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ULTRA LI-IDEALS IN LATTICE IMPLICATION ALGEBRAS AND MTL-ALGEBRAS

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Abstract. A mistake concerning the ultra LI-ideal of a lattice implication algebra is pointed out, and some new sufficient and necessary conditions for an LI-ideal to be an ultra LI-ideal are given. Moreover, the notion of an LI-ideal is extended to MTL-algebras, the notions of a (prime, ultra, obstinate, Boolean) LI-ideal and an ILI-ideal of an MTL-algebra are introduced, some important examples are given, and the following notions are proved to be equivalent in MTL-algebra: (1) prime proper LI-ideal and Boolean LI-ideal, (2) prime proper LI-ideal and ILI-ideal, (3) proper obstinate LI-ideal, (4) ultra LI-ideal.

Keywords: lattice implication algebra, MTL-algebra, (prime, ultra, obstinate, Boolean) LI-ideal, ILI-ideal

MSC 2000: 03G10, 06B10, 54E15

1. Introduction

In order to research a logical system whose propositional value is given in a lattice, Y. Xu proposed the concept of lattice implication algebras, and some researchers have studied their properties and the corresponding logic systems (see [15], [17]). In [7], Y. B. Jun et al. proposed the concept of an LI-ideal of a lattice implication algebra, discussed the relationship between filters and LI-ideals, and studied how to generate an LI-ideal by a set. In [11], K. Y. Qin et al. introduced the notion of ultra LI-ideals in lattice implication algebras, and gave some sufficient and necessary conditions for an LI-ideal to be ultra LI-ideal.

The interest in the foundations of fuzzy logic has been rapidly growing recently and several new algebras playing the role of the structures of truth-values have been

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introduced. P. Hájek introduced the system of basic logic (BL) axioms for the fuzzy propositional logic and defined the class of BL-algebras (see [4]). G. J. Wang proposed a formal deductive system L^* for fuzzy propositional calculus, and a kind of new algebraic structures, called R_0-algebras (see [13], [14]). F. Esteva and L. Godo proposed a new formal deductive system MTL, called the monoidal t-norm-based logic, intended to cope with left-continuous t-norms and their residual. The algebraic semantics for MTL is based on MTL-algebras (see [3], [5]). It is easy to verify that a lattice implication algebra is an MTL-algebra. Varieties of MTL-algebras are described in [10], and some other results concerning MTL-algebras are presented in [19] and [20].

This paper is devoted to a discussion of the ultra LI-ideals, we correct a mistake in [11] and give some new equivalent conditions for an LI-ideal to be ultra. We also generalize the notion of an LI-ideal to MTL-algebras, introduce the notions of a (prime, ultra, obstinate, Boolean) LI-ideal and an ILI-ideal of MTL- algebra, give some important examples, and prove that the following notions are equivalent in an MTL-algebra: (1) prime proper LI-ideal and Boolean LI-ideal, (2) prime proper LI-ideal and ILI-ideal, (3) proper obstinate LI-ideal, (4) ultra LI-ideal.

2. Preliminaries

Definition 2.1 ([17]). By a *lattice implication algebra* L we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with an order-reversing involution $'$ and a binary operation $\rightarrow$ satisfying the following axioms:

(I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,

(I2) $x \rightarrow x = 1$,

(I3) $x \rightarrow y = y' \rightarrow x'$,

(I4) $x \rightarrow y = y \rightarrow x = 1 \Longrightarrow x = y$,

(I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,

(L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$ for all $x, y, z \in L$.

We can define a partial ordering $\leqslant$ on a lattice implication algebra L by

$$x \leqslant y \quad \text{if and only if} \quad x \rightarrow y = 1.$$

For any lattice implication algebra L, $(L, \vee, \wedge)$ is a distributive lattice and the De Morgan law holds, that is

(L3) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(L4) $(x \wedge y)' = x' \vee y'$, $(x \vee y)' = x' \wedge y'$ for all $x, y, z \in L$.

Theorem 2.2 ([17]). *In a lattice implication algebra L, the following relations hold:*

(1) $0 \to x = 1$, $1 \to x = x$ and $x \to 1 = 1$,

(2) $x' = x \to 0$,

(3) $x \to y \leqslant (y \to z) \to (x \to z)$,

(4) $x \vee y = (x \to y) \to y$,

(5) $x \leqslant y$ implies $y \to z \leqslant x \to z$ and $z \to x \leqslant z \to y$,

(6) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(7) $x \to (y \wedge z) = (x \to y) \wedge (y \to z)$,

(8) $(x \to y) \vee (y \to x) = 1$,

(9) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$,

(10) $x \to (y \to z) = y \to (x \to z)$,

(11) $((x \to y) \to y) \to y = x \to y$.

From the above theorem it follows that lattice implication algebras are strictly connected with BCC-algebras and BCK-algebras of the form $(L, \to, 1)$ [2, 21].

For shortness, in the sequel the formula $(x \to y')'$ will be denoted by $x \otimes y$, the formula $x' \to y$ by $x \oplus y$.

Theorem 2.3 ([17]). *In a lattice implication algebra L, the relations*

(12) $x \otimes y = y \otimes x$, $x \oplus y = y \oplus x$,

(13) $x \otimes (y \otimes z) = (x \otimes y) \otimes z$, $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(14) $x \otimes x' = 0$, $x \oplus x' = 1$,

(15) $x \otimes (x \to y) = x \wedge y$,

(16) $x \to (y \to z) = (x \otimes y) \to z$,

(17) $x \leqslant y \to z \Longleftrightarrow x \otimes y \leqslant z$,

(18) $x \leqslant a$ and $y \leqslant b$ imply $x \otimes y \leqslant a \otimes b$ and $x \oplus y \leqslant a \oplus b$

hold for all $x, y, z \in L$.

Definition 2.4 ([7]). A subset A of a lattice implication algebra L is called an *LI-ideal* of L if

(LI1) $0 \in A$,

(LI2) $(x \to y)' \in A$ and $y \in A$ imply $x \in A$ for all $x, y \in L$.

An LI-ideal A of a lattice implication algebra L is said to be *proper* if $A \neq L$.

Theorem 2.5 ([7], [17]). *Let A be an LI-ideal of a lattice implication algebra L, then*

(LI3) $x \in A$, $y \leqslant x$ *imply* $y \in A$,

(LI4) $x, y \in A$ *imply* $x \vee y \in A$.

The least LI-ideal containing a subset A is called the *LI-ideal generated by A* and is denoted by $\langle A \rangle$.

Theorem 2.6 ([7], [17]). *If A is a non-empty subset of a lattice implication algebra L, then*

$$\langle A \rangle = \{x \in L \colon\ a_n' \to (\ldots \to (a_1' \to x')\ldots) = 1 \ \text{ for some } \ a_1, \ldots, a_n \in A\}.$$

Theorem 2.7 ([11]). *Let A be a subset of a lattice implication algebra L. Then A is an LI-ideal of L if and only if it satisfies* (LI3) *and*

(LI5) $x \in A$ *and* $y \in A$ *imply* $x \oplus y \in A$.

Theorem 2.8 ([11]). *If A is a non-empty subset of a lattice implication algebra L, then*

$$\langle A \rangle = \{x \in L \colon\ x \leqslant a_1 \oplus a_2 \oplus \ldots \oplus a_n \ \text{ for some } \ a_1, \ldots, a_n \in A\}.$$

Definition 2.9 ([9]). An LI-ideal A of a lattice implication algebra L is said to be *ultra* if for every $x \in L$, the following equivalence holds:

(LI6) $x \in A \Longleftrightarrow x' \notin A$.

Definition 2.10 ([9]). A non-empty subset A of a lattice implication algebra L is said to be an *ILI-ideal* of L if it satisfies (LI1) and

(LI7) $(((x \to y)' \to y)' \to z)' \in A$ and $z \in A$ imply $(x \to y)' \in A$ for all $x, y, z \in L$.

Theorem 2.11 ([9]). *If A is an LI-ideal of a lattice implication algebra L, then the following assertions are equivalent:*

(i) *A is an ILI-ideal of L,*

(ii) $((x \to y)' \to y)' \in A$ *implies* $(x \to y)' \in A$ *for all* $x, y, z \in L$,

(iii) $((x \to y)' \to z)' \in A$ *implies* $((x \to z)' \to (y \to z)')' \in A$ *for all* $x, y, z \in L$,

(iv) $(x \to (y \to x)')' \in A$ *implies* $x \in A$ *for all* $x, y, z \in L$.

Definition 2.12 ([6]). A proper LI-ideal A of a lattice implication algebra L is said to be a *prime* LI-ideal of L if $x \wedge y \in A$ implies $x \in A$ or $y \in A$ for any $x, y \in L$.

Theorem 2.13 ([9]). *Let A be a proper LI-ideal of a lattice implication algebra L. The following assertions are equivalent:*

(i) *A is a prime LI-ideals of L,*

(ii) *$x \wedge y = 0$ implies $x \in A$ or $y \in A$ for any $x, y \in L$.*

An LI-ideal of a lattice implication algebra L is called *maximal*, if it is proper and not a proper subset of any proper LI-ideal of L.

Theorem 2.14 ([9]). *In a lattice implication algebra L, any maximal LI-ideal must be prime.*

Theorem 2.15 ([9]). *Let L be a lattice implication algebra and A a proper LI-ideal of L. Then A is both a prime LI-ideal and an ILI-ideal of L if and only if $x \in A$ or $x' \in A$ for any $x \in L$.*

Theorem 2.16 ([9]). *Let L be a lattice implication algebra and A a proper LI-ideal. Then A is both a maximal LI-ideal and an ILI-ideal if and only if for any $x, y \in L$, $x \notin A$ and $y \notin A$ imply $(x \to y)' \in A$ and $(y \to x)' \in A$.*

Definition 2.17 ([1], [3]). A *residuated lattice* is an algebra $(L, \wedge, \vee, \otimes, \to, 0, 1)$ with four binary operations and two constants such that

(i) $(L, \wedge, \vee, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leqslant$),

(ii) $(L, \otimes, 1)$ is a commutative semigroup with the unit element 1, i.e., $\otimes$ is commutative, associative, $1 \otimes x = x$ for all x,

(iii) $\otimes$ and $\to$ form an adjoint pair, i.e., $z \leqslant x \to y$ if and only if $z \otimes x \leqslant y$ for all $x, y, z \in L$.

Definition 2.18 ([3]). A residuated lattice L is called an *MTL-algebra*, if it satisfies the pre-linearity equation: $(x \to y) \vee (y \to x) = 1$ for all $x, y \in L$. An MTL-algebra L is called an *$IMTL$-algebra*, if $(a \to 0) \to 0 = a$ for any $a \in L$.

In the sequel x' will be reserved for $x \to 0$, L for $(L, \wedge, \vee, \otimes, \to, 0, 1)$.

Proposition 2.19 ([3], [12]). *Let L be a residuated lattice. Then for all $x, y, z \in L$,*

(R1) $x \leqslant y \iff x \to y = 1$,

(R2) $x = 1 \to x$, $x \to (y \to x) = 1$, $y \leqslant (y \to x) \to x$,

(R3) $x \leqslant y \to z \iff y \leqslant x \to z$,

(R4) $x \to (y \to z) = (x \otimes y) \to z = y \to (x \to z)$,

(R5) $x \leqslant y$ *implies* $z \to x \leqslant z \to y$ *and* $y \to z \leqslant x \to z$, (R6) $z \to y \leqslant (x \to z) \to (x \to y)$, $z \to y \leqslant (y \to x) \to (z \to x)$,

(R7) $(x \to y) \otimes (y \to z) \leqslant x \to z$,

(R8) $x' = x'''$, $x \leqslant x''$,

(R9) $x' \wedge y' = (x \vee y)'$,

(R10) $x \vee x' = 1$ *implies* $x \wedge x' = 0$,

(R11) $\big(\bigvee_{i\in\Gamma} y_i\big) \to x = \bigwedge_{i\in\Gamma} (y_i \to x)$,

(R12) $x \otimes \big(\bigvee_{i\in\Gamma} y_i\big) = \bigvee_{i\in\Gamma} (x \otimes y_i)$,

(R13) $x \to \big(\bigwedge_{i\in\Gamma} y_i\big) = \bigwedge_{i\in\Gamma} (x \to y_i)$,

(R14) $\bigvee_{i\in\Gamma} (y_i \to x) \leqslant \big(\bigwedge_{i\in\Gamma} y_i\big) \to x$,

where Γ *is a finite or infinite index set and we assume that the corresponding infinite meets and joints exist in* L.

Proposition 2.20 ([3], [18]). *Let* L *be an* MTL*-algebra. Then for all* $x, y, z \in L$,

(M1) $x \otimes y \leqslant x \wedge y$,

(M2) $x \leqslant y$ *implies* $x \otimes z \leqslant y \otimes z$,

(M3) $y \to z \leqslant x \vee y \to x \vee z$,

(M4) $x' \vee y' = (x \wedge y)'$,

(M5) $(x \wedge y) \to z = (x \to z) \vee (y \to z)$,

(M6) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(M7) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(M8) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, *i.e., the lattice structure of* L *is distributive.*

Definition 2.21 ([3]). Let L be an MTL-algebra. A *filter* is a nonempty subset F of L such that

(F1) $x \otimes y \in F$ for any $x, y \in F$,

(F2) for any $x \in F$, $x \leqslant y$ implies $y \in F$.

Proposition 2.22 ([3]). *A subset* F *of an* MTL*-algebra* L *is a filter of* L *if and only if*

(F3) $1 \in F$,

(F4) $x \in F$ *and* $x \to y \in F$ *imply* $y \in F$.

3. Ultra LI-ideals of lattice implication algebras

In [11], the following result is presented: *Let A be a subset of a lattice implication algebra L, then A is an ultra LI-ideal of L if and only if A is a maximal proper LI-ideal of L.* The following example shows that this result is not true.

Example 3.1. Let $L = \{0, a, b, 1\}$ be a set with the Cayley table

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

For any $x \in L$, we have $x' = x \rightarrow 0$. The operations $\wedge$ and $\vee$ on L are defined as follows:

$$x \vee y = (x \rightarrow y) \rightarrow y, \quad x \wedge y = ((x' \rightarrow y') \rightarrow y')'.$$

Then $(L, \vee, \wedge, 0, 1)$ is a lattice implication algebra. It is easy to check that $\{0\}$ is a maximal proper LI-ideal of L, but not an ultra LI-ideal of L, because $a' = b \notin \{0\}$, but $a \notin \{0\}$.

Below, we give some new sufficient and necessary conditions for an LI-ideal to be an ultra LI-ideal.

Theorem 3.2. *Let L be a lattice implication algebra and A an LI-ideal of L. Then the following assertions are equivalent:*

(i) *A is an ultra LI-ideal,*

(ii) *A is a prime proper LI-ideal and an ILI-ideal,*

(iii) *A is a proper LI-ideal and $x \in A$ or $x' \in A$ for any $x \in L$,*

(iv) *A is a maximal ILI-ideal,*

(v) *A is a proper LI-ideal and for any $x, y \in L$, $x \notin A$ and $y \notin A$ imply $(x \rightarrow y)' \in A$ and $(y \rightarrow x)' \in A$.*

Proof. (i) $\Longrightarrow$ (ii): A is a proper LI-ideal, because $0 \in A$, and so $1 = 0' \notin A$.

We show that A is a prime LI-ideal. Assume $x \wedge y = 0$ for some $x, y \in L$. We prove that $x \in A$ or $y \in A$. If $x \notin A$ and $y \notin A$, then $x' \in A$ and $y' \in A$, by the definition of an ultra LI-ideal. So, by Theorem 2.5 (LI4), we have $x' \vee y' \in A$, thus $1 = 0' = (x \wedge y)' = x' \vee y' \in A$. This means that $A = L$, a contradiction. Therefore $x \wedge y = 0$ implies $x \in A$ or $y \in A$. So, by Theorem 2.13, A is a prime proper LI-ideal.

Now we show that A is an ILI-ideal. Let $((x \rightarrow y)' \rightarrow y)' \in A$. If $(x \rightarrow y)' \notin A$, then $x \rightarrow y \in A$ by the definition of an ultra LI-ideal. Since $y \leqslant x \rightarrow y$, we have

$y \in A$. From $((x \to y)' \to y)' \in A$ and $y \in A$, we conclude $(x \to y)' \in A$, by Definition 2.4 (LI2). This is a contradiction. Thus, $(x \to y)' \in A$. By Theorem 2.11 (ii), A is an ILI-ideal. This means that (ii) holds.

(ii) $\Longleftrightarrow$ (iii): See Theorem 2.15.

(iii) $\Longrightarrow$ (i): For any $x \in L$, if $x' \notin A$ then $x \in A$ by (iii). If $x \in A$, we prove that $x' \notin A$. Indeed, if $x' \in A$, then $x \oplus x' = 1 \in A$ by Theorem 2.3 (14) and Theorem 2.7 (LI5). This is a contradiction with the fact that A is a proper LI-ideal. This means that A is an ultra LI-ideal.

(iv) $\Longleftrightarrow$ (v): See Theorem 2.16.

(i) $\Longrightarrow$ (v): A is a proper LI-ideal, because $0 \in A$, and so $1 = 0' \notin A$.

If $x \notin A$, from $x \leqslant y \to x$ and Theorem 2.7 (LI3), we have $y \to x \notin A$. Thus, by the definition of an ultra LI-ideal, $(y \to x)' \in A$. Similarly, from $y \notin A$ we obtain $(x \to y)' \in A$. That is, (v) holds.

(v) $\Longrightarrow$ (i): By (v), $1 \notin A$. If $x' \notin A$, by (v) we have $(1 \to x')' \in A$, that is $x \in A$. If $x \in A$, then $x' \notin A$ (see (iii) $\Longrightarrow$ (i)). This means that A is an ultra LI-ideal. The proof is complete. □

Remind [11] that a subset A of a lattice implication algebra L has the *finite additive property* if $a_1 \oplus a_2 \oplus \ldots \oplus a_n \neq 1$ for any finite members $a_1, \ldots, a_n \in A$. $\langle A \rangle$ is a proper LI-ideal of L if and only if A has the finite additive property.

Our theorem proves that the part results formulated in Theorem 3.7 and Corollary 3.8 in [11] is correct. Namely we have

Theorem 3.3. *If a subset A of a lattice implication algebra L has the finite additive property, then there exists a maximal LI-ideal of L containing A. Every proper LI-ideal of a lattice implication algebra can be extended to a maximal LI-ideal.*

4. LI-ideals of MTL-algebras

Definition 4.1. A subset A of an MTL-algebra L is called an *LI-ideal* of L if $0 \in A$ and

(LI8) $(x' \to y')' \in A$ and $x \in A$ imply $y \in A$ for all $x, y \in L$.

Obviously, for a lattice implication algebra L, (LI8) coincides with (LI2). For a MTL-algebra it is not true because $x = x''$ is not true.

An LI-ideal A of an MTL-algebra L is said to be *proper* if $A \neq L$.

Lemma 4.2 ([17], Theorem 4.1.3). *A non-empty subset A of a lattice implication algebra L is a filter of L if and only if $A' = \{a' \colon a \in A\}$ is an LI-ideal of L.*

For MTL-algebras the above lemma is not true.

Example 4.3. Consider the set $L = \{0, a, b, c, d, 1\}$, where $0 < a < b < c < d < 1$, and two operations $\otimes$, $\to$ defined by the following two tables:

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	a	a
b	0	0	0	b	b	b
c	0	0	b	c	c	c
d	0	a	b	c	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	1	1	1	1
b	b	b	1	1	1	1
c	a	a	b	1	1	1
d	0	a	b	c	1	1
1	0	a	b	c	d	1

If we define on L the operations $\wedge$ and $\vee$ as min and max, respectively, then $(L, \wedge, \vee, \otimes, \to, 0, 1)$ will be an MTL-algebra. Obviously, $A = \{0, a, b, c, d, 1\}$ is a filter of L, but $A' = \{0, a, b, c, 1\}$ is not an LI-ideal of L, since

$$(0' \to d')' = 1 \in A \text{ and } 0 \in A, \quad \text{but } d \notin A.$$

Moreover, $B = \{1, c\}$ is not a filter of L, because $c \to d = 1 \in B$ and $c \in B$, $d \notin B$. By the following MATHEMATICA program, we can verify that $B' = \{0, a\}$ is an LI-ideal of L:

```
M1={{6,6,6,6,6,6},{4,6,6,6,6,6},{3,3,6,6,6,6},{2,2,3,6,6,6},
     {1,2,3,4,6,6},{1,2,3,4,5,6}};
a1=0;
For[i=1, i<7, i++, For[j=1, j<7, j++,
    If[(i==1||i==2) &&(M1[[M1[[M1[[i,1]],M1[[j,1]]]],1]]==1||
      M1[[M1[[M1[[i,1]],M1[[j,1]]]],1]]==2)&&(j!=1&&j!=2), a1++]]];
If[a1==0, Print[‘‘true’’], Print[‘‘false’’]]
```

From Example 4.3 we see that LI-ideals have a proper meaning in MTL-algebras.

Theorem 4.4. *Let A be an LI-ideal of an MTL-algebra L, then*
(LI3) *if $x \in A$, $y \leqslant x$, then $y \in A$,*
(LI9) *if $x \in A$, then $x'' \in A$,*
(LI4) *if $x, y \in A$, then $x \vee y \in A$.*

Proof. Assume $x \in A$, $y \leqslant x$. From $y \leqslant x$, by Proposition 2.19 (R5), we have $x \to 0 \leqslant y \to 0$, i.e., $x' \leqslant y'$. By Proposition 2.19 (R1), $x' \to y' = 1$. Then $(x' \to y')' = 1' = 0 \in A$ and $x \in A$, and by (LI8) we get $y \in A$. This means that (LI3) holds.

Suppose $x \in A$. By Proposition 2.19 (R8) we have $(x' \to (x'')')' = (x' \to x')' = 1' = 0 \in A$. Applying (LI8) we get $x'' \in A$, i.e., (LI9) holds.

Assume $x, y \in A$. By Proposition 2.19 (R2) we have $y' \leqslant x' \to y'$. So, $(x' \to y')' \leqslant y''$ by (R5). Whence, by $y \in A$ and (LI9), we obtain $y'' \in A$. From this and $(x' \to y')' \leqslant y''$, using (LI3) we get $(x' \to y')' \in A$. Thus

$$\begin{aligned}(x' \to (x \vee y)')' &= (x' \to (x' \wedge y'))' && \text{(by (R9))}\\ &= ((x' \to x') \wedge (x' \to y'))' && \text{(by (R13))}\\ &= (1 \wedge (x' \to y'))' && \text{(by (R1))}\\ &= (x' \to y')' \in A.\end{aligned}$$

From this and $x \in A$, using (LI8), we deduce $x \vee y \in A$, i.e., (LI4) holds.

The proof is complete. □

Definition 4.5. An LI-ideal A of an MTL-algebra L is said to be an *ILI-ideal* of L if it satisfies

(LI10) $(x \to (y \to x)')' \in A$ implies $x \in A$ for all $x, y, z \in L$.

Example 4.6. Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$, be a set with the Cayley tables:

$\otimes$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Defining the operations $\wedge$ and $\vee$ on L as min and max, respectively, we obtain an MTL-algebra $(L, \wedge, \vee, \otimes, \to, 0, 1)$ in which $A = \{0\}$ is an ILI-ideal of L.

In Example 4.3, $\{0, a\}$ is an LI-ideal, but it is not an ILI-ideal of L, because

$$(b \to (1 \to b)')' = 0 \in \{0, a\}, \quad \text{but} \quad b \notin \{0, a\}.$$

Theorem 4.7. *For each ILI-ideal A of an MTL-algebra L we have*

(LI11) *$x \wedge x' \in A$ for each $x \in L$.*

P r o o f. Indeed, for all $x \in L$ we get

$$\begin{aligned}
&((x \wedge x') \to (1 \to (x \wedge x'))')' \\
&\quad = ((x \wedge x') \to (x \wedge x')')' \\
&\quad = ((x \wedge x') \to (x' \vee x''))' && \text{(by Proposition 2.20 (M4))} \\
&\quad = (((x \wedge x') \to x') \vee ((x \wedge x') \to x''))' && \text{(by Proposition 2.20 (M7))} \\
&\quad = (1 \vee ((x \wedge x') \to x''))' && \text{(by Proposition 2.19 (R1))} \\
&\quad = 1' = 0 \in A.
\end{aligned}$$

From this, applying (LI10), we deduce (LI11). □

Definition 4.8. An LI-ideal A satisfying (LI11) is called *Boolean.*

Theorem 4.9. *If A is a Boolean LI-ideal of an MTL-algebra L, then*
(LI12) $(x \to x')' \in A$ *implies* $x \in A$.

P r o o f. According to the assumption $x \wedge x' \in A$ for all $x \in L$. Let $(x \to x')' \in A$. Then

$$\begin{aligned}
&((x \wedge x')' \to x')' \\
&\quad = (x \to (x \wedge x')'')' && \text{(by Proposition 2.19 (R4))} \\
&\quad = (x \to (x'' \wedge x'''))' && \text{(by Propositions 2.19 (R9) and 2.20 (M4))} \\
&\quad = ((x \to x'') \wedge (x \to x'''))' && \text{(by Proposition 2.19 (R13))} \\
&\quad = (1 \wedge (x \to x'))' && \text{(by Proposition 2.19 (R8), (R1))} \\
&\quad = (x \to x')' \in A.
\end{aligned}$$

Now, applying (LI8) we get $x \in A$, which completes the proof. □

Theorem 4.10. *For LI-ideals of MTL-algebras the conditions* (LI10) *are equivalent* (LI11).

P r o o f. (LI10) $\Longrightarrow$ (LI11): See Theorem 4.7.
(LI11) $\Longrightarrow$ (LI10): Let $(x \to (y \to x)')' \in A$. Then

$$\begin{aligned}
&((x \to (y \to x)')'' \to (x \to x')'')' \\
&\quad = ((x \to x')' \to (x \to (y \to x)')')' && \text{(by Proposition 2.19 (R4), (R8))} \\
&\quad \leqslant ((x \to (y \to x)') \to (x \to x'))' && \text{(by Proposition 2.19 (R6))} \\
&\quad \leqslant ((y \to x)' \to x')' && \text{(by Proposition 2.19 (R6))} \\
&\quad \leqslant (x \to (y \to x))' && \text{(by Proposition 2.19 (R6))} \\
&\quad = 1' = 0 \in A && \text{(by Proposition 2.19 (R2)).}
\end{aligned}$$

This, by (LI8), implies $(x \to x')' \in A$, whence, using (LI12), we obtain $x \in A$. So, (LI11) implies (LI10). □

Definition 4.11. A proper LI-ideal A of an MTL-algebra L is said to be a *prime* if $x \wedge y \in A$ implies $x \in A$ or $y \in A$ for any $x, y \in L$.

Theorem 4.12. *A proper LI-ideal A of a MTL-algebra L is prime if and only if for all $x, y \in L$ we have $(x \to y)' \in A$ or $(y \to x)' \in A$.*

P r o o f. Assume that an LI-ideal A of L is prime. Since

$$(x \to y)' \wedge (y \to x)' = ((x \to y) \vee (y \to x))' = 1' = 0 \in A$$

for all $x, y \in L$, the assumption on A implies $(x \to y)' \in A$ or $(y \to x)' \in A$.

Conversely, let A be a proper LI-ideal of L and let $x \wedge y \in A$. Assume that $(x \to y)' \in A$ or $(y \to x)' \in A$ for $x, y \in L$. If $(x \to y)' \in A$, then

$$\begin{aligned}
((x \wedge y)' \to x')' &= ((x' \vee y') \to x')' && \text{(by Proposition 2.20 (M4))}\\
&= ((x' \to x') \wedge (y' \to x'))' && \text{(by Proposition 2.19 (R11))}\\
&= (1 \wedge (y' \to x'))' && \text{(by Proposition 2.19 (R1))}\\
&= (y' \to x')' \leqslant (x \to y)' \in A && \text{(by Proposition 2.19 (R6)).}
\end{aligned}$$

So, $((x \wedge y)' \to x')' \in A$ (Theorem 4.4 (LI3)), which together with $x \wedge y \in A$ and the definition of an LI-ideal, gives $x \in A$.

Similarly, from $(y \to x)' \in A$ we can obtain $y \in A$.

This means that A is a prime LI-ideal of L. The proof is complete. □

Theorem 4.13. *Let A be an LI-ideal of an MTL-algebra L. Then A is both a prime LI-ideal and a Boolean LI-ideal of L if and only if $x \in A$ or $x' \in A$ for any $x \in L$.*

P r o o f. Assume that for all $x \in L$ we have $x \in A$ or $x' \in A$. At first we show that an LI-ideal A is prime. For this let $x \wedge y \in A$. If $x \notin A$, then $x' \in A$. Hence

$$\begin{aligned}
((x \wedge y)' \to y')' &= ((x' \vee y') \to y')' && \text{(by Proposition 2.20 (M4))}\\
&= ((x' \to y') \wedge (y' \to y'))' && \text{(by Proposition 2.19 (R11))}\\
&= ((x' \to y') \wedge 1)' && \text{(by Proposition 2.19 (R1))}\\
&= (x' \to y')' \leqslant (y \to x)' && \text{(by Proposition P2.19 (R6))}\\
&\leqslant x' \in A && \text{(by Proposition 2.19 (R2)).}
\end{aligned}$$

So, $((x \wedge y)' \to y')' \in A$, by Theorem 4.4 (LI3). From this, $x \wedge y \in A$ and Definition 4.1 we get $y \in A$. This proves that an LI-ideal A is prime. To prove that it is Boolean observe that $x \wedge x' \leqslant x'$ implies $x \wedge x' \leqslant x$, whence, by Theorem 4.4 (LI3), we obtain $x \wedge x' \in A$. Thus A is Boolean.

Conversely, if an LI-ideal A is both prime and Boolean, then by Definition 4.8, for all $x \in L$ we have $x \wedge x' \in A$. Hence $x \in A$ or $x' \in A$, by Definition 4.11. This completes the proof. □

Definition 4.14. An LI-ideal A of an MTL-algebra L is said to be *ultra* if for every $x \in L$

(LI6) $x \in A \Longleftrightarrow x' \notin A$.

It is easy to verify the following proposition is true.

Proposition 4.15. *Each ultra LI-ideal of an MTL-algebra is a proper LI-ideal.*

Definition 4.16. An LI-ideal A of an MTL-algebra L is said to be *obstinate* if for all $x, y \in L$

(LI13) $x \notin A$ and $y \notin A$ imply $(x \to y)' \in A$ and $(y \to x)' \in A$.

Theorem 4.17. *For an LI-ideal A of an MTL-algebra L the following conditions are equivalent:*

(i) *A is an ultra LI-ideal,*
(ii) *A is a proper LI-ideal and for any $x \in L$, $x \in A$ or $x' \in A$,*
(iii) *A is a prime proper LI-ideal and a Boolean LI-ideal,*
(iv) *A is a prime proper LI-ideal and an ILI-ideal,*
(v) *A is a proper obstinate LI-ideal.*

Proof. (i) $\Longrightarrow$ (ii): Obvious.

(ii) $\Longrightarrow$ (i): If $x' \notin A$, then $x \in A$, by (ii). Similarly, if $x \in A$, that must be $x' \notin A$. If not, i.e., if $x' \in A$, then, by Proposition 2.19 (R8), we have

$$(x' \to 1')' = (x' \to 0)' = x''' = x' \in A,$$

which together with $x \in A$ and (LI8) implies $1 \in A$. This, by Theorem 4.4 (LI3), gives $A = L$. This is a contradiction, because an LI-ideal A is proper. Obtained contradiction proves that $x \in A$ implies $x' \notin A$. So, A is an ultra LI-ideal.

(ii) $\Longleftrightarrow$ (iii): See Theorem 4.13.

(iv) $\Longrightarrow$ (iii): See Theorem 4.7.

(iii) $\Longrightarrow$ (iv): See Theorem 4.10.

(v) $\Longrightarrow$ (ii): Since A is a proper LI-ideal, $1 \notin A$. If $x \notin A$, then $(1 \to x)' = x' \in A$, by Definition 4.16. This means that (ii) holds.

(ii) $\Longrightarrow$ (v): It suffices to show that A is obstinate. Let $x \notin A$ and $y \notin A$. Then, according to (ii), we have $x' \in A$ and $y' \in A$. Thus

$$\begin{aligned}
(y'' \to (x \to y)'')' &= ((x \to y)' \to y''')' && \text{(by Proposition 2.19 (R4))} \\
&= ((x \to y)' \to y')' && \text{(by Proposition 2.19 (R8))} \\
&= (y \to (x \to y)'')' && \text{(by Proposition 2.19 (R4))} \\
&\leqslant (y \to (x \to y))' && \text{(by (R8), } x \to y \leqslant (x \to y)'' \text{ and (R5))} \\
&= 1' = 0 \in A && \text{(by Proposition 2.19 (R2)).}
\end{aligned}$$

This together with $y' \in A$ and Definition 4.1 implies $(x \to y)' \in A$.

Similarly, we obtain $(y \to x)' \in A$. So, A is obstinate.

The proof is complete. □

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