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VERTICES CONTAINED IN ALL MINIMUM PAIRED-DOMINATING SETS OF A TREE

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Abstract. A set S of vertices in a graph G is called a paired-dominating set if it dominates V and $\langle S\rangle$ contains at least one perfect matching. We characterize the set of vertices of a tree that are contained in all minimum paired-dominating sets of the tree.

Keywords: domination number, paired-domination number, tree

MSC 2000: 05C69, 05C35

1. Introduction

Graph theory terminology not presented here can be found in [1]. Let $G=(V,E)$ be a graph with $|V|=n$. The *neighborhood and closed neighborhood* of a vertex v in the graph G are denoted by $N(v)$ and $N[v]=N(v)\cup\{v\}$ respectively. For a set $X\subseteq V(G)$, let $N(X)=\bigcup\limits_{x\in X}N(x)$. The *minimum degree and maximum degree of the graph* G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. The graph induced by $S\subseteq V$ is denoted by $\langle S\rangle$. We denote the distance between two vertices u and v by $d(u,v)$. The degree of a vertex v of a graph G is denoted by $d_G(v)$, or simply by $d(v)$. A path on n vertices is denoted by P_n.

A set $S\subseteq V$ is a *dominating set* of G if every vertex $u\in V-S$ is adjacent to a vertex of S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A minimum dominating set of a graph G is called a $\gamma(G)$-set, or simply a γ-set, if the graph G is clear from the context. We use similar notation for other domination parameters.

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A set $S \subseteq V$ is a *total dominating set* if every vertex $u \in V$ is adjacent to a vertex of S. The *total domination number of* G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G.

A *paired-dominating set* S *with matching* M is a dominating set $S = \{v_1, v_2, \ldots, v_{2t-1}, v_{2t}\}$ with independent edge set $M = \{e_1, e_2, \ldots, e_t\}$, where each edge e_i joins two elements of S, that is, M is a perfect matching of $\langle S \rangle$. If $v_j v_k = e_i \in M$ we say that v_j and v_k are *paired* in S. Let $S_p = \{\{v_j, v_k\}\colon\ v_j$ and v_k are paired in $S\}$. The *paired-domination number* $\gamma_p(G)$ is the minimum cardinality of a paired-dominating set S in G.

We define the set $\psi(G)$ of a graph G by $\psi(G) = \{v \in V(G)\colon\ v$ is in every γ_p-set of $G\}$. For ease of presentation, we mostly consider *rooted trees*. For a vertex v in a (rooted) tree T, let $C(v)$ and $F(v)$ denote the set of children and descendants, respectively, of v. The maximal subtree at v is the subtree of T induced by $F(v) \cup \{v\}$, and is denoted by T_v. A *leaf* of T is a vertex of degree 1, while a *support vertex* of T is a vertex that is adjacent to a leaf. The set of leaves in T is denoted by $L(T)$ and the set of support vertices by $S(T)$. Let $L(v)$ denote the set of leaves in T_v distinct from v, i.e., $L(v) = F(v) \cap L(T)$. We define a *branch vertex* as a vertex of degree at least 3. The set of branch vertices of T is denoted by $B(T)$. For $j = 0, 1, 2, 3$, we define $L^j(v) = \{u \in L(v)\colon\ d(u, v) \equiv j (\operatorname{mod} 4)\}$. We sometimes write $L^j_T(v)$ to emphasize the tree (or subtree) concerned.

Paired-domination was introduced by Haynes and Slater[4] and is studied, for example, in [5]. For a survey of domination and variations, see the books by Haynes et al. [6], [7].

Hammer et al. [1] investigated vertices belonging to all or to no maximum stable sets of a graph. Mynhardt [2] characterized the set of vertices that are contained in all or in no minimum dominating sets of a tree. Cockayne et al. [3] characterized the set of vertices that are contained in all or in no minimum total dominating sets of a tree. In this paper, we characterize the set of vertices that are contained in all minimum paired-dominating sets of a tree.

2. Tree pruning

The technique of tree pruning was introduced by Cockayne et al. [3]. Let T denote an arbitrary tree. Given a vertex u of T, we say we *attach* a path of length q to u if we join u to a leaf of the path P_q.

Let v be a vertex of T that is not a support vertex. The pruning of T is performed with respect to the root. Hence, suppose T is rooted at v, i.e., $T = T_v$. If $d(u) \leqslant 2$ for each $u \in V(T_v) - \{v\}$, then let $\overline{T}_v = T$. Otherwise, let u be a branch vertex at maximum distance from v; note that $|C(u)| \geqslant 2$ and $d(x) \leqslant 2$ for each $x \in F(u)$. We now apply the following pruning process:

- If $|L^2(u)| \geqslant 1$, then delete $F(u)$ and attach a path of length 2 to u.
- If $|L^1(u)| \geqslant 1$, $|L^2(u)| = 0$ and $u \in S(T)$, then delete $F(u)$ and attach a path of length 1 to u.
- If $|L^1(u)| \geqslant 1$, $|L^2(u)| = 0$ and $u \notin S(T)$, then delete $F(u)$ and attach a path of length 5 to u.
- If $L^1(u) = L^2(u) = \emptyset$ and $|L^3(u)| \geqslant 1$, then delete $F(u)$ and attach a path of length 3 to u.
- If $L^1(u) = L^2(u) = L^3(u) = \emptyset$, then delete $F(u)$ and attach a path of length 4 to u.

This step of the pruning process, where all the descendants of u are deleted and a path of length $1, 2, 3, 4$, or 5 is attached to u to give a tree in which u has degree 2, is called a pruning of T_v at u. Repeat the above process until a tree $\overline{T}_v$ is obtained with $d(u) \leqslant 2$ for each $u \in V(\overline{T}_v) - \{v\}$. The tree $\overline{T}_v$ is unique and is called the pruning of T_v. To simplify notation, we write $\bar{L}^j(v)$ instead of $L^j_{\overline{T}_v}(v)$.

We shall prove the following two theorems:

Theorem 1. *Let T be a tree rooted at a vertex v such that $d(u) \leqslant 2$ for each $u \in V(T) - \{v\}$. Then $v \in \psi(T)$ if and only if v is a support vertex or $|L^1(v)| \geqslant 1$ and $|L^1(v) \cup L^2(v)| \geqslant 2$.*

Theorem 2. *Let v be a vertex of a tree T. Then $v \in \psi(T)$ if and only if v is a support vertex or $|\bar{L}^1(v)| \geqslant 1$ and $|\bar{L}^1(v) \cup \bar{L}^2(v)| \geqslant 2$.*

3. Preliminary results

It is obvious that the following lemma holds.

Lemma 1. *Let T be a tree with order $n \geqslant 3$. Then every vertex of $S(T)$ is in every minimum paired-dominating set.*

Lemma 2. *Let T be a tree with order $n \geqslant 3$ and $v \in L(T)$. Then there exists a γ_p-set S of T such that $v \notin S$.*

Proof. Suppose that v is in every γ_p-set of T. Let S be a γ_p-set of T. Then $v \in S$. Let u be the support vertex that is adjacent to v. Then $\{v, u\} \in S_p$. Since $n \geqslant 3$, we have $d(u) \geqslant 2$. If there exists a vertex $w \in N(u) \setminus \{v\}$ such that $w \notin S$, then $(S_p - \{\{v, u\}\}) \cup \{\{u, w\}\}$ is a γ_p-set of T that does not contain v, which is a contradiction. Hence, $w \in S$ for every vertex $w \in N(u) \setminus \{v\}$. Without loss of generality, say $t \in N(w) \setminus \{u\}$ and $\{w,t\} \in S_p$. If $t \in L(T)$, then $(S_p - \{\{v,u\},\{w,t\}\}) \cup \{\{u,w\}\}$ would be a paired-dominating set of T with cardinality less than $\gamma_p(T)$, which is a contradiction. So, $d(t) \geqslant 2$. If there exists a vertex $z \in N(t) \setminus \{w\}$ such that $z \notin S$, then $(S_p - \{\{v,u\},\{w,t\}\}) \cup \{\{u,w\},\{t,z\}\}$ is a γ_p-set of T that does not contain v, which is a contradiction. Hence, $z \in S$ for every vertex $z \in N(t) \setminus \{w\}$. Then $(S_p - \{\{v,u\},\{w,t\}\}) \cup \{\{u,w\}\}$ is a paired-dominating set of T with cardinality less than $\gamma_p(T)$, which is a contradiction. □

Lemma 3. *Let T' be a tree with $v, u' \in V(T')$ and $d(v,u') \geqslant 2$. Let T be the tree obtained from T' by attaching a path of length 4 to u'. Then*
(a) $\gamma_p(T) = \gamma_p(T') + 2$;
(b) *$v \in \psi(T')$ if and only if $v \in \psi(T)$.*

Proof. Suppose T is obtained from T' by adding the path u, x, y, z and the edge uu'.

(a) Let S be a γ_p-set of T'. Then $S_p \cup \{\{x,y\}\}$ is a paired-dominating set of T. So, $\gamma_p(T) \leqslant \gamma_p(T') + 2$.

By Lemma 2, let D be a γ_p-set of T that does not contain z. Let $D_p = \{\{v_j, v_k\} \colon v_j$ and v_k are paired in $S, v_i, v_j \in D$ }. Then $\{x,y\} \in D_p$. If $u \notin D$, then $D_p - \{\{x,y\}\}$ is a paired-dominating set of T'. Hence, $\gamma_p(T') \leqslant \gamma_p(T) - 2$. If $u \in D$, then $\{u,u'\} \in D_p$. Furthermore, there exists a vertex $t \in N(u') \setminus \{u\}$ such that $t \notin D$. Otherwise, $D_p - \{\{u,u'\}\}$ would be a paired-dominating set of T, which is a contradiction. Hence, $(D_p - \{\{x,y\},\{u,u'\}\}) \cup \{\{u',t\}\}$ is a paired-dominating set of T'. So, $\gamma_p(T') \leqslant \gamma_p(T) - 2$. Hence, $\gamma_p(T) = \gamma_p(T') + 2$.

(b) Suppose that $v \notin \psi(T')$. Let S' be a γ_p-set of T' that does not contain v. Then $S_p^{'} \cup \{\{x,y\}\}$ is a γ_p-set of T that does not contain v. Hence, $v \notin \psi(T)$.

Conversely, suppose that $v \in \psi(T')$. Let D be an arbitrary γ_p-set of T.

If $z \notin D$, then $\{x,y\} \in D_p$. In a similar way as above, if $u \notin D$, then $D_p - \{\{x,y\}\}$ is a γ_p-set of T'; if $u \in D$, then $(D_p - \{\{x,y\},\{u,u'\}\}) \cup \{\{u',t\}\}$ is a γ_p-set of T', where $t \in N(u') \setminus \{u\}$. Since $v \in \psi(T')$ and $v \neq t$, it follows that $v \in D$.

If $z \in D$, then $\{y,z\} \in D_p$. If $x \notin D$, then $(D_p - \{\{y,z\}\}) \cup \{\{x,y\}\}$ is a γ_p-set of T. In a similar way as above, we can prove that $v \in D$. If $x \in D$, then $\{x,u\} \in D_p$. Furthermore, $t \notin D$ for arbitrary vertex $t \in N[u'] \setminus \{u\}$. Otherwise, $(D_p - \{\{y,z\},\{x,u\}\}) \cup \{\{x,y\}\}$ would be a γ_p-set of T, which is a contradiction. Hence, $(D_p - \{\{y,z\},\{x,u\}\}) \cup \{\{u',t\}\}$ is a γ_p-set of T', where $t \in N(u') \setminus \{u\}$. Since $v \in \psi(T')$ and $v \neq t$, it follows that $v \in D$. Hence, $v \in \psi(T)$.

4. Proof of Theorem 1

If v is a support vertex, then Theorem 1 holds by Lemma 1. Hence we may assume that v is not a support vertex of T. If v is a leaf, then $v \notin \psi(T)$ by Lemma 2. For each $w \in L(v)$, if $d(v,w) \geqslant 5$, then let T^* be the tree obtained from T by replacing the $v-w$ path in T by a $v-w$ path of length j, $j = 4,5,2,3$ if $w \in L^i(v)$, $i = 0,1,2,3$. By repeated application of Lemma 3 it now follows that $v \in \psi(T)$ if and only if $v \in \psi(T^*)$.

To prove Theorem 1 we may therefore assume without loss of generality that $v \notin S(T)$, $d(v) \geqslant 2$ and every leaf of T is at distance $2,3,4$ or 5 from v. We consider the following cases.

Case 1: $|L^1(v)| \geqslant 2$.

Let u_5 and w_5 be two leaves at distance 5 from v in T with $P_u\colon\ v,u_1,\ldots,u_5$ and $P_w\colon\ v,w_1,\ldots,w_5$ the $v-u_5$ and $v-w_5$ paths, respectively. If there exists a γ_p-set S of T such that $v \notin S$, then $|S \cap V(P_u)| = 4$ and $|S \cap V(P_w)| = 4$. Without loss of generality, say $\{u_1,u_2\},\{u_3,u_4\} \in S_p$ and $\{w_1,w_2\},\{w_3,w_4\} \in S_p$. Then $(S_p - \{\{u_1,u_2\},\{w_1,w_2\}\}) \cup \{\{v,u_1\}\}$ is a paired-dominating set of T with cardinality less than $\gamma_p(T)$, which is a contradiction. Hence, $v \in \psi(T)$.

Case 2: $|L^1(v)| = 1$ and $|L^2(v)| \geqslant 1$.

In a similar way as Case 1, it is easy to prove that $v \in \psi(T)$.

Case 3: $|L^1(v)| = 1$ and $|L^2(v)| = 0$.

Let u_5 be the leaf at distance 5 from v in T with $P_u\colon\ v,u_1,\ldots,u_5$ the $v-u_5$ path. Then every leaf distinct from u_5 is at distance 3 or 4 from v. For any γ_p-set S of T, S contains every support vertex and at least one neighbor of every support vertex. In order to dominate u_1, two vertices are necessary. It follows that $\gamma_p(T) \geqslant 2|L(v)|+2$. On the other hand, $D^* = S(T) \cup (N(S(T)) \setminus L(T)) \cup \{u_1,u_2\}$ is a paired-dominating set of T with cardinality $2|L(v)|+2$, and so $\gamma_p(T) = 2|L(v)|+2$. Since $v \notin D^*$, it follows that $v \notin \psi(T)$.

Case 4: $|L^1(v)| = 0$ and $|L^2(v) \cup L^3(v)| \geqslant 1$.

Then every leaf is at distance 2, 3 or 4 from v. Let $A = N(L^2(v))$, $B = N(L^3(v) \cup L^0(v))$ and $C = N(B) \setminus (L^3(v) \cup L^0(v))$. Let S be an arbitrary γ_p-set of T. If there exists a vertex $u \in A$ such that u and v are paired, then w must be paired with its leaf for arbitrary vertex $w \in A \setminus \{u\}$. Since S contains every support vertex and at least one neighbor of every support vertex, it follows that $\gamma_p(T) \geqslant 2|L(v)|$. On the other hand, $D^* = L^2(v) \cup A \cup B \cup C$ is a paired-dominating set of T with cardinality $2|L(v)|$, and so $\gamma_p(T) = 2|L(v)|$. Since $v \notin D^*$, it follows that $v \notin \psi(T)$.

Case 5: $L^1(v) = L^2(v) = L^2(v) = \emptyset$.

In a similar way as Case 4, we can prove that $v \notin \psi(T)$.

5. Proof of theorem 2

For $1 \leqslant i \leqslant j \leqslant 5$, let $P\colon u_i, u_{i-1}, \ldots, u_1, w, z_1, z_2, \ldots, z_j$ be a path in a tree T_1 with $L(P) \subseteq L(T_1)$, $w \in V(P) \cap B(T_1)$ and $d(t) = 2$ for arbitrary vertex $t \in V(P) - (L(P) \cup \{w\})$. Assume $P_u\colon u_1, u_2, \ldots, u_i$ and $P_z\colon z_1, z_2, \ldots, z_j$. Let $v \in V(T_1) - V(P)$. For a set (to be defined) $X \subset V(P) - \{w\}$, let $T_2 = T_1 - X$.

Lemma 4. *If $j = 4$ and $X = V(P_z)$, then $v \in \psi(T_2)$ if and only if $v \in \psi(T_1)$.*

Proof. In a way similar to Lemma 3, we can prove that $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \notin \psi(T_2)$. Let S be a γ_p-set of T_2 that does not contain v. Then $S_p \cup \{\{z_2, z_3\}\}$ is a γ_p-set of T_1 that does not contain v. Hence, $v \notin \psi(T_1)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1.

If $z_4 \notin D$, then $\{z_2, z_3\} \in D_p$. If $z_1 \notin D$, then $D_p - \{\{z_2, z_3\}\}$ is a γ_p-set of T_2. Since $v \in \psi(T_2)$, it follows that $v \in D$. If $z_1 \in D$, then $\{w, z_1\} \in D_p$. Furthermore, $i \neq 2$. Otherwise, $\{u_1, u_2\} \in D_p$ and $(D_p - \{\{u_1, u_2\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. We consider the following cases.

Case 1: $i = 1$. Then $(D_p - \{\{z_2, z_3\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. Since $v \in \psi(T_2)$, it follows that $v \in D$.

Case 2: $i = 3$. Then $|D \cap V(P_u)| = 2$. If $u_1 \notin D$, then $(D_p - \{\{z_2, z_3\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. If $u_1 \in D$, then $\{u_1, u_2\} \in D_p$, and $(D_p - \{\{z_2, z_3\}, \{w, z_1\}, \{u_1, u_2\}\}) \cup \{\{w, u_1\}, \{u_2, u_3\}\}$ is a γ_p-set of T_2. Since $v \in \psi(T_2)$, it follows that $v \in D$.

Case 3: $i = 4$. Then $u_1 \notin D$. Otherwise, if $u_1 \in D$, then $\{u_1, u_2\} \in D_p$ and $\{u_3, u_4\} \in D_p$. So, $(D_p - \{\{u_1, u_2\}, \{u_3, u_4\}\}) \cup \{\{u_2, u_3\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $(D_p - \{\{z_2, z_3\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. Since $v \in \psi(T_2)$, it follows that $v \in D$.

Case 4: $i = 5$. Then $u_1 \notin D$. Otherwise, if $u_1 \in D$, then $|D \cap V(P_u)| = 4$. Without loss of generality, say $\{u_1, u_2\}, \{u_3, u_4\} \in D_p$. So, $D_p - \{\{u_1, u_2\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $(D_p - \{\{z_2, z_3\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. Since $v \in \psi(T_2)$, it follows that $v \in D$.

If $z_4 \in D$, then $\{z_3, z_4\} \in D_p$. If $z_2 \notin D$, then $(D_p - \{\{z_3, z_4\}\}) \cup \{\{z_2, z_3\}\}$ is a γ_p-set of T_1. In a way similar to the above, we can prove that $v \in D$. If $z_2 \in D$, then $\{z_1, z_2\} \in D_p$. Furthermore, $t \notin D$ for arbitrary vertex $t \in N[w] \setminus \{z_1\}$. Otherwise, $(D_p - \{\{z_1, z_2\}, \{z_3, z_4\}\}) \cup \{\{z_2, z_3\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $i \neq 1, 2$. If $i = 3$, then $\{u_2, u_3\} \in D_p$. So, $(D_p - \{\{z_1, z_2\}, \{z_3, z_4\}, \{u_2, u_3\}\}) \cup \{\{z_2, z_3\}, \{u_1, u_2\}\}$

is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. If $i = 4$, then $(D_p - \{\{z_1, z_2\}, \{z_3, z_4\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. Since $v \in \psi(T_2)$, it follows that $v \in D$. If $i = 5$, then $\{u_2, u_3\}, \{u_4, u_5\} \in D_p$. So, $(D_p - \{\{z_1, z_2\}, \{z_3, z_4\}, \{u_2, u_3\}, \{u_4, u_5\}\}) \cup \{\{z_2, z_3\}, \{w, u_1\}, \{u_3, u_4\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. $\square$

Lemma 5. *If $i = 2$ and $X = V(P_z)$, then $v \in \psi(T_2)$ if and only if $v \in \psi(T_1)$.*

P r o o f. We consider the following cases.

Case 1: $j = 1$. By Lemma 2, let S be a γ_p-set of T_2 that does not contain u_2. Then $\{w, u_1\} \in S_p$ and S is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2)$. Let D be a γ_p-set of T_1 that does not contain z_1. Then $w \in D$ and D is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1)$. Hence, $\gamma_p(T_1) = \gamma_p(T_2)$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. If $w \in S$, then S is a γ_p-set of T_1. Hence, $v \in S$. If $w \notin S$, then $\{u_1, u_2\} \in D_p$ and $(S_p - \{\{u_1, u_2\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. Then $w, u_1 \in D$. If $z_1 \in D$, then $\{w, z_1\} \in D_p$ and $\{u_1, u_2\} \in D_p$. So, $(D_p - \{\{u_1, u_2\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $z_1 \notin D$. Then D is a γ_p-set of T_2. Hence, $v \in D$. So, $v \in \psi(T_1)$.

Case 2: $j = 2$. Let S be a γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_2. Then $\{w, z_1\} \in D_p$ and $\{u_1, u_2\} \in D_p$. Furthermore, $(D_p - \{\{u_1, u_2\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. Then $S_p \cup \{z_1, z_2\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_2 \notin D$, then $\{w, z_1\} \in D_p$ and $\{u_1, u_2\} \in D_p$. Then $(D_p - \{\{u_1, u_2\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. So, $v \in D$. If $z_2 \in D$, then $\{z_1, z_2\} \in D_p$ and $D_p - \{\{z_1, z_2\}\}$ is a γ_p-set of T_2. So, $v \in D$. Therefore, $v \in \psi(T_1)$.

Case 3: $j = 3$. Let S be a γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_3. Then $\{z_1, z_2\} \in D_p$ and $D_p - \{\{z_1, z_2\}\}$ is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_3 \notin D$, then $\{z_1, z_2\} \in D_p$ and $D_p - \{\{z_1, z_2\}\}$ is a γ_p-set of T_2. So, $v \in D$.

If $z_3 \in D$, then $\{z_2, z_3\} \in D_p$. Furthermore, $z_1 \notin D$. Otherwise, $\{w, z_1\} \in D_p$, $\{u_1, u_2\} \in D_p$ and $(D_p - \{\{u_1, u_2\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Then $D_p - \{\{z_2, z_3\}\}$ is a γ_p-set of T_2. So, $v \in D$. Therefore, $v \in \psi(T_1)$.

Case 4: $j = 4$. By Lemma 4, Lemma 5 holds.

Case 5: $j = 5$. By Lemma 2, let S be a γ_p-set of T_2 that does not contain u_2. Then $\{w, u_1\} \in S_p$ and $S_p \cup \{\{z_3, z_4\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_5. Then $\{z_3, z_4\} \in D_p$. If $z_2 \in D$, then $\{z_1, z_2\} \in D_p$. So, $w \notin D$. Otherwise, $D_p - \{\{z_1, z_2\}\}$ would be a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $\{u_1, u_2\} \in D_p$. But $(D_p - \{\{u_1, u_2\}, \{z_1, z_2\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction.

Hence, $z_2 \notin D$. If $z_1 \in D$, then $\{w, z_1\}$, $\{u_1, u_2\} \in D_p$. So, $(D_p - \{\{u_1, u_2\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $z_1 \notin D$. So, $D_p - \{\{z_3, z_4\}\}$ is a paired-dominating set of T_2 and $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. If $w \in S$, then $S_p \cup \{\{z_3, z_4\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. If $w \notin S$, then $\{u_1, u_2\} \in S_p$. Then $(S_p - \{\{u_1, u_2\}\}) \cup \{\{w, u_1\}, \{z_3, z_4\}\}$ is a γ_p-set of T_1. So, $v \in S$. Therefore, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_5 \notin D$, then $\{z_3, z_4\} \in D_p$. In a way similar to the above, $D_p - \{\{z_3, z_4\}\}$ is a γ_p-set of T_2. Hence, $v \in D$. If $z_5 \in D$, then $\{z_4, z_5\} \in D_p$. If $z_3 \in D$, then $\{z_2, z_3\} \in D_p$. If $z_1 \notin D$, then $D_p - \{\{z_2, z_3\}, \{z_4, z_5\}\}$ is a paired-dominating set of T_2 with cardinality less than $\gamma_p(T_2)$, which is a contradiction. If $z_1 \in D$, then $\{w, z_1\} \in D_p$ and $(D_p - \{\{z_2, z_3\}, \{z_4, z_5\}\}) \cup \{\{z_3, z_4\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $z_3 \notin D$. Then $(D_p - \{\{z_4, z_5\}\}) \cup \{\{z_3, z_4\}\}$ is a γ_p-set of T_1. In a way similar to the above, we can prove that $v \in D$. So, $v \in \psi(T_1)$. □

Lemma 6. *If $i = 1$, $j = 1, 3, 5$ and $X = V(P_z)$, then $v \in \psi(T_2)$ if and only if $v \in \psi(T_1)$.*

P r o o f. We consider the following cases.

Case 1: $j = 1$. It is easy to prove that the lemma holds.

Case 2: $j = 3$. Let S be a γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_3. Then $\{z_1, z_2\} \in D_p$ and $D_p - \{\{z_1, z_2\}\}$ is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_3 \notin D$, then $\{z_1, z_2\} \in D_p$ and $D_p - \{\{z_1, z_2\}\}$ is a γ_p-set of T_2. So, $v \in D$. If $z_3 \in D$, then $\{z_2, z_3\} \in D_p$. If $z_1 \notin D$, then $D_p - \{\{z_2, z_3\}\}$ is a γ_p-set of T_2. So, $v \in D$. If $z_1 \in D$, then $\{w, z_1\} \in D_p$ and $(D_p - \{\{w, z_1\}, \{z_2, z_3\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. So, $v \in D$. Therefore, $v \in \psi(T_1)$.

Case 3: $j = 5$. Let S be a γ_p-set of T_2. Then $S_p \cup \{\{z_3, z_4\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_5. Then $\{z_3, z_4\} \in D_p$. If $z_2 \in D$, then $\{z_1, z_2\} \in D_p$ and $D_p - \{\{z_1, z_2\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $z_2 \notin D$. If $z_1 \in D$, then $\{w, z_1\} \in D_p$ and $(D_p - \{\{z_3, z_4\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a paired-dominating set of T_2. If $z_1 \notin D$, then $D_p - \{\{z_3, z_4\}\}$ is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. Then $S_p \cup \{\{z_3, z_4\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_5 \notin D$, then $\{z_3, z_4\} \in D_p$. In a way similar to the above, $D_p - \{\{z_3, z_4\}\}$ or $(D_p - \{\{z_3, z_4\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. Hence, $v \in D$.

If $z_5 \in D$, then $\{z_4, z_5\} \in D_p$. If $z_3 \in D$, then $\{z_2, z_3\} \in D_p$. If $z_1 \in D$, then $D_p - \{\{z_2, z_3\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. If $z_1 \notin D$, then $D_p - \{\{z_2, z_3\}, \{z_4, z_5\}\}$ is a paired-dominating set of T_2 with cardinality less than $\gamma_p(T_2)$, which is a contradiction. Hence, $z_3 \notin D$. Then $(D_p - \{\{z_4, z_5\}\}) \cup \{\{z_3, z_4\}\}$ is a γ_p-set of T_1. In a way similar to the above, we can prove that $v \in D$. So, $v \in \psi(T_1)$.

Lemma 7. *If $i = 3$, $j = 3$ and $X = V(P_z)$, then $v \in \psi(T_2)$ if and only if $v \in \psi(T_1)$.*

P r o o f. Let S be a γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_3. Then $\{z_1, z_2\} \in D_p$. If $w \in D$, then $D_p - \{\{z_1, z_2\}\}$ is a paired-dominating set of T_2. If $w \notin D$, then $|D \cap V(P_u)| = 2$. Without loss of generality, say $\{u_1, u_2\} \in D_p$. Then $D_p - \{\{z_1, z_2\}\}$ is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_3 \notin D$, then $\{z_1, z_2\} \in D_p$. In a way similar to the above, $D_p - \{\{z_1, z_2\}\}$ is a γ_p-set of T_2.

So, $v \in D$. If $z_3 \in D$, then $\{z_2, z_3\} \in D_p$. If $z_1 \notin D$, then $D_p - \{\{z_2, z_3\}\}$ is a γ_p-set of T_2. If $z_1 \in D$, then $\{w, z_1\} \in D_p$ and $|D \cap V(P_u)| = 3$. Without loss of generality, say $\{u_1, u_2\} \in D_p$. Then $(D_p - \{\{w, z_1\}, \{z_2, z_3\}, \{u_1, u_2\}\}) \cup \{\{w, u_1\}, \{u_2, u_3\}\}$ is a γ_p-set of T_2. So, $v \in D$. Therefore, $v \in \psi(T_1)$. □

Lemma 8. *If $i = 5$, $j = 3, 5$ and $X = V(P_z)$, then $v \in \psi(T_2)$ if and only if $v \in \psi(T_1)$.*

Proof. We consider the following cases.

Case 1: $j = 3$. Let S be a γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_3. Then $\{z_1, z_2\} \in D_p$. If $w \in D$, then $D_p - \{\{z_1, z_2\}\}$ is a paired-dominating set of T_2. If $w \notin D$, then $|D \cap V(P_u)| = 4$. Without loss of generality, say $\{u_1, u_2\}, \{u_3, u_4\} \in D_p$. Then $D_p - \{\{z_1, z_2\}\}$ is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. Then $S_p \cup \{\{z_1, z_2\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_3 \notin D$, then $\{z_1, z_2\} \in D_p$. In a way similar to the above, $D_p - \{\{z_1, z_2\}\}$ is a γ_p-set of T_2. So, $v \in D$. If $z_3 \in D$, then $\{z_2, z_3\} \in D_p$. If $z_1 \notin D$, then $D_p - \{\{z_1, z_2\}\}$ is a γ_p-set of T_2. If $z_1 \in D$, then $\{w, z_1\} \in D_p$. If $u_1 \in D$, then $|D \cap V(P_u)| = 4$. Without loss of generality, say $\{u_1, u_2\}, \{u_3, u_4\} \in D_p$. Then $D_p - \{\{u_1, u_2\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $u_1 \notin D$. Then $(D_p - \{\{w, z_1\}, \{z_2, z_3\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. So, $v \in D$. Therefore, $v \in \psi(T_1)$.

Case 2: $j = 5$. By Lemma 2, let S be a γ_p-set of T_2 that does not contain u_5. If $w \in S$, then $S_p \cup \{\{z_3, z_4\}\}$ is a paired-dominating set of T_1. If $w \notin S$, without loss of generality let us assume that $\{u_1, u_2\}$, $\{u_3, u_4\} \in D_p$. It follows that $(S_p - \{\{u_1, u_2\}\}) \cup \{\{w, u_1\}, \{z_3, z_4\}\}$ is a paired-dominating set of T_1. So, $\gamma_p(T_1) \leqslant \gamma_p(T_2) + 2$. Let D be a γ_p-set of T_1 that does not contain z_5. Then $\{z_3, z_4\} \in D_p$. If $z_2 \in D$, then $\{z_1, z_2\} \in D_p$. Furthermore, $w \notin D$, otherwise $D_p - \{\{z_1, z_2\}\}$ would be a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $|D \cap V(P_u)| = 4$. Without loss of generality, say $\{u_1, u_2\}, \{u_3, u_4\} \in D_p$. Then $(D_p - \{\{z_1, z_2\}, \{u_1, u_2\}\}) \cup \{\{w, u_1\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Hence, $z_2 \notin D$. If $z_1 \in D$, then $\{w, z_1\} \in D_p$. If $u_1 \in D$, then $|D \cap V(P_u)| = 4$. Without loss of generality, say $\{u_1, u_2\}, \{u_3, u_4\} \in D_p$. Then $D_p - \{\{u_1, u_2\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. If $u_1 \notin D$, then $(D_p - \{\{z_3, z_4\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a paired-dominating set of T_2. If $z_1 \notin D$, then $D_p - \{\{z_3, z_4\}\}$ is a paired-dominating set of T_2. So, $\gamma_p(T_2) \leqslant \gamma_p(T_1) - 2$. Hence, $\gamma_p(T_1) = \gamma_p(T_2) + 2$.

Suppose that $v \in \psi(T_1)$. Let S be an arbitrary γ_p-set of T_2. If $w \in S$, then $S_p \cup \{\{z_3, z_4\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. If $w \notin S$, then $|S \cap V(P_u)| = 4$. Without loss of generality, say $\{u_1, u_2\}, \{u_3, u_4\} \in S_p$. So, $(S_p - \{\{u_1, u_2\}\}) \cup \{\{w, u_1\}, \{z_3, z_4\}\}$ is a γ_p-set of T_1. Hence, $v \in S$. So, $v \in \psi(T_2)$.

Conversely, suppose that $v \in \psi(T_2)$. Let D be an arbitrary γ_p-set of T_1. If $z_5 \notin D$, then $\{z_3, z_4\} \in D_p$. In a way similar to the above, $D_p - \{\{z_3, z_4\}\}$ or $(D_p - \{\{z_3, z_4\}, \{w, z_1\}\}) \cup \{\{w, u_1\}\}$ is a γ_p-set of T_2. Hence, $v \in D$. If $z_5 \in D$, then $\{z_4, z_5\} \in D_p$. If $z_3 \in D$, then $\{z_2, z_3\} \in D_p$. Suppose that $z_1 \in D$. Then $D_p - \{\{z_2, z_3\}\}$ is a paired-dominating set of T_1 with cardinality less than $\gamma_p(T_1)$, which is a contradiction. Suppose that $z_1 \notin D$. Then $D_p - \{\{z_2, z_3\}, \{z_4, z_5\}\}$ is a paired-dominating set of T_2 with cardinality less than $\gamma_p(T_2)$, which is a contradiction. Hence, $z_3 \notin D$. Then $(D_p - \{\{z_4, z_5\}\}) \cup \{\{z_3, z_4\}\}$ is a γ_p-set of T_1. In a way similar to the above, we can prove that $v \in D$. So, $v \in \psi(T_1)$. □

By Theorem 1 and Lemmas 3–8, Theorem 2 holds.

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