then $\overline{f}$ is a morphism in $\mathbf{SetF}(\Omega)_{\mathrm{comp}}$. In fact it is clear that $\overline{f}(s)$ is a normal extensional subobject in (B,γ). For any $b, c \in B$ we have

$$\begin{aligned}
\overline{f}(s)(b) \otimes \overline{f}(s)(c) &= \bigvee_{x,y \in A} s(x) \otimes s(y) \otimes \gamma(f(x), b) \otimes \gamma(f(y), c) \\
&\leqslant \bigvee_{x,y \in A} \delta(x,y) \otimes \gamma(f(x), b) \otimes \gamma(f(x), c) \\
&\leqslant \bigvee_{x,y \in A} \gamma(f(x), f(y)) \otimes \gamma(f(x), b) \otimes \gamma(f(y), c) \\
&\leqslant \bigvee_{x,y \in A} \gamma(b, f(y)) \otimes \gamma(f(y), c) \leqslant \gamma(b,c).
\end{aligned}$$

Hence $\overline{f}(s)$ is a singleton in (B,γ). We show further that $\overline{f}\colon (\mathrm{singl}(A,\delta), \tau_{(A,\delta)}) \to (\mathrm{singl}(B,\gamma), \tau_{(B,\gamma)})$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, we have

$$\begin{aligned}
\tau_{(B,\gamma)}(\overline{f}(s), \overline{f}(t)) &= \bigvee_{b \in B} \overline{f}(s)(b) \otimes \overline{f}(t)(b) \\
&= \bigvee_{b \in B} \Big(\bigvee_{x \in A} s(x) \otimes \gamma(f(x), b) \Big) \otimes \Big(\bigvee_{y \in A} t(y) \otimes \gamma(f(y), b) \Big) \\
&\geqslant \bigvee_{b \in B} \bigvee_{x \in A} s(x) \otimes \gamma(f(x), b) \otimes t(x) \otimes \gamma(f(x), b) \\
&\geqslant \bigvee_{x \in A} s(x) \otimes \gamma(f(x), f(x)) \otimes t(x) \otimes \gamma(f(x), f(x)) \\
&= \bigvee_{x \in A} s(x) \otimes t(x) = \tau_{(A,\delta)}(s,t).
\end{aligned}$$

Let us now consider the singleton map

$$(A,\delta) \xrightarrow{\{-\}} C(A,\delta) = (\mathrm{singl}(A,\delta), \tau_{(A,\delta)}).$$

We show that C is a quasi-reflector. Since $\tau(\{x\},\{y\}) = \delta(x,y)$, it is clear that $\{-\}$ is a morphism in $\mathbf{SetF}(\Omega)$. Moreover let (B,γ) be a complete object and let

$f\colon (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{SetF}(\Omega)$. Then there exists a morphism $\tilde{f}\colon (\mathrm{singl}(A,\delta),\tau) \to (B,\gamma)$ such that the following diagram commutes:

$$\begin{tikzcd}
(A,\delta) \arrow[r, "\{-\}"] \arrow[d, "f"'] & (\mathrm{singl}(A,\delta), \tau_{(A,\delta)}) \arrow[d, "\tilde{f}"] \\
(B,\gamma) \arrow[r, equal] & (B,\gamma)
\end{tikzcd}$$

The map $\tilde{f}$ is defined as follows. Let $s \in \mathrm{singl}(A,\delta)$. Then $C(f)(s) = \overline{f}(s)\colon B \to \Omega$ is a singleton in (B,γ). Since (B,γ) is complete, there exists the unique element $b \in B$ such that $\overline{f}(s) = \{b\}$. We set $\tilde{f}(s) = b$. We show that $\tilde{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, let $\tilde{f}(s) = b$, $\tilde{f}(t) = c$. Then we have $\gamma(\tilde{f}(s), \tilde{f}(t)) = \gamma(b,c) = \tau_{(B,\gamma)}(\{b\},\{c\}) = \tau_{(B,\gamma)}(\overline{f}(s), \overline{f}(t)) \geqslant \tau_{(A,\delta)}(s,t)$ since $\overline{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. We show that the above mentioned diagram commutes. In fact, let $a \in A$, then we have $\tilde{f}(\{a\}) = b$, where $\overline{f}(\{a\}) = \{b\}$. But we have $\overline{f}(\{a\})(y) = \bigvee_{x\in A} \{a\}(x) \otimes \gamma(f(x),y) = \bigvee_{x\in A} \delta(a,x) \otimes \gamma(f(x),y) \geqslant \delta(a,a) \otimes \gamma(f(a),y) = \gamma(f(a),y) = \{f(a)\}(y)$. On the other hand we have $\overline{f}(\{a\})(y) \leqslant \bigvee_{x\in A} \gamma(f(a),f(x)) \otimes \gamma(f(x),y) \leqslant \bigvee_{x\in A} \gamma(f(a),y) = \gamma(f(a),y) = \{f(a)\}(y)$. Hence we have $\{b\} = \overline{f}(\{a\}) = \{f(a)\}$ and it follows that $b = f(a)$. $\square$

Theorem 2. *There exists a functor $D\colon \mathbf{SetF}(\Omega) \to \Omega_{\leftrightarrow}^{\mathbf{SetF}(\Omega)}$ which is a quasi-reflector.*

Proof. Let (A,δ) be an object of $\mathbf{SetF}(\Omega)$ and let $f\colon (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{SetF}(\Omega)$. We define a functor $D\colon \mathbf{SetF}(\Omega) \to \Omega_{\leftrightarrow}^{\mathbf{SetF}(\Omega)}$ such that

$$\begin{aligned}
&D(A,\delta) = (\Omega^{(A,\delta)}, \sigma_{(A,\delta)}), \quad D(f)\colon D(A,\delta) \to D(B,\gamma),\\
&(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad D(f)(s)(b) = \bigvee_{x\in A} s(x) \otimes \gamma(b, f(x)).
\end{aligned}$$

It is clear that this definition is correct.

Now let (A,δ) be an object in $\mathbf{SetF}(\Omega)$. We consider the map

$$(A,\delta) \xrightarrow{\{-\}} D(A,\delta) = (\Omega^{(A,\delta)}, \sigma_{(A,\delta)}).$$

We show that this map is a morphism in $\mathbf{SetF}(\Omega)$. In fact, for $x,y \in A$ we have

$$\begin{aligned}
\sigma_{(A,\delta)}(\{x\},\{y\}) &= \bigwedge_{a\in A} \delta(a,x) \leftrightarrow \delta(a,y)\\
&\geqslant \bigwedge_{a\in A} (\delta(a,x) \to \delta(x,y) \otimes \delta(a,x)) \wedge (\delta(a,y) \to \delta(a,y) \otimes \delta(x,y))\\
&\geqslant \delta(x,y).
\end{aligned}$$

On the other hand we have $\sigma(\{x\},\{y\}) \leqslant \delta(x,y)$ and it follows that $\sigma(\{x\},\{y\}) = \delta(x,y)$.

Finally, let $f\colon (A,\delta) \to (\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)})$ be a morphism in $\mathbf{SetF}(\Omega)$. Then there exists a morphism $\hat{f}$ such that the following diagram commutes.

$$\begin{tikzcd}
(A,\delta) \arrow[r, "\{-\}"] \arrow[d, "f"'] & D(A,\delta) \arrow[d, "\hat{f}"] \\
(\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)}) \arrow[r, equal] & (\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)})
\end{tikzcd}$$

The morphism $\hat{f}\colon \Omega^{(A,\delta)} \to \Omega^{(B,\gamma)}$ is defined as follows.

$$(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad \hat{f}(s)(b) = \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, s).$$

This definition is correct. In fact, we show first that $\hat{f}(s) \in \Omega^{(B,\gamma)}$. Let $b, c \in B$, then we have

$$\begin{aligned}
\hat{f}(s)(b) \otimes \gamma(b,c) &= \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, s) \otimes \gamma(b,c) \\
&\leqslant \bigvee_{x\in A} f(x)(c) \otimes \sigma_{(A,\delta)}(\{x\}, s) = \hat{f}(s)(c).
\end{aligned}$$

Further, $\hat{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, we have

$$\sigma_{(B,\gamma)}(\hat{f}(s), \hat{f}(t)) = \bigwedge_{b\in B} \hat{f}(s)(b) \leftrightarrow \hat{f}(t)(b).$$

According to Lemma 4 and Lemma 5 we have

$$\begin{aligned}
\hat{f}(s)(b) \leftrightarrow \hat{f}(t)(b) &= \left(\bigvee_{a\in A} f(a)(b) \otimes \sigma_{(A,\delta)}(\{a\}, s)\right) \leftrightarrow \left(\bigvee_{c\in A} f(c)(b) \otimes \sigma_{(A,\delta)}(\{c\}, t)\right) \\
&\geqslant \bigwedge_{x\in A} (f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, s) \leftrightarrow f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\}, t)) \\
&\geqslant \bigwedge_{a\in A} \sigma_{(A,\delta)}(\{a\}, s) \leftrightarrow \sigma_{(A,\delta)}(\{a\}, t) \\
&= \bigwedge_{a\in A} \left(\left(\bigwedge_{x\in A} \delta(a,x) \leftrightarrow s(x)\right) \leftrightarrow \left(\bigwedge_{y\in A} \delta(a,y) \leftrightarrow t(y)\right)\right) \\
&\geqslant \bigwedge_{a\in A} \bigwedge_{x\in A} (\delta(a,x) \leftrightarrow s(x)) \leftrightarrow (\delta(a,x) \leftrightarrow t(x)) \\
&\geqslant \bigwedge_{a\in A} \bigwedge_{x\in A} (\delta(a,x) \leftrightarrow \delta(a,x)) \otimes (s(x) \leftrightarrow t(x)) \\
&= \bigwedge_{x\in A} s(x) \leftrightarrow t(x) = \sigma_{(A,\delta)}(s,t).
\end{aligned}$$

It follows that $\sigma_{(B,\gamma)}(\hat{f}(s), \hat{f}(t)) \geqslant \sigma_{(A,\delta)}(s,t)$. We show that the diagram commutes. In fact, let $a \in A, b \in B$. Then

$$\begin{aligned}
\hat{f}(\{a\})(b) &= \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(A,\delta)}(\{x\},\{a\}) \\
&= \bigvee_{x\in A} f(x)(b) \otimes \delta(x,a) \leqslant \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(B,\gamma)}(f(x), f(a)) \\
&= \bigvee_{x\in A} f(x)(b) \otimes \Big(\bigwedge_{y\in B} f(x)(y) \leftrightarrow f(a)(y)\Big) \\
&= \bigvee_{x\in A}\bigwedge_{y\in B} f(x)(b) \otimes (f(x)(y) \leftrightarrow f(a)(y)) \leqslant \bigvee_{x\in A} f(x)(b) \otimes (f(x)(b) \leftrightarrow f(a)(b)) \\
&\leqslant \bigvee_{x\in A} f(x)(b) \otimes (f(x)(b) \to f(a)(b)) \leqslant f(a)(b).
\end{aligned}$$

On the other hand we have

$$\hat{f}(\{a\})(b) = \bigvee_{x\in A} f(x)(b) \otimes \delta(x,a) \geqslant f(a)(b).$$

Hence the diagram commutes and $D\colon \mathbf{SetF}(\Omega) \to \Omega_{\leftrightarrow}^{\mathbf{SetF}(\Omega)}$ is a quasi-reflector. $\square$

It should be observed that $\hat{f}$ is the smallest morphism for which the above diagram commutes. In fact, let $g\colon D(A,\delta) \to (\Omega^{(B,\gamma)}, \sigma_{(B,\gamma)})$ be a morphism in $\mathbf{SetF}(\Omega)$ such that the diagram commutes. Then for any $s \in \Omega^{(A,\delta)}, b \in B$ we have

$$\begin{aligned}
\hat{f}(s)(b) &= \bigvee_{a\in A} f(a)(b) \otimes \sigma_{(A,\delta)}(\{a\}, s) \\
&\leqslant \bigvee_{a\in A} f(a)(b) \otimes \sigma_{(B,\gamma)}(g(\{a\}), g(s)) = \bigvee_{a\in A} f(a)(b) \otimes \sigma_{(B,\gamma)}(f(a), g(s)) \\
&\leqslant \bigvee_{a\in A} f(a)(b) \otimes \bigwedge_{x\in B} f(a)(x) \to g(s)(x) \\
&\leqslant \bigvee_{a\in A} f(a)(b) \otimes (f(a)(b) \to g(s)(b)) \leqslant g(s)(b).
\end{aligned}$$

Theorem 3. *There exists a functor $E\colon \mathbf{SetF}(\Omega) \to \Omega_{\otimes}^{\mathbf{SetF}(\Omega)}$ which is a quasi-reflector.*

Proof. Let (A,δ) be an object in $\mathbf{SetF}(\Omega)$ and let $f\colon (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{SetF}(\Omega)$. We define a functor E such that

$$\begin{aligned}
&E(A,\delta) = (\Omega^{(A,\delta)}, \tau_{(A,\delta)}), \quad E(f)\colon E(A,\delta) \to E(B,\gamma), \\
&(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad D(f)(s)(b) = \bigvee_{x\in A} s(x) \otimes \gamma(b, f(x)).
\end{aligned}$$

It is clear that $\tau_{(A,\delta)}(s,t) \leqslant \tau_{(B,\gamma)}(E(f)(s), E(f)(t))$ for any $s,t \in \Omega^{(A,\delta)}$ and it follows that E is defined correctly. We consider a map

$$(A,\delta) \xrightarrow{\{-\}} (\Omega^{(A,\delta)}, \tau_{(A,\delta)})$$

such that $\{-\}(a)(b) = \tau_{(A,\delta)}(\{a\},\{b\})$. Since $\tau_{(A,\delta)}(\{a\},\{b\}) = \delta(a,b)$, the definition is correct.

Finally, let $f\colon (A,\delta) \to (\Omega^{(B,\gamma)}, \tau_{(B,\gamma)})$ be a morphism in $\mathbf{SetF}(\Omega)$. Then there exists a morphism $\hat{f}$ such that the following diagram commutes.

$$\begin{tikzcd}
(A,\delta) \arrow[r, "\{-\}"] \arrow[d, "f"'] & E(A,\delta) \arrow[d, "\hat{f}"] \\
(\Omega^{(B,\gamma)}, \tau_{(B,\gamma)}) \arrow[r, equal] & (\Omega^{(B,\gamma)}, \tau_{(B,\gamma)})
\end{tikzcd}$$

The morphism $\hat{f}\colon E(A,\delta) \to \Omega^{(B,\gamma)}$ is defined as follows.

$$(\forall s \in \Omega^{(A,\delta)})(\forall b \in B) \quad \hat{f}(s)(b) = \bigvee_{x\in A} f(x)(b) \otimes \tau_{(A,\delta)}(\{x\}, s) = \bigvee_{x\in A} f(x)(b) \otimes s(x).$$

We show that $\hat{f}$ is a morphism in $\mathbf{SetF}(\Omega)$. In fact, let $s,t \in \Omega^{(A,\delta)}$. Then we have

$$\begin{aligned}
\tau_{(B,\gamma)}(\hat{f}(s), \hat{f}(t)) &= \bigvee_{y\in B} \bigvee_{x,z\in A} f(x)(y) \otimes f(z)(y) \otimes s(x) \otimes t(z) \\
&= \bigvee_{x,z\in A} \left(\bigvee_{y\in B} f(x)(y) \otimes f(z)(y) \right) \otimes s(x) \otimes t(z) \\
&= \bigvee_{x,z\in A} \tau_{(B,\gamma)}(f(x), f(z)) \otimes s(x) \otimes t(z) \geqslant \bigvee_{x,z\in A} \delta(x,z) \otimes s(x) \otimes t(z) \\
&\geqslant \bigvee_{x\in A} s(x) \otimes t(x) = \tau_{(A,\delta)}(s,t).
\end{aligned}$$

The above mentioned diagram commutes. In fact, let $x \in A, b \in B$. Then we have

$$\begin{aligned}
\hat{f}(\{a\})(b) &= \bigvee_{x\in A} f(x)(b) \otimes \{a\}(x) \geqslant f(a)(b), \\
\hat{f}(\{a\})(b) &= \bigvee_{x\in A} f(x)(b) \otimes \delta(a,x) \\
&\leqslant \bigvee_{x\in A} f(x)(b) \otimes \tau_{(B,\gamma)}(f(x), f(a)) \\
&\leqslant \bigvee_{x\in A} f(x)(b) \otimes \sigma_{(B,\gamma)}(f(x), f(a)) \leqslant f(a)(b).
\end{aligned}$$

Hence, $E\colon \mathbf{SetF}(\Omega) \to \Omega_{\otimes}^{\mathbf{SetF}(\Omega)}$ is a quasi-reflector. $\square$

Theorem 4. *There exists a functor $G\colon \mathbf{Set}(\Omega) \to \Omega_{1,\otimes}^{\mathbf{Set}(\Omega)}$ which is a quasi-reflector.*

Proof. The functor G is defined by $G(A,\delta) = (\Omega_1^{(A,\delta)}, \tau_{(A,\delta)})$ which is considered as an object of a category $\mathbf{Set}(\Omega)$. Let $f\colon (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{Set}(\Omega)$. The morphism $G(f)$ will be defined later. First let us define a morphism $v_{(A,\delta)} = v\colon (A,\delta) \to G(A,\delta)$ in $\mathbf{Set}(\Omega)$ such that $v = F(\{-\})$, where $\{-\}\colon (A,\delta) \to (\Omega_1^{(A,\delta)}, \tau_{(A,\delta)})$ is a singleton morphism in $\mathbf{SetF}(\Omega)$ and F is the functor from Lemma 1. This definition is correct since for any $a, b \in A$ we have $\tau_{(A,\delta)}(\{a\},\{b\}) = \delta(a,b)$ as can easily be proved. Then for any object (B,γ) and any morphism $f\colon (A,\delta) \to (\Omega_1^{(B,\gamma)}, \tau_{(B,\gamma)})$ in $\mathbf{Set}(\Omega)$ there exists a morphism $\tilde{f}$ in $\mathbf{Set}(\Omega)$ such that the following diagram commutes.

$$\begin{tikzcd}
(A,\delta) \arrow[r, "v_{(A,\delta)}"] \arrow[d, "f"'] & G(A,\delta) \arrow[d, "\tilde{f}"] \\
(\Omega_1^{(B,\gamma)}, \tau_{(B,\gamma)}) \arrow[r, equal] & (\Omega_1^{(B,\gamma)}, \tau_{(B,\gamma)})
\end{tikzcd}$$

In fact, we set

$$(\forall s \in \Omega_1^{(A,\delta)})(\forall t \in \Omega_1^{(B,\gamma)}) \quad \tilde{f}(s,t) = \bigvee_{x\in A} f(x,t) \otimes v_{(A,\delta)}(x,s).$$

Then we have

$$\begin{aligned}
\bigvee_{t\in\Omega_1^{(B,\gamma)}} \tilde{f}(s,t) &= \bigvee_{t\in\Omega_1^{(B,\gamma)}} \bigvee_{x\in A} f(x,t) \otimes v(x,s) \\
&= \bigvee_{x\in A} \Bigl(\bigvee_{t\in\Omega_1^{(B,\gamma)}} f(x,t) \Bigr) \otimes v(x,s) = \bigvee_{x\in A} v(x,s) = \bigvee_{x\in A} \tau_{(A,\delta)}(\{x\},s) \\
&\geqslant \bigvee_{x\in A} s(x) = 1.
\end{aligned}$$

Further,

$$\begin{aligned}
\tilde{f}(s,t) \otimes \tilde{f}(s,t') &= \bigvee_{a,b\in A} f(a,t) \otimes f(b,t') \otimes v(a,s) \otimes v(b,s) \\
&\leqslant \bigvee_{a,b\in A} f(a,t) \otimes f(b,t') \otimes \delta(a,b) \leqslant \bigvee_{b\in A} f(b,t) \otimes f(b,t') \leqslant \tau_{(A,\delta)}(t,t').
\end{aligned}$$

The other properties of a morphism in $\mathbf{Set}(\Omega)$ can be proved analogously.

We show that the diagram commutes. In fact, let $a \in A, t \in \Omega_1^{(B,\gamma)}$. Then we have

$$\begin{aligned}
(\tilde{f} \circ v)(a,t) &= \bigvee_{s \in \Omega_1^{(A,\delta)}} v(a,s) \otimes \tilde{f}(s,t) \\
&= \bigvee_{s \in \Omega_1^{(A,\delta)}} v(a,s) \otimes \Big(\bigvee_{x \in A} f(x,t) \otimes v(x,s) \Big) \\
&\geqslant v(a,\{a\}) \otimes \bigvee_{x \in A} f(x,t) \otimes v(x,\{a\}) \\
&= \bigvee_{x \in A} f(x,t) \otimes v(x,\{a\}) \geqslant f(a,t),
\end{aligned}$$

and on the other hand we have

$$\begin{aligned}
\tilde{f} \circ v(a,t) &= \bigvee_{s \in \Omega_1^{(A,\delta)}} \bigvee_{x \in A} \tau_{(A,\delta)}(\{a\},s) \otimes \tau_{(A,\delta)}(\{x\},s) \otimes f(x,t) \\
&\leqslant \bigvee_{s \in \Omega_1^{(A,\delta)}} \bigvee_{x \in A} \tau_{(A,\delta)}(\{a\},\{x\}) \otimes f(x,t) = \bigvee_{x \in A} \delta(a,x) \otimes f(x,t) \leqslant f(a,t).
\end{aligned}$$

The morphism $G(f)\colon G(A,\delta) \to G(B,\gamma)$ will be defined by $G(f) = \widetilde{v_{(B,\gamma)} \circ f}$. Hence, more explicitly, we have

$$G(f)(s,t) = \bigvee_{x \in A} \bigvee_{y \in B} f(x,y) \otimes v_{(B,\gamma)}(y,t) \otimes v_{(A,\delta)}(x,s).$$

$\square$

Theorem 5. *There exists a functor $H\colon \mathbf{Set}(\Omega) \to \Omega^{\mathbf{Set}(\Omega)}$ which is a quasi-reflector.*

P r o o f. Let (A,δ) be an object in $\mathbf{Set}(\Omega)$. We set $H(A,\delta) = (\text{w-singl}(A,\delta), \varrho_{(A,\delta)})$. For an element $a \in A$ we define a morphism $[a]\colon (A,\delta) \to (\Omega, \leftrightarrow)$ in the category $\mathbf{Set}(\Omega)$ such that $[a](x,\alpha) = F(\{a\})(x,\alpha)$, for any $x \in A, \alpha \in \Omega$, where F is the functor from Lemma 1 and $\{a\}\colon (A,\delta) \to (\Omega,\leftrightarrow)$ is a morphism in the category $\mathbf{SetF}(\Omega)$ such that $\{a\}(x) = \delta(a,x)$. Then $[a]$ is a normal extensional subobject of (A,δ) in a category $\mathbf{Set}(\Omega)$. Since $[a](x,1) \otimes [a](y,1) \leqslant \delta(x,y)$ we obtain that $[a]$ is a weak singleton. Moreover, since $\varrho_{(A,\delta)}([a],[b]) = \delta(a,b)$ we obtain that $[-]\colon (A,\delta) \to (\text{w-singl}(A,\delta), \varrho_{(A,\delta)})$ is a morphism in $\mathbf{SetF}(\Omega)$.

We define a morphism $u = u_{(A,\delta)}\colon (A,\delta) \to H(A,\delta) = (\text{w-singl}(A,\delta), \varrho_{(A,\delta)})$ in $\mathbf{Set}(\Omega)$ such that $u(a,s) = F([-])(a,s)$ for all $a \in A, s \in \text{w-singl}(A,\delta)$, where F is the functor from Lemma 1.

Let $f\colon (A,\delta) \to (\text{w-singl}(B,\gamma), \varrho_{(B,\gamma)})$ be a morphism in $\mathbf{Set}(\Omega)$. Then we define a morphism $\tilde{f}\colon (\text{w-singl}(A,\delta), \varrho_{(A,\delta)}) \to (\text{w-singl}(B,\gamma), \varrho_{(B,\gamma)})$ by

$$\tilde{f}(s,t) = \bigvee_{x\in A} f(x,t) \otimes u(x,s),$$

for all $s \in \text{w-singl}(A,\delta), t \in \text{w-singl}(B,\gamma)$. Then $\tilde{f}$ is a morphism in $\mathbf{Set}(\Omega)$. In fact, we have

$$\begin{aligned}\bigvee_{t\in\text{w-singl}(B,\gamma)} \tilde{f}(s,t) &= \bigvee_{x\in A}\Bigg(\bigvee_{t\in\text{w-singl}(B,\gamma)} f(x,t)\Bigg) \otimes u(x,s)\\ &= \bigvee_{x\in A} u(x,s) = \bigvee_{x\in A}\bigwedge_{y\in A} \delta(x,y) \leftrightarrow s(y,1) \geqslant \bigvee_{x\in A} s(x,1) = 1.\end{aligned}$$

Moreover, the following diagram commutes.

$$\begin{tikzcd}
(A,\delta) \arrow[r, "u"] \arrow[d, "f"'] & (\text{w-singl}(A,\delta), \varrho_{(A,\delta)}) \arrow[d, "\tilde{f}"] \\
(\text{w-singl}(B,\gamma), \varrho_{(B,\gamma)}) \arrow[r, equal] & (\text{w-singl}(B,\gamma), \varrho_{(B,\gamma)})
\end{tikzcd}$$

In fact, we have

$$\begin{aligned}\tilde{f}\circ u(a,t) &= \bigvee_{s\in\text{w-singl}(A,\delta)} u(a,s)\otimes \tilde{f}(s,t)\\ &= \bigvee_{s\in\text{w-singl}(A,\delta)} u(a,s)\otimes \bigvee_{x\in A} u(x,s)\otimes f(x,t)\\ &\geqslant u(a,[a])\otimes \bigvee_{x\in A} u(x,[a])\otimes f(x,t)\\ &= \bigvee_{x\in A} u(x,[a])\otimes f(x,t) \geqslant f(a,t),\end{aligned}$$

and on the other hand

$$\tilde{f}\circ u(a,t) \leqslant \bigvee_{x\in A} \varrho_{(A,\delta)}([a],[x])\otimes f(x,t) = \bigvee_{x\in A} \delta(x,a)\otimes f(x,t) \leqslant f(a,t).$$

Let $f\colon (A,\delta) \to (B,\gamma)$ be a morphism in $\mathbf{Set}(\Omega)$. The morphism $H(f)$ will be defined as $u_{(B,\gamma)}\circ f$. □

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Author's address: Jiří Močkoř, University of Ostrava, Department of Mathematics and Institute for Research and Applications of Fuzzy Modeling, 30. dubna 22, 701 03 Ostrava 1, Czech Republic, e-mail: `Jiri.Mockor@osu.cz`.