Gen Qiang Wang; Sui-Sun Cheng

Even periodic solutions of higher order duffing differential equations

Czechoslovak Mathematical Journal, Vol. 57 (2007), No. 1, 331–343

Persistent URL: http://dml.cz/dmlcz/128174

Czechoslovak Mathematical Journal, 57 (132) (2007), 331–343

EVEN PERIODIC SOLUTIONS OF HIGHER ORDER DUFFING DIFFERENTIAL EQUATIONS

Genqiang Wang, Guangzhou, Sui Sun Cheng, Hsinchu

(Received June 6, 2005)

Abstract. By using Mawhin's continuation theorem, the existence of even solutions with minimum positive period for a class of higher order nonlinear Duffing differential equations is studied.

Keywords: high order Duffing equation, even periodic solution, continuation theorem

MSC 2000: 334K15, 34C25

1. Introduction

In [1], even and periodic solutions are found for the Duffing equation

$$x''(t) + g(x(t)) = p(t), \tag{1}$$

where g and p are real continuous functions defined on $\mathbb{R}$, g is Lipschitz, p is periodic with minimum period 2π and even, that is, $p(-t) = p(t)$ for $t \in \mathbb{R}$. A typical example of such an equation is

$$x'' + x^3 = 0.04\cos t,$$

which has been studied and many of its even and periodic solutions are observed numerically. Although there are a number of other studies which are related to even and periodic solutions (see e.g. [1]–[7]), this number is insignificant when compared with the large number of studies related to general periodic solutions.

In this paper, we consider a more general even order differential equations of the form

$$x^{(2k)}(t) + g(x(t)) = p(t), \tag{2}$$

where k is positive integer; g and p are real continuous functions defined on $\mathbb{R}$, p is periodic with minimum positive period T and even.

As mentioned in [1], the study of existence of periodic solutions of (1) is usually reduced to the one of existence of fixed points of the Poincaré mapping, and in the case of even and periodic solutions, it is reduced to the one of a simple boundary value problem in the phase plane. In our case, the concept of boundary value problem is still useful but since we are dealing with higher order equations, we find the phase plane analysis difficult. For this reason, we use Mawhin's continuation theorem and several sharp inequalities for finding even T-periodic solutions of (2).

For the sake of completeness, we first state Mawhin's continuation theorem [7] in the following manner. Let X and Y be two Banach spaces and $L\colon \operatorname{Dom} L \subset X \to Y$ is a linear mapping and $N\colon X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$, and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P\colon X \to X$ and $Q\colon Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$. It follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P}\colon (I-P)X \to \operatorname{Im} L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I-Q)N(\overline{\Omega})$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$ there exists an isomorphism $J\colon \operatorname{Im} Q \to \operatorname{Ker} L$.

Theorem A (Mawhin's continuation theorem [7]). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose that*

(i) *for each $\lambda \in (0,1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$ and*

(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$ and $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

2. Existence criteria

We will establish existence criteria based on combinations of the following conditions, where D and r are positive constants:

(a_1) $\operatorname{sgn}(x)g(x) > \max\limits_{0\leqslant t\leqslant T} |p(t)|$, for $|x| \geqslant D$,

(a_2) $\operatorname{sgn}(-x)g(x) > \max\limits_{0\leqslant t\leqslant T} |p(t)|$, for $|x| \geqslant D$,

(b) $\lim\limits_{x\to+\infty} \sup |g(x)/x| \leqslant r$.

Our main result is the following.

Theorem 1. *Suppose either one of the following sets of conditions holds:*

(i) (a_1) *and* (b), *or,*

(ii) (a_2) *and* (b).

Then for $r < 2^{3k}/T^{2k}$, (2) *has an even solution with minimum positive period* T.

We only give the proof for the case of (a_1) and (b), since the other case can be treated in a similar manner. Let Y be the Banach space of all real T-periodic and even continuous functions of the form $y = y(t)$ defined on $\mathbb{R}$ and endowed with the usual linear structure as well as the norm $\|y\|_0 = \max_{0\leqslant t\leqslant T} |y(t)|$. Let X be the Banach space of all real T-periodic, even and continuous $2k$ times continuously differentiable functions of the form $x = x(t)$ defined on $\mathbb{R}$ and endowed with the usual linear structure as well as the norm $\|x\|_1 = \max\{\|x\|_0, \|x'\|_0, \dots, \|x^{(2k-1)}\|_0\}$. Define the mappings $L\colon X \to Y$ and $N\colon X \to Y$ respectively by

$$Lx(t) = x^{(2k)}(t), \quad t \in \mathbb{R}, \tag{3}$$

and

$$Nx(t) = -g(x(t)) + p(t), \quad t \in \mathbb{R}. \tag{4}$$

Clearly,

$$\operatorname{Ker} L = \{x \in X\colon\ x(t) = c \in \mathbb{R}\} \tag{5}$$

and

$$\operatorname{Im} L = \left\{y \in Y\colon\ \int_0^T y(t)\,\mathrm{d}t = 0\right\} \tag{6}$$

is closed in Y. Thus L is a Fredholm mapping of index zero. Let us define $P\colon X \to X$ and $Q\colon Y \to Y/\operatorname{Im} L$ respectively by

$$Px(t) = x(0), \quad t \in \mathbb{R}, \tag{7}$$

for $x = x(t) \in X$ and

$$Qy(t) = \frac{1}{T}\int_0^T y(t)\,\mathrm{d}t, \quad t \in \mathbb{R} \tag{8}$$

for $y = y(t) \in Y$. It is easy to see that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I-Q)$. It follows that $L|_{\operatorname{Dom} L\cap \operatorname{Ker} P}\colon\ (I-P)X \to \operatorname{Im} L$ has an inverse which will be denoted by K_P. Furthermore for any $y = y(t) \in \operatorname{Im} L$, if $k = 1$, it is easy to check that

$$K_P y(t) = -\frac{t}{T}\int_0^T \mathrm{d}v \int_0^v y(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y(s)\,\mathrm{d}s, \tag{9}$$

while if $k > 1$, we let $y_0(t) = y(t)$, then

$$\begin{aligned} y_1(t) &= \Big(\frac{1}{2} - \frac{t}{T}\Big) \int_0^T \mathrm{d}v \int_0^v y_0(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y_0(s)\,\mathrm{d}s \\ &\quad - \frac{1}{T} \int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v y_0(s)\,\mathrm{d}s, \\ y_i(t) &= \Big(\frac{1}{2} - \frac{t}{T}\Big) \int_0^T \mathrm{d}v \int_0^v y_{i-1}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y_{i-1}(s)\,\mathrm{d}s \\ &\quad - \frac{1}{T} \int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v y_{i-1}(s)\,\mathrm{d}s, \end{aligned}$$

for $i = 1, 2, \dots, k-1$, and

$$y_k(t) = -\frac{t}{T} \int_0^T \mathrm{d}v \int_0^v y_{k-1}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y_{k-1}(s)\,\mathrm{d}s. \tag{10}$$

Indeed, let $x(t) \in \operatorname{Dom} L \cap \operatorname{Ker} P$ be such that $K_P y(t) = x(t)$. Then $x^{(2k)}(t) = y(t)$,

$$x^{(2k-1)}(t) = x^{(2k-1)}(0) + \int_0^t x^{(2k)}(s)\,\mathrm{d}s,$$

and

$$x^{(2k-2)}(t) = x^{(2k-2)}(0) + x^{(2k-1)}(0)t + \int_0^t \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s. \tag{11}$$

Since $x^{(2k-2)}(T) = x^{(2k-2)}(0)$, we have $x^{(2k-1)}(0)T + \int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s = 0$ or

$$x^{(2k-1)}(0) = -\frac{1}{T} \int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s.$$

By (11), we have

$$x^{(2k-2)}(t) = x^{(2k-2)}(0) - \frac{t}{T} \int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s. \tag{12}$$

If $k = 1$, then $x^{(2k-2)}(t) = x(t) \in \operatorname{Dom} L \cap \operatorname{Ker} P$, so $x^{(2k-2)}(0) = x(0) = 0$. From (12),

$$x^{(2k-2)}(t) = -\frac{t}{T} \int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s,$$

that is,

$$K_P y(t) = -\frac{t}{T}\int_0^T \mathrm{d}v \int_0^v y(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y(s)\,\mathrm{d}s.$$

If $k > 1$, since $\int_0^T x^{(2k-2)}(s)\,\mathrm{d}s = 0$, from (12) we have

$$x^{(2k-2)}(0)T - \frac{T}{2}\int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s + \int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s = 0,$$

or

$$x^{(2k-2)}(0) = \frac{1}{2}\int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s. \tag{13}$$

From (12) and (13),

$$\begin{aligned} x^{(2k-2)}(t) &= \frac{1}{2}\int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s - \frac{1}{T}\int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s \\ &\quad - \frac{t}{T}\int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s \\ &= \Big(\frac{1}{2} - \frac{t}{T}\Big)\int_0^T \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s \\ &\quad - \frac{1}{T}\int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v x^{(2k)}(s)\,\mathrm{d}s. \end{aligned} \tag{14}$$

Let $y_0(t) = y(t) = x^{(2k)}(t)$ and $y_1(t) = x^{(2k-2)}(t)$. Then from (14),

$$\begin{aligned} y_1(t) &= \Big(\frac{1}{2} - \frac{t}{T}\Big)\int_0^T \mathrm{d}v \int_0^v y_0(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y_0(s)\,\mathrm{d}s \\ &\quad - \frac{1}{T}\int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v y_0(s)\,\mathrm{d}s, \\ y_i(t) &= \Big(\frac{1}{2} - \frac{t}{T}\Big)\int_0^T \mathrm{d}v \int_0^v y_{i-1}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y_{i-1}(s)\,\mathrm{d}s \\ &\quad - \frac{1}{T}\int_0^T \mathrm{d}w \int_0^w \mathrm{d}v \int_0^v y_{i-1}(s)\,\mathrm{d}s, \end{aligned}$$

for $i = 1, 2, \ldots, k-1$, and

$$y_k(t) = y_k(0) - \frac{t}{T}\int_0^T \mathrm{d}v \int_0^v y_{k-1}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y_{k-1}(s)\,\mathrm{d}s.$$

Note that $y_k(t) = x(t) \in \operatorname{Dom} L \cap \operatorname{Ker} P$. Thus $y_k(0) = x(0) = 0$, and

$$y_k(t) = -\frac{t}{T}\int_0^T \mathrm{d}v \int_0^v y_{k-1}(s)\,\mathrm{d}s + \int_0^t \mathrm{d}v \int_0^v y_{k-1}(s)\,\mathrm{d}s.$$

Thus we have $K_P y(t) = y_k(t) \in \operatorname{Dom} L \cap \operatorname{Ker} P$.

Let Ω be an open and bounded subset of X. In view of (4), (8) and (9) (or (10)), we can easily see that $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I-Q)N(\overline{\Omega})}$ is compact. Thus the mapping N is L-compact on $\overline{\Omega}$. That is, we have the following result.

Lemma 1. *Let L, N, P and Q be defined by* (3), (4), (7) *and* (8) *respectively. Then L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$, where Ω is any open and bounded subset of X.*

Let

$$x^{(2k)}(t) + \lambda g(x(t)) = \lambda p(t), \tag{15}$$

where $\lambda \in (0,1)$.

Lemma 2. *Suppose the condition* (a_1) *is satisfied. Let $x(t)$ be any one solution of* (15) *in X. Then there is a $\xi_x \in [0,T]$ such that*

$$|x(\xi_x)| < D. \tag{16}$$

Proof. Let $x(t)$ be any one solution of (15) in X. From (15), we have

$$\int_0^T \{g(x(s)) - p(s)\}\,\mathrm{d}s = 0. \tag{17}$$

In view of the integral mean value theorem, we see that there is a $\xi_x \in [0,T]$ such that

$$g(x(\xi_x)) - p(\xi_x) = 0. \tag{18}$$

Thus

$$|g(x(\xi_x))| = |p(\xi_x)| \leqslant \|p\|_0. \tag{19}$$

From the condition (a_1) and (19) we see that

$$|x(\xi_x)| < D.$$

The proof is complete. □

Lemma 3. *Let $C_T^{(n)}$ be the set of all real T-periodic continuous n-times continuously differentiable functions which are defined on $\mathbb{R}$. Then for any $x \in C_T^{(n)}$, we have*

$$\sup_{0\leqslant t_1,\, t_2\leqslant T} |x^{(i)}(t_1) - x^{(i)}(t_2)| \leqslant \frac{1}{2}\int_0^T |x^{(i+1)}(s)|\,\mathrm{d}s, \quad i = 0, 1, \ldots, n-1, \tag{20}$$

where the constant factor $1/2$ is the best possible.

Proof. For any $x = x(t) \in C_T^{(n)}$ and any $t_1, t_2 \in [0, T]$, without loss of generality, we may assume that $t_2 \in [t_1, t_1 + T]$. By the fundamental theorem of Calculus, we get

$$x^{(i)}(t_2) - x^{(i)}(t_1) = \int_{t_1}^{t_2} x^{(i+1)}(s)\,\mathrm{d}s, \tag{21}$$

and

$$x^{(i)}(t_2) - x^{(i)}(t_1) = x^{(i)}(t_2) - x^{(i)}(t_1 + T) = -\int_{t_2}^{t_1+T} x^{(i+1)}(s)\,\mathrm{d}s. \tag{22}$$

From (21) and (22), we see that

$$x^{(i)}(t_2) - x^{(i)}(t_1) = \frac{1}{2}\biggl\{\int_{t_1}^{t_2} x^{(i+1)}(s)\,\mathrm{d}s - \int_{t_2}^{t_1+T} x^{(i+1)}(s)\,\mathrm{d}s\biggr\}. \tag{23}$$

It follows that

$$\sup_{0\leqslant t_1,\, t_2\leqslant T} |x^{(i)}(t_1) - x^{(i)}(t_2)| \leqslant \frac{1}{2}\int_{t_1}^{t_1+T} |x^{(i+1)}(s)|\,\mathrm{d}s = \frac{1}{2}\int_0^T |x^{(i+1)}(s)|\,\mathrm{d}s. \tag{24}$$

Now we assert that if α is a constant and $\alpha < \frac{1}{2}$, then there is $x \in C_T^{(n)}$ such that

$$\sup_{0\leqslant t_1, t_2\leqslant T} |x^{(i)}(t_1) - x^{(i)}(t_2)| > \alpha \int_0^T |x^{(i+1)}(s)|\,\mathrm{d}s. \tag{25}$$

Indeed, let $x(t) = \bigl(\frac{1}{2}T\pi\bigr)^i \sin\bigl(2\pi t/T - \frac{1}{2}i\pi\bigr)$. Then $x \in C_T^{(n)}$, $x^{(i)}(t) = \sin(2\pi t/T)$ and $x^{(i+1)}(t) = (2\pi/T)\cos(2\pi t/T)$. Furthermore,

$$\begin{aligned} \alpha\int_0^T |x^{(i+1)}(s)|\,\mathrm{d}s &= \alpha\frac{2\pi}{T}\int_0^T \Bigl|\cos\frac{2\pi}{T}t\Bigr|\,\mathrm{d}s = 4\alpha < 2 \\ &= \sup_{0\leqslant t_1,\, t_2\leqslant T} |x^{(i)}(t_1) - x^{(i)}(t_2)| \end{aligned} \tag{26}$$

as required. This shows that the constant $\frac{1}{2}$ in (20) is the best possible. The proof is complete. □

Lemma 4. *Let $x \in X$. Then for $j = 1, \ldots, k$, we have*

$$\|x^{(2j-1)}\|_0 \leqslant \frac{1}{4} \int_0^T |x^{(2j)}(s)| \,\mathrm{d}s, \tag{27}$$

where the constant factor $\frac{1}{4}$ is the best possible.

Proof. For any $x \in X$, since $x(t)$ is an even function, we see that $x^{(2j-1)}(t)$ is an odd function. Thus $\max\limits_{0 \leqslant t \leqslant T} x^{(2j-1)}(t) \geqslant 0$ and

$$\|x^{(2j-1)}\|_0 = \max_{0 \leqslant t \leqslant T} x^{(2j-1)}(t) = -\min_{0 \leqslant t \leqslant T} x^{(2j-1)}(t). \tag{28}$$

From Lemma 3 and (28), we see that

$$\begin{aligned} \|x^{(2j-1)}\|_0 &= \max_{0 \leqslant t \leqslant T} x^{(2j-1)}(t) \\ &= \frac{1}{2}\Big\{ \max_{0 \leqslant t \leqslant T} x^{(2j-1)}(t) - \min_{0 \leqslant t \leqslant T} x^{(2j-1)}(t) \Big\} \\ &\leqslant \frac{1}{2} \sup_{0 \leqslant t_1,\, t_2 \leqslant T} |x^{(2j-1)}(t_1) - x^{(2j-1)}(t_2)| \\ &\leqslant \frac{1}{4} \int_0^T |x^{(2j)}(s)| \,\mathrm{d}s. \end{aligned} \tag{29}$$

Now we assert that if β is a constant and $\beta < \frac{1}{4}$, then there is $x \in X$ such that

$$\|x^{(2j-1)}t\|_0 > \beta \int_0^T |x^{(2j)}(s)| \,\mathrm{d}s. \tag{30}$$

Indeed, let $x(t) = \cos \frac{2\pi}{T} t$, then

$$x^{(2j-1)}(t) = \Big(\frac{2\pi}{T}\Big)^{2j-1} \cos\Big(\frac{2\pi}{T} t + \frac{(2j-1)\pi}{2}\Big) \tag{31}$$

and

$$x^{(2j)}(t) = \Big(\frac{2\pi}{T}\Big)^{2j} \cos\Big(\frac{2\pi}{T} t + j\pi\Big), \tag{32}$$

so that

$$\begin{aligned} \beta \int_0^T |x^{(2j)}(s)| \,\mathrm{d}s &= \beta \Big(\frac{2\pi}{T}\Big)^{2j} \int_0^T \Big|\cos\Big(\frac{2\pi}{T} t + j\pi\Big)\Big| \,\mathrm{d}s \\ &= 4\beta \Big(\frac{2\pi}{T}\Big)^{2j-1} < \Big(\frac{2\pi}{T}\Big)^{2j-1} = \|x^{(2j-1)}\|_0 \end{aligned} \tag{33}$$

as required. This shows that the constant $\frac{1}{4}$ in (33) is the best possible. The proof is complete. □

Lemma 5. *For any $x \in X$ and any $\xi \in [0,T]$, we have*

$$\|x^{(2j-1)}\|_0 \leqslant \frac{T^{2(k-j)}}{2^{3(k-j)+2}} \int_0^T |x^{(2k)}(s)|\,\mathrm{d}s, \quad j = 1,2,\ldots,k, \tag{34}$$

$$\|x^{(2i)}\|_0 \leqslant \frac{T^{2(k-i)-1}}{2^{3(k-i)}} \int_0^T |x^{(2k)}(s)\|\,\mathrm{d}s, \quad i = 1,2,\ldots,k-1, \;\; k \geqslant 2, \tag{35}$$

and

$$\|x\|_0 \leqslant |x(\xi)| + \frac{T^{2k-1}}{2^{3k}} \int_0^T |x^{(2k)}(s)|\,\mathrm{d}s. \tag{36}$$

Proof. First, we prove (34). If $j = k$, then we see from Lemma 4 that (34) holds. If $j = k-1$ where $k \geqslant 2$, then by Lemma 4, we get

$$\|x^{(2j-1)}\|_0 = \|x^{(2(k-1)-1)}\|_0 \leqslant \frac{1}{4}\int_0^T |x^{(2(k-1))}(s)|\,\mathrm{d}s \leqslant \frac{T}{4}\|x^{(2(k-1))}\|_0. \tag{37}$$

Since $x^{(2(k-1)-1)}(0) = x^{(2(k-1)-1)}(T)$, there is $\xi_1 \in [0,T]$ such that $x^{(2(k-1))}(\xi_1) = 0$. It is easy to see from Lemma 3 that

$$\begin{aligned} \|x^{(2(k-1))}\|_0 &= \max_{0\leqslant t\leqslant T} |x^{(2(k-1))}(t) - x^{(2(k-1))}(\xi_1)| \\ &\leqslant \sup_{0\leqslant t_1,\, t_2\leqslant T} |x^{(2(k-1))}(t_1) - x^{(2(k-1))}(t_2)| \\ &\leqslant \frac{1}{2}\int_0^T |x^{(2k-1)}(s)|\,\mathrm{d}s. \end{aligned} \tag{38}$$

In view of (37), (38) and Lemma 4, we have

$$\begin{aligned} \|x^{(2j-1)}\|_0 &\leqslant \frac{T}{2^3}\int_0^T |x^{(2k-1)}(s)|\,\mathrm{d}s \\ &\leqslant \frac{T^2}{2^3}\|x^{(2k-1)}\|_0 \leqslant \frac{T^2}{2^5}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s \\ &= \frac{T^{2(k-j)}}{2^{3(k-j)+2}}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s. \end{aligned} \tag{39}$$

Similarly, by induction, we know that (34) holds.

Next, we prove (35). Similar to the derivation of (38), we may obtain

$$\|x^{(2i)}\|_0 \leqslant \frac{1}{2}\int_0^T |x^{(2i+1)}(s)|\,\mathrm{d}s \leqslant \frac{T}{2}\|x^{(2i+1)}\|_0. \tag{40}$$

From (34) and (40), we see that

$$\|x^{(2i)}\|_0 \leqslant \frac{T}{2}\|x^{(2i+1)}\|_0 \leqslant \frac{T}{2}\frac{T^{2(k-i-1)}}{2^{3(k-i-1)+2}}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s = \frac{T^{2(k-i)-1}}{2^{3(k-i)}}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s. \tag{41}$$

Thus (35) holds.

Finally, we prove (36). For any $x \in X$ and any $t, \xi \in [0, T]$, from Lemma 3 we know that

$$|x(t)| - |x(\xi)| \leqslant \sup_{0\leqslant t_1,t_2\leqslant T} |x(t_1) - x(t_2)| \leqslant \frac{1}{2}\int_0^T |x'(s)|\,\mathrm{d}s \leqslant \frac{T}{2}\|x'\|_0. \tag{42}$$

From (34) and (42), we have

$$\begin{aligned} \|x\|_0 &\leqslant |x(\xi)| + \frac{1}{2}T\|x'\|_0 \\ &\leqslant |x(\xi)| + \frac{1}{2}T\frac{T^{2(k-1)}}{2^{3(k-1)+2}}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s \\ &= |x(\xi)| + \frac{T^{2k-1}}{2^{3k}}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s. \end{aligned} \tag{43}$$

Thus (36) holds. The proof is complete. □

We now turn to the proof of Theorem 1. We first assert that there exist constants $M_0, M_1, \ldots, M_{2k-1}$ such that for any solution $x(t)$ of (15) in X,

$$\|x^{(i)}\|_0 \leqslant M_i, \quad i = 0, \ldots, 2k-1. \tag{44}$$

Indeed, in view of Lemma 2, we can find a $\xi \in [0, T]$ such that

$$|x(\xi)| \leqslant D. \tag{45}$$

From (36) and (45), we see that

$$\|x\|_0 \leqslant |x(\xi)| + \frac{T^{2k-1}}{2^{3k}}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s \leqslant D + \frac{T^{2k-1}}{2^{3k}}\int_0^T |x^{(2k)}(s)|\,\mathrm{d}s. \tag{46}$$

By the condition (b), given the constant $\varepsilon = \frac{1}{2}(2^{3k}/T^{2k} - r)$, there is constant $A_1 \geqslant D$ such that for $|x(t)| \geqslant A_1$,

$$|g(x(t))| \leqslant (r + \varepsilon)|x(t)|. \tag{47}$$

Let

$$C_0 = \max_{|x| \leqslant A_1} |g(x)|, \tag{48}$$

$$E_1 = \{t\colon\ t \in [0, T],\ |x(t)| < A_1\}, \tag{49}$$

and

$$E_2 = \{t\colon\ t \in [0, T], |x(t)| \geqslant A_1\}. \tag{50}$$

In view of (15), (46), (47), (48), (49) and (50), we have

$$\begin{aligned} \int_0^T |x^{(2k)}(s)| \, \mathrm{d}s &\leqslant \int_0^T |g(x(t))| \, \mathrm{d}t + \int_0^T |p(t)| \, \mathrm{d}t \\ &\leqslant \int_{E_1} |g(x(t))| \, \mathrm{d}t + \int_{E_2} |g(x(t))| \, \mathrm{d}t + \int_0^T |p(t)| \, \mathrm{d}t \\ &\leqslant (r + \varepsilon) T \|x\|_0 + C_0 T + T \|P\|_0 \\ &\leqslant (r + \varepsilon) T \left\{ D + \frac{T^{2k-1}}{2^{3k}} \int_0^T |x^{(2k)}(s)| \, \mathrm{d}s \right\} + C_0 T + T \|P\|_0 \\ &\leqslant \frac{T^{2k}}{2^{3k}} (r + \varepsilon) \int_0^T |x^{(2k)}(s)| \, \mathrm{d}s + C, \end{aligned} \tag{51}$$

for some positive constant C. Thus

$$\int_0^T |x^{(2k)}(s)| \, \mathrm{d}s \leqslant \sigma_2, \tag{52}$$

where $\sigma_2 = C/(1 - \sigma_1)$, $\sigma_1 = T^{2k} 2^{-3k} (r + \varepsilon)$. It is easy to see from (46) and (52) that

$$\|x\|_0 \leqslant D + \frac{T^{2k-1}}{2^{3k}} \int_0^T |x^{(2k)}(s)| \, \mathrm{d}s \leqslant M_0, \tag{53}$$

where $M_0 = D + T^{2k-1} 2^{-3k} \sigma_2$. By (34) and (52), we know that

$$\|x^{(2j-1)}\|_0 \leqslant \frac{T^{2(k-j)}}{2^{3(k-j)+2}} \int_0^T |x^{(2k)}(s)| \, \mathrm{d}s \leqslant M_{2j-1}, \quad j = 1, 2, \ldots, k, \tag{54}$$

where

$$M_{2j-1} = \frac{T^{2(k-j)}}{2^{3(k-j)+2}} \sigma_2.$$

In view of (35) and (52),

$$\|x^{(2i)}\|_0 \leqslant \frac{T^{2(k-i)-1}}{2^{3(k-i)}} \int_0^T |x^{(2k)}(s)|\,\mathrm{d}s \leqslant M_{2i}, \quad i = 1, 2, \ldots, k-1, \ k \geqslant 2 \tag{55}$$

where

$$M_{2i} = \frac{T^{2(k-i)-1}}{2^{3(k-i)}}\,\sigma_2.$$

From (53), (54) and (56), we see that (44) holds.

Now choose a positive number $\overline{D} > \max\limits_{0 \leqslant i \leqslant 2k-1} \{M_i\} + D$ and let

$$\Omega = \{x \in X \colon \ \|x\|_1 < \overline{D}\}.$$

From Lemma 1, we know that L is a Fredholm mapping of index zero and N is L-compact on $\overline{\Omega}$. In terms of (44), we see that for any $\lambda \in (0,1)$ and any $x \in \partial\Omega$, $Lx \neq \lambda Nx$. Since for any $x \in \partial\Omega \cap \operatorname{Ker} L$, $x = \overline{D}$ ($> D$) or $x = -\overline{D}$, in view of (a_1), we have

$$QNx = \frac{1}{T}\int_0^T (-g(x(t)) + p(t))\,\mathrm{d}s = \frac{1}{T}\int_0^T (-g(x) + p(t))\,\mathrm{d}s \neq 0. \tag{56}$$

In particular, we see that

$$\frac{1}{T}\int_0^T (-g(-\overline{D}) + p(t))\,\mathrm{d}s > 0 \quad \text{and} \quad \frac{1}{T}\int_0^T (-g(\overline{D}) + p(t))\,\mathrm{d}s < 0. \tag{57}$$

This shows that $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$. In view of Theorem A, there exists an even solution $x(t)$ of (2) which is T-periodic.

Next, we will prove that T is the minimum period of $x(t)$. Suppose to the contrary that there is a positive number $T_1 < T$ such that $x(t)$ is T_1-periodic. Then from (2), we see that

$$x^{(2k)}(t + T_1) + g(x(t + T_1)) = p(t + T_1). \tag{58}$$

By noting that $x(t+T_1) = x(t)$ and $x^{(2k)}(t+T_1) = x^{(2k)}(t)$, it is easy to see from (2) and (58) that for any $t \in \mathbb{R}$,

$$p(t + T_1) = p(t). \tag{59}$$

But this is contrary to our assumption on p. The proof is complete. □

Example. Consider the Duffing equation

$$x^{(4)}(t) + \frac{(x(t))^3}{49(1+(x(t))^2)} = \cos t. \tag{60}$$

Since $k = 1$, $g(x) = \frac{1}{6}x^3(1+x^2)$, $p(t) = \cos t$ and $T = 2\pi$. Taking $D > 0$ and $r = \frac{1}{49}$, it is easy to see that the conditions (a_1) and (b) are satisfied and $r = \frac{1}{49} < 2^6/(2\pi)^4$. Thus from Theorem 1 we see that (60) has an even solution which has minimum period T.

References

[1] *F. Nakajima*: Even and periodic solutions of the equation $u'' + g(u) = e(t)$. J. Diff. Equations *83* (1990), 277–299. Zbl 0693.34045

[2] *T. R. Ding*: Nonlinear oscillations at a point of resonance. Sci. Sinica Ser. A *25* (1982), 918–931. Zbl 0509.34043

[3] *P. Omari, P. Zanolin*: A note on nonliner oscillations at resonance. Acta Math. Sinica *3* (1987), 351–361. Zbl 0648.34040

[4] *X. K. Huang, Z. G. Xiang*: On the existence of 2π-periodic solution for delay Duffing equation $x''(t) + g(x(t-r)) = p(t)$. Chinese Science Bulletin *39* (1994), 201–203. Zbl 0810.34066

[5] *W. S. Loud*: Periodic solutions of nonlinear differential equation of Duffing types. In: Differential and Functional Equations. Benjami, New York, 1967, pp. 199–224. Zbl 0162.12302

[6] *W. B. Liu*: The Existence of periodic solutions for high order Duffing equations. Acta Math. Sinica *46* (2003), 49–56. Zbl 1036.34052

[7] *R. E. Gaines, J. L. Mawhin*: Coincidence Degree and Nonlinear Differential Equations. Lecture Notes in Math. Vol. 586. Springer-Verlag, Berlin, New York, 1977. Zbl 0339.47031

Authors' addresses: Genqiang Wang, Department of Computer Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510665, P. R. China; Sui Sun Cheng, Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China.